The complexity of function approximation on Sobolev spaces with bounded mixed derivative by linear Monte Carlo methods

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Abstract
We study the information-based complexity of the approximation problem on the multivariate Sobolev space with bounded mixed derivative $\mathcal{M}_{p,q}^{r}$ in the norm of $L_q$ by linear Monte Carlo methods. Applying the Maiorov’s discretization technique and some properties of pseudo-s-scale, we determine the exact orders of this problem for $1 < p, q < \infty$.

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1. Introduction

One of the most intensively disputed questions of computational mathematics is the following: What is the use of Monte Carlo methods, i.e., can it be of help to involve chance, randomness into numerical processes, and if yes, in which situations is this advisable? The first results about the analysis of efficiency of randomized (Monte Carlo) methods were due to Bakhvalov [2] in 1959, while an intensive wider research started only after the theory of information-based complexity [22] was established. In particular, recently many authors have investigated the complexity of

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the problems of approximation, quadrature formulae, approximate solutions of differential and integral equations in the randomized setting [6–10,12–15,17,22].

Information-based complexity provides the notions to give a precise mathematical meaning to the efficiency issues both for deterministic and randomized methods, and the complexity of some basic numerical problems can be determined by this theory. In this way some comparison between the worst case setting and the randomized settings becomes possible.

A basic problem of information-based complexity theory is to compute asymptotic degrees of complexity on various numerical problems, and give an optimal algorithm in different computational settings. In particular, Traub, Wasilkowski, Woźniakowski [22], Novak [14], Mathé [12,13] and Heinrich [8,9] studied the approximation problem on the classical multivariate Sobolev space \( W^p_\alpha([0,1]^d) \) in the norm of \( L_q([0,1]^d) \), \( 1 \leq p, q \leq \infty \) by Monte Carlo methods, and they found that the optimal errors in the randomized setting depend exponentially on the number \( d \) of variables as those in the deterministic setting.

In this paper we are mainly concerned with the approximation problem on the multivariate Sobolev space of periodic functions with bounded mixed derivative \( MW^{p,\alpha}([\mathbb{T}^d]) \) in the norm of \( L_q(\mathbb{T}^d) \), \( 1 < p, q < \infty \) by linear randomized (Monte Carlo) methods. This problem is crucial for numerical analysis, since it includes, in particular, interpolation and optimal recovery of functions, and also has close relation to some more complicated numerical problems such as approximate solutions of integral and differential equations. Moreover, it is also a typical problem in the research fields of the information-based complexity theory [3,22].

The paper is organized as follows: In section 2, we provide the necessary notions of approximation theory and information-base complexity and state our main result. In section 3, using the Maiorov’s discretization technique, we give the proof of our main result.

2. Main result

First, we recall some fundamental notions from the information-based complexity theory [9,22], and start with the deterministic setting. Given Banach spaces \( X \) and \( Y \), let \( L(X, Y) \) denote the space of all bounded linear operators from \( X \) to \( Y \). Let \( S \) be a continuous (possibly nonlinear) mapping from a closed bounded subset \( X_0 \) of a Banach space \( X \) to a Banach space \( Y \). Here \( X_0 \) is interpreted as the set of problem elements, that is, the collection of problem instances for which we want to solve the given numerical problem. Throughout this paper, we assume that \( X_0 = B_X \) is the unit ball of \( X \), and \( S \) is an operator mapping the problem elements to the exact solution of the problem which is usually called a solution operator. We seek to approximate \( S \) by mappings of the form

\[
u = \varphi \circ N,
\]

where

\[
N : X_0 \to \mathbb{R}^k, \quad \varphi : N(X_0) \to Y.
\]

\( N \) and \( \varphi \) describe a numerical method. The mapping \( N \), called information operator, stands for the process of gaining information about \( f \in X_0 \) (e.g., computing values of the function \( f \) at certain points or Fourier coefficients of \( f \)), and the mapping \( \varphi \), called algorithm, represents the computational process in the way that \( \varphi(N(f)) \) is the outcome of the numerical operations performed on \( N(f) \) in order to obtain an approximation to \( S(f) \). The natural number \( k \) is called the cardinality of a method \( u \). Since we are only concerned with linear methods, we assume that
\( N \) is a continuous linear mapping from \( X_0 \) to \( \mathbb{R}^k \) and \( \phi \) is a linear mapping from \( \mathbb{R}^k \) to \( Y \). Let \( A_k(X_0, Y) \) denote the set of linear methods with cardinality \( k \), and let \( A^n(X_0, Y) \) and \( A(X_0, Y) \) denote the sets given by

\[
A^n(X_0, Y) = \bigcup_{k=0}^{n} A_k(X_0, Y)
\]

and

\[
A(X_0, Y) = \bigcup_{n \in \mathbb{N}} A^n(X_0, Y),
\]

respectively.

The worst case error of any single linear method \( u \in A(X_0, Y) \) is measured by

\[
e_{\text{wor}}(S, u, X, Y) := \sup \{ \| S(f) - u(f) \|_Y : f \in B_X \}.
\]

Minimizing the errors with respect to the choice of methods within the given class, we get the \( n \)-th approximation number of \( S \)

\[
a_n(S, X, Y) := \inf \{ e_{\text{wor}}(S, u, X, Y) : u \in A^{n-1}(X_0, Y) \},
\]

for any \( n \in \mathbb{N} \).

Next we pass to the setting of randomization, or Monte Carlo methods. As compared to deterministic procedures, the randomized methods, and hence also the approximation results, depend on chance, or on a random parameter.

**Definition 1.** The \( n \)-th Monte Carlo approximation number of \( S \) is defined as

\[
a^{MC}_n(S, X, Y) := \inf_{f \in B_X} \left( \int_{\Omega} \| S(f) - u_\omega(f) \|_Y^2 \ d\mu(\omega) \right)^{1/2},
\]

where the infimum is taken over all measure spaces \((\Omega, \Sigma, \mu)\) and families \( u_\omega \in A(X_0, Y) \) \((\omega \in \Omega)\) with the properties:

(i) The mapping \((f, \omega) \mapsto u_\omega(f)\) is \((B(X_0) \times \Sigma, B(Y))\) measurable, and

(ii) there is a measurable mapping \( k : \Omega \mapsto \mathbb{N} \) such that \( u_\omega \in A^{k(\omega)}(X_0, Y) \) for all \( \omega \in \Omega \) and

\[
\int_{\Omega} k(\omega) \ d\mu(\omega) \leq n - 1,
\]

where \( B(X_0), B(Y) \) denote the \( \sigma \)-algebras of Borel subsets of \( X_0 \) and \( Y \). Thus \( a^{MC}_n(S, X, Y) \) describes the best possible error of a linear randomized approximation to \( S \) which uses not more than \( n - 1 \) information functionals.

It is clear that if \( S \in L(X, Y) \) then the following inequality holds

\[
a^{MC}_n(S, X, Y) \leq a_n(S, X, Y).
\]  \(1\)

In this paper, we study the approximation problem in which a solution operator \( S \) is an embedding operator \( I \) from \( X \) to \( Y \). Note that the notation \( I \) will stand for different embedding operators and can be determined according to the context.
Now we introduce the space of functions $MW_{p,\varpi}^r$ which will be studied in this paper. Denote by $L_p(\mathbb{T}^d)$, $1 < p < \infty$, the classical space of $p$-th powers integrable $2\pi$-periodic functions defined on the $d$-dimensional torus $\mathbb{T}^d := [0, 2\pi)^d$ with the usual norm $\| \cdot \|_p := \| \cdot \|_{L_p(\mathbb{T}^d)}$. Consider the Bernoulli kernel

$$F_r(x, \varpi) = 1 + 2 \sum_{k=1}^{\infty} k^{-r} \cos \left( kx - \frac{\varpi}{2} \right), \quad r > 0, \quad \varpi \in \mathbb{R},$$

which converges in the sense of $L_1$ and let

$$F_r(x, \varpi) = \prod_{j=1}^{d} F_{r_j}(x_j, \varpi_j), \quad \varpi = (\varpi_1, \ldots, \varpi_d),$$

be its multidimensional analogue, where the vector $\varpi = (r_1, \ldots, r_d)$ has the form

$$0 < r = r_1 = \cdots = r_v < r_{v+1} \leq \cdots \leq r_d, \quad 1 \leq v \leq d.$$

We associate the vector $\gamma = (\gamma_1, \ldots, \gamma_d)$, $\gamma_j = r_j/r$, $j = 1, \ldots, d$ with the vector $\varpi = (r_1, \ldots, r_d)$.

The multivariate Sobolev space with bounded mixed derivative is defined by

$$MW_{p,\varpi}^r = \{ f(x) : f(x) = F_r(x, \varpi) \ast \varphi, \quad \| \varphi \|_p < \infty \}, \quad 1 < p < \infty,$$

where $f^{(r)}(x) := \varphi(x)$ is the $(r, \varpi)$-derivative of $f(x)$. If the space is endowed with the norm $\| f \|_{MW_{p,\varpi}^r} := \| f \|_p + \| f^{(r)} \|_p$, it is not difficult to verify that $MW_{p,\varpi}^r$ is a Banach space. For more information on the Bernoulli kernel and the spaces with bounded mixed derivative, one can refer to the monographs [20, 21].

The purpose of this paper is to study the complexity of the approximation problem by linear Monte Carlo methods on the space $MW_{p,\varpi}^r$ in the norm of $L_q$ for $1 < p, q < \infty$. In other words, we study the approximation complexity of the solution operator $S = I$, where $I$ denotes Sobolev embedding from $MW_{p,\varpi}^r$ to $L_q$. In this paper, we are only concerned with the linear approximations in the deterministic and the randomized settings. For the deterministic setting, the asymptotic orders of the $n$-th approximation numbers on this space have already been obtained (see [7, 19]). While for the randomized setting, the problem of the asymptotic orders is still open. It’s the aim of this paper to solve this problem.

Now we are in a position to state our main result.

**Theorem 1.** For $1 < p, q < \infty$ and $0 < r = r_1 = \cdots = r_v < r_{v+1} \leq \cdots \leq r_d$, $1 \leq v \leq d$, we have

$$a_n^{MC}(I, MW_{p,\varpi}^r, L_q) \asymp \begin{cases} 
(n^{-1} \log^{r-1} n)^r, & q \leq p, \quad (a) \\
(n^{-1} \log^{r-1} n)^r, & 2 \leq p < q, \quad r > 1/2, \quad (b) \\
(n^{-1} \log^{r-1} n)^r(1/p-1/q), & p < q < 2, \quad r > 1/p-1/q, \quad (c) \\
(n^{-1} \log^{r-1} n)^r(1/p-1/2), & p < 2 \leq q, \quad r > 1/p. \quad (d)
\end{cases}$$

In this paper, we have used the following notations $\preceq$ and $\asymp$. For two sequences $\{a_n\}_{n \in \mathbb{N}}$ and $\{b_n\}_{n \in \mathbb{N}}$ of positive real numbers we write $a_n \preceq b_n$ provided that $a_n \leq c b_n$ for a certain $c > 0$. If, furthermore, also $b_n \preceq a_n$, then we write $a_n \asymp b_n$. 

To compare the Monte Carlo approximation numbers in Theorem 1 with its deterministic
counterpart, we summarize the results obtained by Romanyuk [19], Galeev [5] and some known
results in [20, 21] as

Theorem 2. Let $1 < p, q < \infty$, $1/p + 1/p' = 1$ and $0 < r = r_1 = \cdots = r_v < r_{v+1} \leq \cdots \leq r_d$, $1 \leq v \leq d$. Then

$$a_n(I, MW_{p,v}^r, L_q) \approx \begin{cases} 
(n^{-1} \log^{v-1} n)^r, & q \leq p, \\
(n^{-1} \log^{v-1} n)^r(1/p-1/q), & 2 \leq p < q, \ p < q < 2, \\
(n^{-1} \log^{v-1} n)^r(1/p-1/2), & p < 2 \leq q, \ p' > q, \ r > 1/p, \\
(n^{-1} \log^{v-1} n)^r(1/2-1/q), & p < 2 \leq q, \ p' < q, \ r > 1 - 1/q.
\end{cases}$$

Comparing Theorem 2 with Theorem 1, one can see that there are situations, in which linear
Monte Carlo methods provide better rates than those of linear deterministic ones (if $\max\{p', p\} < q$). The difference can reach a factor

$$(n^{-1} \log^{v-1} n)^{1/p-1/q}$$

for $2 \leq p < q < \infty$ and

$$(n^{-1} \log^{v-1} n)^{1-(1/p+1/q)}$$

for $1 < p < 2 \leq q < \infty$, $p' < q$. In both cases, the exponent can almost reach $\frac{1}{2}$ when $p = 2$, $q \to \infty$ or $p \to 2$, $q \to \infty$. This indicates that the linear randomized methods have a potential
superiority over the linear deterministic ones at least in some cases, and gives an example to
answer the questions posed at the beginning of this paper.

3. Proof of main result

In this section, we shall prove our main result, and start with the definition of pseudo-s-scale [16].

Definition 2. A map $s$ assigning to every $S \in L(X, Y)$ a sequence $\{s_n(S)\}_{n \in \mathbb{N}}$ is called a pseudo-
s-scale if the following properties are satisfied:

(i) $s_1(S) = \|S\| \geq s_2(S) \geq \cdots \geq 0$,
(ii) $s_{m+n}(S + T) \leq s_m(S) + s_n(T)$ for $m, n \in \mathbb{N}$, and $T \in L(X, Y)$,
(iii) $s_n(\text{VsU}) \leq \|V\| s_n(S) \|U\|$ for all Banach spaces $\tilde{X}, \tilde{Y}$ and operators $U \in L(\tilde{X}, X)$, $V \in L(\tilde{Y}, Y)$.

It is readily checked that the quantities $a_n, a_n^{MC}$ are pseudo-s-scales. We first establish the lower
bounds in Theorem 1. Associate every vector $s = (s_1, \ldots, s_d)$ whose coordinates are nonnegative integers with the set

$$\rho(s) = \{k \in \mathbb{Z}^d : \lfloor 2^{s_j-1} \rfloor \leq |k_j| < 2^{s_j}, j = 1, \ldots, d\},$$
Lemma 1. \( \text{finite-dimensional spaces. The following three known lemmas are crucial for carrying out a} \)

Let \( \delta_s(f, x) \) denote the “blocks” of the Fourier series for \( f(x) \), namely

\[
\delta_s(f, x) = \sum_{k \in \rho(s)} \hat{f}(k)e^{i(k, x)}.
\]

We set

\[
S^* = \{ s = (s_1, \ldots, s_d, 1, \ldots, 1) \in \mathbb{N}^d : (s, 1) = k \},
F_{S^*} = \text{span}\{e^{i(k, x)} : k \in \rho(s), s \in S^* \},
\]

where \( 1 = (1, \ldots, 1) \in \mathbb{N}^d \) and \( k \) will be chosen later on. The space of trigonometric polynomials \( F_{S^*} \) will play an important role in the lower estimates. Denoting \( \|S^*\| := \sum_{s \in S^*} |\rho(s)| \), we have \( \|S^*\| = 2^k|S^*| \), where \( |A| \) denotes the cardinality of a set \( A \). It’s easy to see that \( \|S^*\| \) equals to the dimension of the space \( F_{S^*} \).

We shall use the discretization technique due to Maiorov (see [11]), which is based on the reduction of the approximation problem of Sobolev embedding to those of identities between finite-dimensional spaces. The following three known lemmas are crucial for carrying out a discretization.

**Lemma 1 (Galeev [4]).** Let \( s \in \mathbb{N}^d \). Then the space of trigonometric polynomials

\[
\text{span}\{e^{i(k, x)} : k \in \rho(s)\}
\]

is isomorphic to the space \( \mathbb{R}^{2^{(s,1)}} \) via the mapping

\[
f(x) \to \{f_s, m(e^j)\}_{m \in M, j \in J} \in \mathbb{R}^{2^{(s,1)}}, \quad f_{s, m}(x) = \sum_{s_{nk1} = s_{nm1}} \hat{f}(k)e^{i(k, x)},
\]

\[
m = (\pm 1, \ldots, \pm 1) \in \mathbb{R}^d, \quad e^j = (\pi 2^{s_1-j_1} 1, \ldots, \pi 2^{s_d-j_d} d),
\]

\[
j_i = 1, \ldots, 2^{s_i-1}, \quad i = 1, \ldots, d,
\]

and the following order equality is valid:

\[
\|\delta_s f\|_p \asymp \left( 2^{-(s,1)} \sum_{m \in M, j \in J} |f_{s, m(e^j)}|^p \right)^{1/p}, \quad 1 < p < \infty,
\]

where \( M = \{m : m = (\pm 1, \ldots, \pm 1) \in \mathbb{R}^d\} \), \( J = \{j : j = (j_1, \ldots, j_d), j_i = 1, \ldots, 2^{s_i-1}, i = 1, \ldots, d\} \), and the asymptotic constants in (2) may depend on \( p, d \).

For any \( f(x) \in F_{S^*} \), we have

\[
f(x) = \sum_{s \in S^*} \sum_{k \in \rho(s)} \hat{f}(k)e^{i(k, x)} := \sum_{s \in S^*} f_s(x),
\]

where \( f_s(x) = \sum_{k \in \rho(s)} \hat{f}(k)e^{i(k, x)} \). Denote by \( I_{S^*} \) the isomorphism from \( F_{S^*} \) to \( \mathbb{R}^{\|S^*\|} \) and define

\[
I_{S^*}(f(x)) := \{f_{s, m(e^j)}\}_{s \in S^*, m \in M, j \in J}.
\]
Lemma 2 (Galeev [4]). Suppose that $Q \subset \mathbb{N}^n$ and is finite, $f = \sum_{s \in Q} \delta_s f$, $\alpha \in \mathbb{R}^n$, and $1 < p < \infty$. Then

$$
|Q|^{(1/2-1/p)} \left( \sum_{s \in Q} \|2^{(x,s)} \delta_s f\|_p^p \right)^{1/p} \lesssim \|f\|_p \lesssim |Q|^{(1/2-1/p)} \left( \sum_{s \in Q} \|2^{(x,s)} \delta_s f\|_p^p \right)^{1/p},
$$

where $a_- = \min\{0, a\}$, $b_+ = \max\{0, b\}$, and $f^{(x)}$ is the derivative in the sense of Weyl, that is, for $f(x) = \sum_k c_k e^{i(k,x)} \in L_p$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ with nonnegative components,

$$
f^{(x)}(x) = \sum_k (ik_1)^{\alpha_1} \cdots (ik_n)^{\alpha_n} c_k e^{i(k,x)},
$$

where $(ik_j)^{\alpha_j} = |k_j|^{\alpha_j} \exp\{\frac{1}{2} \pi i \alpha_j \text{sign} k_j\}$, $j = 1, \ldots, n$.

Lemma 3 (Temlyakov [21, p.17]). Let $G$ be a finite set of vectors $s$, and let the operator $S_G$ map a function $f \in L_p$, $p > 1$, to a function

$$
S_G(f) = \sum_{s \in G} \delta_s(f).
$$

Then

$$
\|S_G\|_{L_p \rightarrow L_p} \leq C(d, p), \quad 1 < p < \infty.
$$

To get the lower estimates, we need also the following lemmas. The first lemma is the lower estimates of the $n$-th Monte Carlo approximation numbers of finite dimension sequence spaces $\ell^m_p$, where $\ell^m_p$ is an $m$-dimensional normed space of vectors $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$, with the norm

$$
\|x\|_{\ell^m_p} := \left\{ \begin{array}{ll}
\left( \sum_{i=1}^m |x_i|^p \right)^{1/p}, & 1 \leq p < \infty, \\
\max\{|x_i|, 1 \leq i \leq m\}, & p = \infty.
\end{array} \right.
$$

Lemma 4 (Mathé [12]). Let $1 \leq p, q \leq \infty$. Then

$$
a_n^{MC}(I, \ell^2_p, \ell^2_q) \geq a_n^{avg}(I, \ell^2_p, \ell^2_q) \geq C \phi(n, p, q),
$$

where

$$
\phi(n, p, q) := \left\{ \begin{array}{ll}
1 & \text{if } p \leq q \leq 2, \\
n^{1/q - 1/p} & \text{if } q \leq p, 2 \leq p \leq q, \\
n^{1/q - 1/2} & \text{if } p \leq 2 \leq q,
\end{array} \right.
$$

and $a_n^{avg}$ denotes the average approximation number (see [12]).

Lemma 5 (Temlyakov [21, p. 100]). Let $1 < p < q \leq \infty$ and $f \in L_p$. Then

$$
\|f\|_{L_p} \geq C(q, p, d) \left( \sum_s \|\delta_s(f)\|_{L_q}^p 2\|\delta_s(f)\|_{1(p/q-1)} \right)^{1/p}.
$$
Lower Estimates of Theorem 1: We will use the discretization technique. To this end, we need to decompose the identity operator $I$ from $\ell_p^{\|S^\ast\|}$ to $\ell_q^{\|S^\ast\|}$ as follows:

$$\ell_p^{\|S^\ast\|} \xrightarrow{I_{S^\ast}} MW_{p,x} \xrightarrow{P} F_{S^\ast} \cap L_q \xrightarrow{I_{S^\ast}} \ell_q^{\|S^\ast\|},$$

where $P$ denotes the orthogonal projection from $L_q$ onto the space $F_{S^\ast} \cap L_q$. From the above decomposition, the property (iii) of pseudo-s-scales for the quantity $a_{MC}^n(I, \ell_p^{\|S^\ast\|}, \ell_q^{\|S^\ast\|})$ yields

$$a_{MC}^n(I, \ell_p^{\|S^\ast\|}, \ell_q^{\|S^\ast\|}) \leq \|I_S\| \|P\| \|I_{S^\ast}\| a_{MC}^n(I, MW_{p,x}^r, L_q).$$

(3)

Next, we estimate the norms of the operators in (3). By Lemma 2 and the Bernstein inequality (see [21, p.96]), we have

$$\|I_{S^\ast}\| \leq 2^r k |S^\ast|^{1/2-1/p} + 2^{-k/p},$$

and

$$\|I_S\| \leq |S^\ast|^{-(1/2-1/q)} - 2^{k/q},$$

which together with Lemma 3 and the inequality (3) imply

$$a_{MC}^n(I, MW_{p,x}^r, L_q) \geq 2^{-rk+k(1/p-1/q)} |S^\ast|^{-(1/2-1/p)} \times |S^\ast|^{1/(2-1/q)} - a_{MC}^n(I, \ell_p^{\|S^\ast\|}, \ell_q^{\|S^\ast\|}).$$

(4)

Now we choose a number $k$ such that

$$\|S^\ast\| = |S^\ast|^{2^k} \geq c k^{v-1} 2^k \geq 2n.$$

By the choice of $k$, we have the relation $2^k k^{v-1} \asymp n \asymp |S^\ast|^{2^k}$.

We continue to estimate the lower bounds. First let $1 < q \leq p < \infty$. In this case, by the embedding relation $MW_{p,x}^r \hookrightarrow MW_{q,x}^r$, clearly it suffices to prove the lower bounds for $1 < q \leq p < \infty$. By the relation (4) and Lemma 4, we have

$$a_{MC}^n(I, MW_{p,x}^r, L_q) \geq 2^{-rk}.$$

(5)

Now let $1 < p < q < \infty$. We split our considerations into three cases $1 < p < 2 \leq q < \infty$, $2 \leq p < q < \infty$, and $1 < p < q < 2$. First, let $1 < p < 2 \leq q < \infty$. In this situation, we only need to consider $1 < p < 2$ and $q = 2$. Therefore the relation (4) and Lemma 4 again imply

$$a_{MC}^n(I, MW_{p,x}^r, L_2) \geq 2^{-rk+k(1/p-1/2)}.$$

(6)

Second, we deal with the case $2 \leq p < q < \infty$. Now the relation (5) gives the required lower bounds. Finally, it remains to estimate the lower bounds for $1 < p < q < 2$. In fact, we have proved that

$$a_{MC}^n(I, MW_{p,x}^r \cap F_{S^\ast}, L_2) \geq 2^{-k(r-1/p+1/2)}.$$

(7)

By Lemmas 3, 5 and (7), we obtain

$$a_{MC}^n(I, MW_{p,x}^r, L_q) \geq a_{MC}^n(I, MW_{p,x}^r \cap F_{S^\ast}, L_q) \geq a_{MC}^n(I, MW_{p,x}^r \cap F_{S^\ast}, L_q \cap F_{S^\ast}) \geq 2^{k(1/2-1/q)} a_{MC}^n(I, MW_{p,x}^r \cap F_{S^\ast}, L_2 \cap F_{S^\ast})$$

By Lemmas 3, 5 and (7), we obtain

$$a_{MC}^n(I, MW_{p,x}^r, L_q) \geq a_{MC}^n(I, MW_{p,x}^r \cap F_{S^\ast}, L_q) \geq a_{MC}^n(I, MW_{p,x}^r \cap F_{S^\ast}, L_q \cap F_{S^\ast}) \geq 2^{k(1/2-1/q)} a_{MC}^n(I, MW_{p,x}^r \cap F_{S^\ast}, L_2 \cap F_{S^\ast})$$
\[ Q^\gamma_m = \left\{ k : k \in \bigcup_{(s, \gamma) < m} \rho(s) \right\} \]

which together with (5), (6) and the relation \( n \asymp 2^k k^{r-1} \) gives the required lower bounds.

Now we turn to the proof of the upper bounds in Theorem 1. For the upper estimates, we still adopt the Maiorov’s discretization technique. To do this, we need the special series representation of a function \( f \) from the Sobolev space \( MW^r_{p, x} \), which is crucial for the discretization. First, we introduce some necessary notations. For \( m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \), let

\[ Q^\gamma_m = \left\{ k : k \in \bigcup_{(s, \gamma) < m} \rho(s) \right\} \]

be a step hyperbolic cross, where \( \gamma = (\gamma_1, \ldots, \gamma_d) \), \( \gamma_i = r_i / r_i \), \( i = 1, \ldots, d \), and \( 0 < r = r_1 = \cdots = r_v < r_{v+1} \leq \cdots \leq r_d \), \( 1 \leq v \leq d \). Consider the Fourier partial sum operators \( S_{Q^\gamma_m} \) (see [20])

\[ S_{Q^\gamma_m}(f) = f * D_{Q^\gamma_m}, \quad f \in L_1, \]

and define a sequence of operators

\[ T_0 = S_{Q^\gamma_0} = 0, \quad T_m = S_{Q^\gamma_m} - S_{Q^\gamma_{m-1}} \quad \text{for } m \geq 1, \]

where \( D_{Q^\gamma_m}(x) = \sum_{k \in Q^\gamma_m} e^{i(k, x)} \). Let

\[ S_{m, k} = \{ s \in \mathbb{N}^d : m - 1 \leq (s, \gamma) < m, (s, 1) = k \}, \]

\[ F_{m, k} = \text{span}\{ e^{i(k, x)} : k \in \rho(s), s \in S_{m, k} \}, \quad (8) \]

and let \( \| S_{m, k} \| := \sum_{s \in S_{m, k}} |\rho(s)| \). Then it is clear that \( S_{m, k} = \emptyset \) if \( k < d \) or \( k \geq m \), and \( \| S_{m, k} \| \) equals to the dimension of the space \( F_{m, k} \). Define operators \( T_{m, k} \) as follows:

\[ T_{m, k}(f) := \begin{cases} \sum_{s \in S_{m, k}} \delta_s(f), & S_{m, k} \neq \emptyset, \\ 0 & \text{otherwise}. \end{cases} \]

Then \( T_m(f) = \sum_{k=d}^m T_{m, k}(f) \), for \( f \in MW^r_{p, x} \). According to the inequality below

\[ \| f - S_{Q^\gamma_m}(f) \|_p \leq 2^{-rm} \| f \|_{MW^r_{p, x}}, \quad 1 < p < \infty \]

(see [20, p.36]) and the definitions of the operators \( T_m \) and \( T_{m, k} \), we obtain

\[ f = \sum_{m=d}^{\infty} T_m(f) = \sum_{m=d}^{\infty} \sum_{k=d}^{m} T_{m, k}(f), \quad f \in MW^r_{p, x} \]

in the sense of convergence in \( L_p \) space.

**Lemma 6** (Mathé [12]). There is a constant \( C < \infty \) such that for \( 1 \leq p \leq 2 \) and \( 1 \leq n \leq m \), we have the estimate

\[ a^M_n(1, \ell_{p_1}^m, \ell_{q_1}^m) \leq C m^{1/q} / n^{1/2}, \quad 2 \leq q < \infty. \]
Lemma 7 (Romanyuk [18]). For any \(d, u \in \mathbb{N}, \beta > 0\), there is a constant \(c > 0\) such that
\[
\sum_{m=d}^{\infty} \sum_{k=d}^{m} j_{m,k} \leq 2^uu^{\beta - 1},
\]
where
\[
j_{m,k} := \begin{cases} \|S_{m,k}\|, & d \leq k \leq m, \ m \leq u, \\ |S_{m,k}|2u^2\beta u^{2+\beta}, & d \leq k \leq m, \ m > u, \end{cases}
\]
and \(m, k, S_{m,k}\) as those in (8).

Upper Estimates of Theorem 1: By the relation
\[
a_{n}^{MC}(I, MW_{p,r}^{2}, L_{q}) \leq a_{n}(I, MW_{p,r}^{2}, L_{q}),
\]
and Theorem 2, obviously it remains to estimate the upper bounds for \(2 \leq p < q < \infty\) and \(1 < p < 2 \leq q < \infty\), \(p' \leq q\).

First we consider \(2 \leq p < q < \infty\). In this case, it is sufficient to estimate the upper bounds for \(p = 2, \ 2 < q < \infty\). For \(f \in MW_{2,r}^{2}\), according to the above analysis, we have
\[
f = \sum_{m=d}^{\infty} T_{m}(f) = \sum_{m=d}^{\infty} \sum_{k=d}^{m} T_{m,k}(f)
\]
in the sense of convergence in \(L_{2}\), which together with the property (ii) of pseudo-s-scales implies
\[
a_{n}^{MC}(I, MW_{2,r}^{2}, L_{q}) \leq \sum_{m=d}^{\infty} \sum_{k=d}^{m} a_{j_{m,k}}^{MC}(T_{m,k}, MW_{2,r}^{2}, L_{q}),
\]
where \(\sum_{m=d}^{\infty} \sum_{k=d}^{m} j_{m,k} \leq n\). In order to use the discretization technique, we factor further the operators \(T_{m,k} : MW_{2,r}^{2} \rightarrow L_{q}\) as follows:
\[
MW_{2,r}^{2} \xrightarrow{T_{m}} F_{m} \cap L_{2} \xrightarrow{T_{m,k}} F_{m,k} \cap L_{2} \xrightarrow{I_{S_{m,k}}} \ell_{2} \xrightarrow{\|S_{m,k}\|} I \xrightarrow{\ell_{q}} \ell_{q} \xrightarrow{I_{S_{m,k}^{-1}}} F_{m,k} \cap L_{q},
\]
where
\[
F_{m} := \text{span}\{e^{i(k,s)k}, \ k \in \rho(s), \ m - 1 \leq (s, r) < m\},
\]
and \(I_{S_{m,k}}\) denotes the isomorphism from \(F_{m,k}\) to \(\ell_{2}\). Thus we obtain
\[
T_{m,k} = I_{S_{m,k}^{-1}} \circ I \circ I_{S_{m,k}} \circ T_{m,k} \circ T_{m},
\]
where \(T_{m,k}\) are the operators \(T_{m,k}\) restricted on \(F_{m} \cap L_{2}\). To proceed the upper estimates, we need to estimate the norms of the operators in (11). First, by Lemma 2, we have
\[
\|I_{S_{m,k}^{-1}}\| \leq |S_{m,k}|^{1/2-1/q}2^{-k/q}, \quad \|I_{S_{m,k}}\| \leq 2^{k/2}.
\]
Then from the definitions of \(T_{m}, \ T_{m,k}\), it’s clear that
\[
\|T_{m}\| \leq 2^{-rm}, \quad \|T_{m,k}\| \leq 1.
\]
For any \( n \in \mathbb{N} \), we can choose a number \( u \) such that \( 2^u u^{v-1} \leq n < C 2^u u^{v-1} \), and let the numbers \( j_{m,k} \) in (10) be defined as those in Lemma 7 with this \( u \). It follows from the property (iii) of pseudo-s-scales, and the relations (10)–(13) that

\[
a_{n}^{MC}(I, MW_{2,2}^{x}, L_q) \leq \sum_{m=d}^{\infty} \sum_{k=d}^{m} 2^{-rm+k(1/2-1/q)} |S_{m,k}|^{1/2-1/q} a_{j_{m,k}}^{MC}(I, \ell^2_{2} \| S_{m,k} \|, \ell^q_{q} \| S_{m,k} \|)
\]

\[
\leq \sum_{m=0}^{\infty} \sum_{k=0}^{m} 2^{-rm+k(1/2-1/q)} |S_{m,k}|^{1/2-1/q} a_{j_{m,k}}^{MC}(I, \ell^2_{2} \| S_{m,k} \|, \ell^q_{q} \| S_{m,k} \|)
\]

\[
= \sum_{m=0}^{\infty} 2^{-rm+k(1/2-1/q)} |S_{m,k}|^{1/2-1/q} a_{j_{m,k}}^{MC}(I, \ell^2_{2} \| S_{m,k} \|, \ell^q_{q} \| S_{m,k} \|)
\]

\[
\sum_{m,k} j_{m,k} > 0 2^{-rm+k(1/2-1/q)} |S_{m,k}|^{1/2-1/q} a_{j_{m,k}}^{MC}(I, \ell^2_{2} \| S_{m,k} \|, \ell^q_{q} \| S_{m,k} \|)
\]

\[
= I_1 + I_2.
\] (14)

In the course of the above proof, we have used the simple fact that

\[
a_{j_{m,k}}^{MC}(I, \ell^2_{2} \| S_{m,k} \|, \ell^q_{q} \| S_{m,k} \|) = 0, \quad d \leq k \leq m, m \leq u.
\]

Now we estimate the two terms on the right side of (14) separately, and start with the term \( I_1 \).

By virtue of Lemma 6 and the choice of \( j_{m,k} \), we have

\[
I_1 \leq 2^{(-u-\beta u)/2} \sum_{m=0}^{\infty} 2^{-rm+\beta m} \sum_{d}^{m} 2^{k(1-\beta)/2}.
\]

Since \( r > 1/2 \), so we can choose a constant \( \beta \) such that \( 0 < \beta < \min\{1, 2r - 1\} \). By a simple computation, we get

\[
I_1 \leq 2^{(-u-\beta u)/2} \sum_{m=0}^{\infty} 2^{-(r-\beta/2-1/2)m}
\]

\[
\leq 2^{-ru}.
\] (15)

We turn to estimate the term \( I_2 \). By \( j_{m,k} = 0 \), i.e., \( |S_{m,k}| < 2^{2\beta m - u - \beta u - \beta k} \) and the above chosen \( \beta \), we have

\[
I_2 \leq 2^{(-u-\beta u)(1/2-1/q)} \sum_{m=0}^{\infty} 2^{-rm+2\beta m(1/2-1/q)} \sum_{d}^{m} 2^{k(1-\beta)(1/2-1/q)}
\]

\[
\leq 2^{-ru}.
\] (16)

In virtue of the relations (15),(16) and \( n \approx 2^u u^{v-1} \), we obtain

\[
a_{n}^{MC}(I, MW_{2,2}^{x}, L_q) \leq (n^{-1} \log^{v-1} n)^r.
\]

Now we pass to the case \( 1 < p < 2 \leq q < \infty \). By the embedding relation \( MW_{p,2}^{x} \hookrightarrow MW_{2,2}^{x-eta 1} \), where \( \beta = 1/p - 1/2 \), and the result which has already been proved, we obtain the required upper bounds, and the proof of Theorem 1 is complete.
Remark 1. The linear Monte Carlo algorithms that yields the upper bounds in Theorem 1 are still unclear. We conjecture that the lower bounds are also valid for non-linear Monte Carlo methods in special cases.

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