Cells with many facets in arrangements of hyperplanes

Jean-Pierre Roudneff

Université P. et M. Curie, U.F.R. 21 (Mathématiques), E.R. 175, 4, place Jussieu, 75005 Paris, France

Received 8 July 1987
Revised 12 February 1990

Abstract

For every $n$, $d$, $n > 2d + 155$, we prove the existence of an arrangement $\mathcal{H}$ of $n$ hyperplanes in the real projective space $\mathbb{P}^d$, such that exactly $\sum_{i=1}^{d} \binom{n-1}{i}$ cells of $\mathcal{H}$ are bounded by every hyperplane of $\mathcal{H}$. In the particular case $d = 3$, this disproves a conjecture of Edelsbrunner and Haussler. We also prove that in any arrangement of $n$ hyperplanes in $\mathbb{P}^d$, the average number of hyperplanes bounding the cells of $\mathcal{H}$ is always less than $2d + 1$.

1. Introduction

An Euclidean (resp. projective) $d$-arrangement of hyperplanes $\mathcal{H}$ is a finite collection of hyperplanes in the Euclidean space $E^d$ (resp. the real projective space $\mathbb{P}^d$) such that no point belongs to every hyperplane of $\mathcal{H}$. Any arrangement $\mathcal{H}$ decomposes $E^d$ (resp. $\mathbb{P}^d$) into a $d$-dimensional cell complex $\mathcal{H}$. For the sake of simplicity we call cells of $\mathcal{H}$ the $d$-cells of $\mathcal{H}$, and facets of $\mathcal{H}$ the $(d-1)$-cells of $\mathcal{H}$. Clearly, any cell $c$ of $\mathcal{H}$ has at most $n$ facets, where $n$ denotes the number of hyperplanes in $\mathcal{H}$. We say that $c$ is a complete cell of $\mathcal{H}$ if $c$ has exactly $n$ facets, i.e., $c$ is bounded by each hyperplane of $\mathcal{H}$.

Edelsbrunner and Haussler have shown in [3] that for every $n \geq 4$, there is an Euclidean 3-arrangement of $n$ planes having 5 complete cells. They have conjectured the following (in an equivalent form).

Conjecture 1.1 [3, Conjecture 2.4]. An arrangement of sufficiently many planes in $E^3$ cannot have 6 complete cells.

In Section 2, we disprove Conjecture 1.1. More precisely, we shall prove the following result.
Theorem 1.2. For all $d$, $n$ such that $2 \leq d \leq n - 1$, there exists an arrangement of $n$ hyperplanes in $E^d$ with at least \( \Sigma_{i=0}^{d-2} (\binom{n}{i} - 1) \) complete cells.

The arrangements used in the proof of Theorem 1.2 are derived from the cyclic arrangements introduced by Shannon. To the analytical description of these arrangements presented in [9], we shall prefer a combinatorial description, using oriented matroids. For basic results on oriented matroid theory, we refer the reader to [1, 4] (the notation of [1] being employed here). The interpretation of cyclic arrangements in terms of oriented matroids is also given in [8].

In Section 3, we consider the average number of hyperplanes bounding the cells in projective $d$-arrangements of $n$ hyperplanes. Answering a question of Duchet (private communication), and using again oriented matroid theory, we show that this value is always less than $2d + 1$ (Theorem 3.2). It should be noted that this upper bound only depends on the dimension.

2. Cyclic arrangements

Cyclic arrangements have been introduced by Shannon as examples of projective arrangements with a minimum number of simplicial cells [9], see also [8]. Cyclic arrangements have many other extremal properties, due to the fact that they are dual to the well-known cyclic polytopes (see e.g., [7]). Also, cyclic arrangements of $n$ hyperplanes in $\mathbb{P}^d$ are equivalent to alternating oriented matroids of rank $r = d + 1$ on $n$ elements, by the representation of Folkman and Lawrence [1, 4, 8]. The (uniform) alternating oriented matroid $M(r, n)$ of rank $r$ on $n$ elements is defined as follows, see [1, Example 3.8]: Let $E$ denote an $n$-element set with $n \geq r + 1$, together with a total order $<$. The signed circuits of $M(r, n)$ are the subsets $C = \{e_1, e_2, \ldots, e_{r+1}\}$, $e_1 < e_2 < \cdots < e_{r+1}$, of $E$ with the signature $C^+ = \{e_i, \ i \text{ odd}\}$ and $C^- = \{e_i, \ i \text{ even}\}$.

We define a cell (resp. a complete cell) of $M(r, n)$ as any pair $(A, E \setminus A)$ such that $\partial M(r, n)$ is acyclic (resp. convex), i.e. every circuit $C$ of $\partial M(r, n)$ satisfies $|C^+| \geq 1$ and $|C^-| \geq 1$ (resp. $|C^+| \geq 2$ and $|C^-| \geq 2$). By the results of [1, 4], the cells (resp. complete cells) of the cyclic arrangement $\mathcal{K}(d, n)$ of $n$ hyperplanes in $\mathbb{P}^d$ are in 1-1 correspondence with the cells (resp. complete cells) of $M(d + 1, n)$.

Theorem 2.1. The cyclic arrangement of $n \geq d + 1$ hyperplanes in $\mathbb{P}^d$, $d \geq 2$ has at least $\Sigma_{i=0}^{d-2} (\binom{n}{i} - 1)$ complete cells. Moreover, this bound is tight for all $n \geq 2d + 1$.

Proof. Let $r = d + 1$. Abbreviating $M(r, n)$ by $M$, it suffices to prove that $M$ has at least $\Sigma_{i=0}^{d-3} (\binom{n}{i} - 1)$ convex reorientations, and exactly this number if $n \geq 2r - 1$. For simplicity, we shall take $E = \{1, 2, \ldots, n\}$ together with the natural order. As usual, an interval of $E$ denotes any subset $\{p, p + 1, \ldots, q\}$ of $E$ with $1 \leq p \leq q \leq n$. For every subset $A$ of $E$, we denote by $i(A)$ the smallest integer $i$
such that $A$ is the union of $i$ intervals. In particular, we have $i(\emptyset) = 0$, with the
convention that $\emptyset$ is the union of no sets. Consider the following two assertions:

(2.1.1) $i(A) + i(E \setminus A) \leq r - 2$;

(2.1.2) $(A, E \setminus A)$ is a convex cell of $M$;

First, we show that (2.1.1) implies (2.1.2) when $n \geq r + 1 \geq 4$. To that aim, we
prove by induction on $n$ ($r$ being fixed) that for every circuit $C = \{e_1, e_2, \ldots, e_{r+1}\}$ of $M$, with $e_1 < e_2 < \cdots < e_{r+1}$, we have $|\tilde{\alpha}C^+| \geq 2$ and $|\tilde{\alpha}C^-| \geq 2$.

If $n = r + 1$, $C^+$ and $C^-$ are the only signed circuits in $M$. If $\tilde{\alpha}C^+$ or $\tilde{\alpha}C^-$ is
empty, then $A$ is either $\{e_i, i \text{ even}\}$ or $\{e_i, i \text{ odd}\}$. In both cases, we have
$i(A) + i(E \setminus A) = n$, a contradiction. If $\tilde{\alpha}C^+$ or $\tilde{\alpha}C^-$ is a singleton $\{e_j\}$, then
$A$ is either $\{e_i, i \text{ even} < j\} \cup \{e_i, i \text{ odd} > j\}$ or $\{e_i, i \text{ odd} < j\} \cup \{e_i, i \text{ even} > j\}$ or the
complement in $E$ of one of these two sets. As is easily verified, we have
$i(A) + i(E \setminus A) = r - 1$ or $r$ in both cases, a contradiction.

If $n > r + 1$, let $x \in E \setminus C$. Since $M \setminus x$ is alternating, we immediately get
$|\tilde{\alpha}C^+| \geq 2$ and $|\tilde{\alpha}C^-| \geq 2$ by applying the induction hypothesis.

Now, we prove that (2.1.2) implies (2.1.1) for all $r$, such that $n \geq 2r - 1 \geq 5$.
Let $A$ be a subset of $E$ which satisfies (2.1.2) but does not satisfy (2.1.1).
Changing $A$ into $E \setminus A$ if necessary, we may assume that $1 \in A$. We denote by
$I_1, I_2, \ldots, I_k$ the maximal pairwise disjoint intervals of $\{1, 2, \ldots, n\}$ defined by $A$, i.e., $A = I_1 \cup I_2 \cup I_3 \cdots$, $E \setminus A = I_2 \cup I_3 \cup I_4 \cdots$. We have $1 \in I_1$, $n \in I_k$ and
$k = i(A) + i(E \setminus A) \geq r - 1$.

If there exists $j$, $1 \leq j \leq k$, such that $|I_j| \geq 3$, then we can select $r - 1$ consecutive
intervals $I_p, \ldots, I_j, \ldots, I_{p+r-2}$. Choosing three points $a, b, c$ (with $a < b < c$) in
$I_j$ and one point in each other interval, we get a circuit $C$ whose signature in $\tilde{\alpha}M$ is
$C^+ = \{b\}$ and $C^- = B \setminus b$ (or $C^+ = C \setminus b$ and $C^- = \{b\}$), and $\tilde{\alpha}M$ cannot be a
convex reorientation of $M$.

If $|I_j| = 2$ for all $j$, $1 \leq j \leq k$, then $k \geq r$ since $n \geq 2r - 1$. Moreover, if $k = r$,
there is exactly one $I_j$ with cardinality 1. Assume that $|I_1| = 2$ (the proof is similar
if $|I_1| = 2$). Taking the two elements 1 and 2 in $I_1$ and one point in each $I_j$, $j > 2$,
we get a circuit $C$ which signature in $\tilde{\alpha}M$ is $C^- = \{1\}$ and $C^+ = C \setminus 1$ (or
$C^+ = \{1\}$ and $C^- = C \setminus 1$). Thus $\tilde{\alpha}M$ cannot be a convex reorientation of $M$.
Finally, if $k \geq r + 1$, the circuit $C$ obtained by choosing one point in the $r + 1$ first
intervals easily satisfies $C^+ = \emptyset$ or $C^- = \emptyset$, and $\tilde{\alpha}M$ is not even acyclic.

To complete the proof of Theorem 2.1, it suffices to count the number $I(r, n)$
of subsets of $E$ that satisfy (2.1.1), for fixed $r$ and $n$. The relation $I(r, n) = I(r, n - 1) + I(r - 1, n - 1)$, for all $n \geq 2$ and $r \geq 3$, is left to the reader as an easy
combinatorial exercise. It is then straightforward to derive by induction that

$$I(r, n) = 2 \cdot \sum_{k=0}^{r-3} \binom{n-1}{k}.$$

As $A$ and $E \setminus A$ define the same cell, we have to divide $I(r, n)$ by 2 to get the
number of cells of $M$ (if $n \geq 2r - 1$), or a lower bound of this number (if we only have $n \geq r$). \hfill \Box

We now use Theorem 2.1 to prove Theorem 1.2 and consequently show the inexactitude of Conjecture 1.1. To that purpose, we consider a cyclic arrangement $\mathcal{H}$ of $n + 1$ hyperplanes in $\mathbb{P}^d$ and a hyperplane $H$ of $\mathcal{H}$. Identifying $\mathbb{P}^d \setminus H$ with the Euclidean space $\mathbb{E}^d$, $\mathcal{H} \setminus H$ can be interpreted as an Euclidean arrangement $\mathcal{H}'$ of $n$ hyperplanes. This proves Theorem 1.2. By Theorem 2.1, $\mathcal{H}'$ has exactly $\sum_{i=0}^{d-2} \binom{n-1}{i}$ complete cells if $n \geq 2d$ (and at least this number if $d + 1 \leq n \leq 2d - 1$), which shows that Conjecture 1.1 does not hold.

We conjecture that cyclic arrangements have the maximum number of complete cells.

**Conjecture 2.2.** Every arrangement of $n \geq 2d + 1 \geq 5$ (pseudo)hyperplanes in $\mathbb{P}^d$ has at most $\sum_{k=0}^{d-2} \binom{n-1}{k}$ complete cells.

A projective $d$-arrangement $\mathcal{H}$ is said to be **simple** if no $d + 1$ hyperplanes of $\mathcal{H}$ have a point in common, i.e., the hyperplanes of $\mathcal{H}$ are in general position.

**Proposition 2.3.** To prove Conjecture 2.2, it suffices to verify it for all simple arrangements of $n = 2d + 1$ (pseudo)hyperplanes in $\mathbb{P}^d$.

**Proof.** First observe that we can derive from every arrangement $\mathcal{H}$ a simple arrangement $\mathcal{H}'$ with at least the same number of complete cells as $\mathcal{H}$. This can be done in the following way. Add to $\mathcal{H}$ a hyperplane $H_x$ in general position, thought of as a hyperplane at infinity. Then, slide each hyperplane of $\mathcal{H}$ a bit, parallel to itself. The arrangement $\mathcal{H}'$ is obtained by removing $H_x$. Such a construction can also be done with pseudohyperplanes, using suitable principal extensions of oriented matroids, see [6]. This shows that we can restrict ourselves to simple arrangements.

In order to show that it suffices to consider the case $n = 2d + 1$, let us take an oriented matroid $M$ of rank $r = d + 1$ on $E$ with $n = |E| \geq 2d + 1$, and an element $x$ of $E$. Calling $\mathcal{P}(M)$ the set of convex reorientations of $M$, we remark that $\mathcal{M} \in \mathcal{P}(M)$ implies $\mathcal{M}(M \setminus x) \in \mathcal{P}(M \setminus x)$. Conversely, let $A \subseteq E \setminus x$ be such that $\mathcal{M}(M \setminus x) \in \mathcal{P}(M \setminus x)$. If $\mathcal{M}$ and $\mathcal{M}(M \setminus x)$ both belong to $\mathcal{P}(M)$, then $\mathcal{M}(M \setminus x) \in \mathcal{P}(M \setminus x)$, as is easily seen. The preceding observations show that $|\mathcal{P}(M)| \leq |\mathcal{P}(M \setminus x)| + |\mathcal{P}(M \setminus x)|$. The inequality

$$|\mathcal{P}(M)| \leq \sum_{k=0}^{d-2} \binom{n-1}{k}$$

then follows by induction, once one knows that it is verified for $n = 2d + 1$. \hfill \Box
3. Average number of facets in the cells of an arrangement

The following conjecture, due to Las Vergnas, is stated in an equivalent form at the very end of [5].

**Conjecture 3.1** (Las Vergnas [5]). Every arrangement of pseudohyperplanes has at least one simplicial cell.

Conjecture 3.1 is proved for arrangements of hyperplanes, see [8–9] and for arrangements of pseudohyperplanes with an additional property [6]. The following theorem gives some credit to this conjecture by showing that the average number of facets in the cells of an arrangement is bounded by a function of $d$.

**Theorem 3.2.** Let $\mathcal{H}$ be an arrangement of $n$ pseudohyperplanes in $\mathbb{P}^d$. Then the average number of facets in the cells of $\mathcal{H}$ is always less than $2d + 1$.

Before proving Theorem 3.2, we shall recall some results on the Tutte polynomial $t(M; \xi, \eta)$ of a matroid $M$.

First, let $M$ denote an oriented matroid associated with the arrangement $\mathcal{H}$, by the representation of Folkman and Lawrence. Then the number of cells in $\mathcal{H}$ is equal to $\frac{1}{2}t(M; 2, 0)$, see [5]. In a similar way, the number of $(d - 1)$-dimensional faces of $\mathcal{H}$ is equal to $\frac{1}{2} \sum_{x \in E} t(M/x; 2, 0)$. As every $(d - 1)$-dimensional face of $\mathcal{H}$ belongs to exactly two cells, the average number of facets in the cells of $\mathcal{H}$ is exactly

$$2 \cdot \frac{\sum_{x \in E} t(M/x; 2, 0)}{t(M; 2, 0)}.$$

Theorem 3.2 is obtained as a corollary of the following result.

**Theorem 3.3.** Let $M$ be a loopless (non-oriented) matroid of rank $r \geq 2$ on $E$. Then

$$\sum_{x \in E} t(M/x; 2, 0) \leq (r - \frac{1}{2}) \cdot t(M; 2, 0) - (r - 1)2^{r-1}.$$

Let $\prec$ be a total order on $E$, $B$ be a base of $M$ and $e \in E \setminus B$. We shall say that $e$ is externally active for $B$ (with respect to $\prec$) if $e$ is the smallest element in the unique circuit included in $B \cup e$. The set of bases in $M$ which have no externally active element is denoted by $B_0(M)$. Note that if $B \in B_0(M)$, then $B$ always contains the smallest element of $M$. Now, if $e \in B$, we say that $e$ is internally active in $B$ (with respect to $\prec$) if $e$ is the smallest element in the unique cocircuit included in $(E \setminus B) \cup e$. The number of internally active elements in $B$ is denoted by $i(B)$. The following lemma is a particular case of a theorem due to Crapo.
Lemma 3.4 (Crapo [2]). \( t(M; 2, 0) = \sum_{B \in \mathcal{B}_0(M)} 2^{i(B)} \).

Proof of Theorem 3.3. We choose the ordering \( x_1 < x_2 < \cdots < x_n \) of the elements of \( E \) in such a way that \( B_0 = \{x_1, x_2, \ldots, x_r\} \) is a base of \( E \). Notice that \( B_0 \in \mathcal{B}_0(M) \). Let \( B \) denote a base of \( E \) and \( x \) be an element of \( B \). Let \( M' = M/x \) and \( B' = B \setminus x \). The internal activity in \( M \) (resp. in \( M' \)) is denoted by \( i \) (resp. \( i' \)), the order on \( E \setminus x \) being that induced by \( < \). Assume that \( B' \in \mathcal{B}_0(M') \).

For every \( e \in E \setminus B \), \( e \) is not the smallest element in the unique circuit \( C' \) of \( M' \) contained in \( B' \cup e \). As the unique circuit \( C \) of \( M \) contained in \( B \cup e \) is either \( C' \) or \( C' \cup x \), it follows that \( e \) is not the smallest element in \( C \), thus \( B \in \mathcal{I}_0(M) \). We deduce that

\[
\sum_{x \in E} t(M/x; 2, 0) = \sum_{x \in E} \sum_{B' \in \mathcal{B}_0(M/x)} 2^{i(B')} \leq \sum_{B \in \mathcal{B}_0(M)} \sum_{x \in B} 2^{i(B/x)}
\]

For every \( e \in B' \), the unique cocircuit of \( M' \) contained in \( (E \setminus B) \cup e \) is equal to the unique cocircuit of \( M \) contained in \( (E \setminus B) \cup e \). As a consequence, \( e \) is internally active in \( B \) (for \( M \)) if and only if it is internally active in \( B' \) (for \( M' \)) hence \( i(B) = i'(B') \) or \( i'(B') + 1 \). As \( x_1 \) belongs to \( B \) for every \( B \in \mathcal{B}_0(M) \), the equality \( i_M(B) = i_{M/x}(B/x) + 1 \) is obtained for at least one element \( x \in B \), hence.

\[
\sum_{x \in B} 2^{i_M(B/x)} \leq \left( \sum_{x \in B} 2^{i_M(B)} \right) - 2^{i_M(B) - 1}
\]

\[
= (r - \frac{1}{2}) \sum_{x \in B} 2^{i_M(B)}.
\]

In the particular case \( B = B_0 \), \( i_M(B) = i_{M/x}(B/x) + 1 \) holds for every \( x \in B_0 \), thus

\[
\sum_{x \in E} t(M/x; 2, 0) \leq \sum_{B \in \mathcal{B}_0(M)} (r - \frac{1}{2}) \sum_{x \in B} 2^{i_M(B)} - (r - 1) 2^{r-1}
\]

\[
= (r - \frac{1}{2}) \sum_{B \in \mathcal{B}_0(M)} 2^{i_M(B)} - (r - 1) 2^{r-1}
\]

\[
= (r - \frac{1}{2}) \cdot t(M; 2, 0) - (r - 1) 2^{r-1}.
\]

In the case of uniform matroids, Theorem 3.3 can be slightly improved.

Proposition 3.5. Let \( M = U_{r,n} \) be the rank \( r \) uniform matroid on \( n \) elements, \( n > r \geq 3 \). Then

\[
\sum_{x \in E} t(M/x; 2, 0) < (r - 1) \cdot t(M; 2, 0).
\]

Proof. As \( M \) is uniform, we have

\[
\sum_{x \in E} t(M/x; 2, 0) = n \cdot t(M/x; 2, 0)
\]
for any element $x$ of $E$. Let

$$S(r, n) = (r - 1) \cdot t(U_{r,n}; 2, 0) - n \cdot t(U_{r-1,n-1}; 2, 0).$$

It is easily checked that $t(U_{r,n}; 2, 0) = 2 \cdot \sum_{i=0}^{r-1} \binom{n-1}{i}$, which implies

$$S(r, n) = S(r - 1, n) + 2 \cdot \sum_{i=0}^{r-2} \binom{n-1}{i} - 2\binom{n-2}{r-2}.$$

Thus, $S(r, n) > S(r - 1, n)$ and $S(r, n) > 0$ follows by induction. □

**Corollary 3.6.** Let $\mathcal{H}$ be a simple arrangement of $n$ pseudohyperplanes in $\mathbb{P}^d$, $d \geq 2$. Then, the average number of facets in the cells of $\mathcal{H}$ is always less than $2d$.

**Remark 3.7.** The value $2d$ is asymptotically best possible, for fixed $d$, as $n$ tends to infinity.

**Acknowledgement**

The author wishes to thank P. Duchet for valuable discussions which led to the theorems presented in Section 3.

**References**