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Existence of Positive Solutions for Elliptic Systems—Degenerate and Nondegenerate Ecological Models

ANTHONY LEUNG AND GUANGWEI FAN

Department of Mathematical Sciences, University of Cincinnati, Cincinnati, Ohio 45221

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1. INTRODUCTION

This article is primarily concerned with positive solutions for degenerate elliptic systems of the form

$$\Delta\psi(w_i) + f_i(x, w_1, w_2) = 0 \qquad \text{in} \quad D, i = 1, 2, \tag{1.1}$$

with homogeneous Dirichlet conditions $w_i = 0$ on ∂D. The function $\psi(s)$ satisfies the conditions $\psi \in C^1[0, \infty)$, $\psi(0) = 0$, and $\psi'(s) > 0$ for $s > 0$. Problems of this nature are of interest in reaction–diffusion processes in biology and chemistry. For example, the case for $\psi(u) = u^m$, $m > 1$, or $m \in (0, 1)$ for single parabolic equations (i.e., $u_t = \Delta u^m + f(x, u)$) has been studied recently for porous medium analysis and population dynamics (cf. [2, 7, 15, 18, 19]). As $t \to \infty$ these solutions tend to a solution of the corresponding elliptic scalar equation. Studies have also been carried out in these and other papers (e.g., [3, 4, 16, 17]) with u^m replaced by $\psi(u)$ satisfying the conditions described above.

In this article, however, the hypotheses on f are quite different from those in the papers mentioned above. For example, f is not necessarily Lipschitz in u and may depend discontinuously on x. Thus even for the scalar case, the results here are not covered by the other papers, although there are some overlaps. Moreover, our emphasis here is on degenerate systems (Section 3) and their applications (Section 4). These results are quite different and generalize the papers mentioned above to practical interesting cases. Other relevant results for the nondegenerate cases can be found in [5, 9, 11].

In Section 2, we discuss some existence and uniqueness theorems for

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scalar equations, which we use in Section 4 for ecological models. Monotone iteration is used to obtain a sequence which converges in $W^{2,p}(D) \cap W_0^{1,p}(D)$ to a maximal solution. This procedure is used again in the proof of Theorem 4.2. In Section 3, we adapt the hypotheses in Section 2 to deduce an existence theorem for systems. We use Schauder's fixed point theorem to find a positive solution in $W^{2,p}(D) \cap W_0^{1,p}(D)$ for the system between appropriate upper and lower solutions. In Section 4, we apply the results to simple ecological prey–predator models of interest. Comparing the results with those for the nondegenerate case ($m = 1$) in [10], we find that (4.3) in Theorem 4.1 is a much less stringent sufficient condition for coexistence in the degenerate case. For example, we do not assume that the intrinsic growth rates of the species are larger than the principal eigenvalue of the domain. In Theorem 4.2, although the equation is nondegenerate, we allow the intrinsic growth rate $a(x)$ to be discontinuous and to have negative values somewhere. We obtain a sufficient condition for the existence of a nonnegative solution for this model, which is relevant in ecology (cf. [8, 13, 14]). Finally, we note that when $\psi(u) = u$, our results in Sections 2 and 3 include the case of nondegenerate diffusion.

2. Existence and Uniqueness of Solutions for Scalar Equations with Spatial Discontinuities

Before discussing systems of equations, we study the existence and uniqueness problems for scalar equations of the type

$$\Delta\psi(w) + f(x, w) = 0 \qquad \text{in } D, \tag{2.1a}$$

$$w = 0 \qquad \text{on } \partial D. \tag{2.1b}$$

Here D is a bounded connected domain in R^N ($N \geqslant 2$) with boundary $\partial D \in C^2$; $\Delta = \sum_{i=1}^{N} \partial^2/\partial x_i^2$ is the Laplacian operator. The functions ψ: $[0, \infty) \to [0, \infty)$, f: $D \times [0, \infty) \to R^1$ are assumed to satisfy the following hypotheses:

(H1) $\psi \in C^1[0, \infty)$, $\psi(0) = 0$, and $\psi'(s) > 0$ for $s > 0$.

(H2) There is a bounded interval $[0, b]$ such that

(i) $f \in L^\infty(D \times [0, b])$;

(ii) for any fixed $x \in D$ a.e., the function $f(x, y)$ is continuous in y for all $y \in [0, b]$;

(iii) there is a constant $M > 0$ such that $f(x, y_2) - f(x, y_1) \geqslant -M(\psi(y_2) - \psi(y_1))$ for $x \in D$ a.e., $0 \leqslant y_1 \leqslant y_2 \leqslant b$.

(H3) For each fixed $x \in D$ a.e., the function $f(x, y)/\psi(y)$ is a strictly monotonic increasing or decreasing function in y for $y \in [0, b]$.

The hypotheses and results in this section will lead to insights for the study of systems in the next section. For fixed ideas, we give the following definitions.

DEFINITION 2.1. Let u be an integrable function in D and α any multi-index. Then a locally integrable function v in D is called the αth weak derivative of u if it satisfies

$$\int_D v\phi \, dx = (-1)^{|\alpha|} \int_D uD^\alpha \phi \, dx, \qquad \text{for all} \quad \phi \in C_0^{|\alpha|}(D).$$

We write $v = D^\alpha u$ and note that $D^\alpha u$ is uniquely determined up to sets of measure zero.

DEFINITION 2.2. Let k be a nonnegative integer and let $1 \leqslant p < \infty$. The space $W^{k,p}(D)$ consists of all functions u in the real space $L^p(D)$ whose weak derivatives of all order $\leqslant k$ exist and belong to $L^p(D)$. The space $W^{k,p}(D)$ is normed by

$$\|u\|_{k,p} = \left\{ \sum_{|\alpha| \leqslant k} \int_D |D^\alpha u(x)|^p \, dx \right\}^{1/p}.$$

Denote by $W_0^{k,p}(D)$ the completion in the space $W^{k,p}(D)$ of the subset $C_0^\infty(D)$. It is well known that both $W^{k,p}(D)$ and $W_0^{k,p}(D)$ are Banach spaces.

DEFINITION 2.3. A function $w \in C(\bar{D})$ is called a nonnegative solution of (2.1) if $w(x) \geqslant 0$ in D and $u = \psi(w) \in W^{2,p}(D) \cap W_0^{1,p}(D)$ $(p > N)$ satisfies

$$\Delta u + f(x, \psi^{-1}(u)) = 0 \qquad \text{a.e. in } D, \tag{2.2a}$$

$$u = 0 \qquad \text{on } \partial D, \tag{2.2b}$$

where the derivatives of u are taken in the weak sense. A function w is called a positive solution of (2.1) if, in addition, $w(x) > 0$ in D.

We first prove an existence result for a nonnegative solution between the "upper" and "lower" solutions in the sense of (2.3) below.

LEMMA 2.1. *Suppose that* (H1), (H2, i) *to* (H2, iii) *are satisfied. Assume* *that there are functions* $\underline{w}$, $\bar{w}$ *in* $C(\bar{D})$ *with* $0 \leqslant \underline{w} \leqslant \bar{w} \leqslant b$ *in* $\bar{D}$ *and that* $\psi(\underline{w})$, $\psi(\bar{w})$ *are in* $W^{1,p}(D)$ $(p > N)$ *satisfying the inequalities*

$$-\int_D \nabla\psi(\underline{w})\nabla\phi\, dx + \int_D f(x, \underline{w})\phi\, dx \geqslant 0, \qquad \underline{w} = 0 \qquad \text{on } \partial D, \quad (2.3\text{a})$$

$$-\int_D \nabla\psi(\bar{w})\nabla\phi\, dx + \int_D f(x, \bar{w})\phi\, dx \leqslant 0 \qquad (2.3\text{b})$$

for all $\phi \in C_0^1(D)$, $\phi \geqslant 0$. *Then there exists at least one nonnegative solution* w *of* (2.1) *satisfying* $\underline{w} \leqslant w \leqslant \bar{w}$ *in* $\bar{D}$.

Proof. For any given $u \in C(\bar{D})$ with $0 \leqslant u \leqslant \psi(b)$, by hypothesis (H1), we have $0 \leqslant \psi^{-1}(u) \leqslant b$, $\psi^{-1}(u) \in C(\bar{D})$; and $f(x, \psi^{-1}(u)) \in L^\infty(D)$ by hypothesis (H2, i). Since D is a bounded domain, we obtain $L^\infty(D) \subset L^p(D)$ for all $1 \leqslant p < \infty$, and

$$Mu + f(x, \psi^{-1}(u)) \in L^p(D).$$

Here M is given in hypothesis (H2, iii). It follows from the linear elliptic L^p-theory that the problem

$$\Delta v - Mv + Mu + f(x, \psi^{-1}(u)) = 0 \qquad \text{a.e. in } D, \qquad (2.4\text{a})$$

$$v = 0 \qquad \text{on } \partial D \qquad (2.4\text{b})$$

has a unique solution v, say $S(u)$, in $W^{2,p}(D) \cap W_0^{1,p}(D) \subset C(\bar{D})$ satisfying

$$\|S(u)\|_{2,p} \leqslant \bar{C}\, \|Mu + f(x, \psi^{-1}(u))\|_p, \qquad (2.5)$$

where $\bar{C}$ is a positive constant which depends only on D and p.

Letting $\underline{u} = \psi(\underline{w})$, $\bar{u} = \psi(\bar{w})$, we have by hypothesis (H1) that $0 \leqslant \underline{u} \leqslant \bar{u} \leqslant \psi(b)$ in $\bar{D}$, and $\underline{u}$, $\bar{u} \in C(\bar{D})$. Hence, as above, we obtain $S(\underline{u})$, $S(\bar{u})$ in $W^{2,p}(D) \cap W_0^{1,p}(D)$ as the unique solution of (2.4) corresponding to $\underline{u}$ and $\bar{u}$, respectively.

We now construct a monotonic sequence $\{u_i\}$ which will converge in $W^{2,p}(D)$ to a solution of (2.2). First, define $u_0 = \bar{u}$ in $\bar{D}$. From the arguments above, we can define u_{i+1}, $i = 1, 2, \ldots$, iteratively as the solution of

$$\Delta v - Mv + Mu_i + f(x, \psi^{-1}(u_i)) = 0 \qquad \text{a.e. in } D, \qquad (2.6\text{a})$$

$$v = 0 \qquad \text{on } \partial D \qquad (2.6\text{b})$$

provided that each successive $u_i \geqslant 0$ in D so that $f(x, \psi^{-1}(u_i))$ is defined. We then have $u_{i+1} = S(u_i) \in W^{2,p}(D) \cap W_0^{1,p}(D) \subset C(\bar{D})$ for $i = 1, 2, \ldots$.

We first show that these u_i are properly defined and that

$$0 \leqslant \underline{u} \leqslant \cdots \leqslant u_2 \leqslant u_1 \leqslant u_0 = \bar{u} \qquad \text{in } \bar{D}. \tag{2.7}$$

Since $u_0 \geqslant 0$, Eq. (2.6) is meaningful for $i = 0$. Multiplying (2.6a) by $\phi \in C_0^1(D)$ and integrating on D, we obtain for $i = 0$ that

$$\int_D (\Delta u_{i+1} - M u_{i+1})\phi \, dx + \int_D [M u_i + f(x, \psi^{-1}(u_i))]\phi \, dx = 0 \tag{2.8}$$

for all $\phi \in C_0^1(D)$. Since

$$\int_D \Delta u_{i+1}\phi \, dx = -\int_D \nabla u_{i+1} \nabla \phi \, dx \qquad \text{for all } \phi \in C_0^1(D),$$

(2.8) yields

$$-\int_D \nabla u_{i+1} \nabla \phi \, dx + \int_D [-M u_{i+1} + M u_i + f(x, \psi^{-1}(u_i))]\phi \, dx = 0 \tag{2.9}$$

for all $\phi \in C_0^1(D)$. From the definition of $u_0 = \bar{u} = \psi(\bar{w})$ and hypothesis (2.3b), we have

$$-\int_D \nabla u_0 \nabla \phi \, dx + \int_D f(x, \psi^{-1}(u_0))\phi \, dx \leqslant 0 \tag{2.10}$$

for all $\phi \in C_0^1(D)$ with $\phi \geqslant 0$. Setting $i = 0$ (2.9) and subtracting (2.10), we obtain

$$-\int_D \nabla(u_1 - u_0) \nabla \phi \, dx - M \int_D (u_1 - u_0)\phi \, dx \geqslant 0 \tag{2.11}$$

for all $\phi \in C_0^1(D)$ with $\phi \geqslant 0$. It follows from the weak maximum principle (see [6, p. 179]) that

$$\sup_D (u_1 - u_0) \leqslant \sup_{\partial D} (u_1 - u_0)^+ = 0,$$

hence $u_1 \leqslant u_0 = \bar{u}$ in D. Similarly, using (2.9) with $i = 0$ and hypothesis (2.3a), we deduce that $\underline{u} \leqslant u_1$. We next inductively assume that

$$\underline{u} \leqslant u_j \leqslant u_{j-1} \leqslant \bar{u} \qquad \text{in } \bar{D} \tag{2.12}$$

for $j \geqslant 1$. Thus Eqs. (2.6), (2.8), and (2.9) are meaningful for $i=j$ and $j-1$. Letting $i=j$ and $j-1$ in (2.6), we subtract to obtain

$$\begin{aligned} &\Delta(u_{j+1}-u_j)-M(u_{j+1}-u_j)+M(u_j-u_{j-1})+f(x,\psi^{-1}(u_j)) \\ &\quad -f(x,\psi^{-1}(u_{j-1}))=0 \qquad \text{a.e. in } D \\ &\qquad u_{j+1}-u_j=0 \qquad \text{on } \partial D. \end{aligned} \tag{2.13}$$

Since $0 \leqslant \psi^{-1}(u_j) \leqslant \psi^{-1}(u_{j-1}) \leqslant b$, we obtain from (H2, iii) that

$$M(u_{j-1}-u_j)+f(x,\psi^{-1}(u_{j-1}))-f(x,\psi^{-1}(u_j)) \geqslant 0 \qquad \text{a.e. in } D,$$

and (2.13) yields

$$\begin{aligned} \Delta(u_{j+1}-u_j)-M(u_{j+1}-u_j) &\geqslant 0 \qquad \text{a.e. in } D, \\ u_{j+1}-u_j &= 0 \qquad \text{on } \partial D. \end{aligned} \tag{2.14}$$

It follows from the maximum principle (see [6, p. 225]) that $u_{j+1} \leqslant u_j$ in D. Analogously, using (2.9) for $i=j$ and (2.3a) as before, we obtain by the maximum principle that $\underline{u} \leqslant u_{j+1}$ in $\bar{D}$. By induction, we have

$$\underline{u} \leqslant \cdots \leqslant u_{i+1} \leqslant u_i \leqslant \cdots \leqslant u_2 \leqslant u_1 \leqslant u_0 \qquad \text{in } \bar{D}.$$

We can therefore define by pointwise convergence in $\bar{D}$

$$u(x)=\lim_{i\to\infty} u_i(x) \qquad \text{in } \bar{D}.$$

By the Lebesgue Convergence Theorem, $\{Mu_i+f(x,\psi^{-1}(u_i))\}$ must be a Cauchy sequence in $L^p(D)$. From the equations satisfied by $u_{i+1}-u_{j+1}$, we obtain the estimate as in (2.5) that

$$\|u_{i+1}-u_{j+1}\|_{2,p} \leqslant \bar{C}\,\|M(u_j-u_i)+f(x,\psi^{-1}(u_j))-f(x,\psi^{-1}(u_i))\|_p. \tag{2.15}$$

Consequently, $\{u_i\}$ is a Cauchy sequence in $W^{2,p}(D)$, and $u_i \to u$ in $W^{2,p}(D)$ as $i \to \infty$. Passing to the limit in (2.6), we have

$$\begin{aligned} \Delta u+f(x,\psi^{-1}(u)) &= 0 \qquad \text{a.e. in } D, \\ u &= 0 \qquad \text{on } \partial D, \end{aligned}$$

where the derivatives are taken in the weak sense and $u \in W^{2,p}(D)$ (note that $v=u_{i+1}$ in (2.6)). Furthermore, since the u_i are in $W_0^{1,p}(D)$, which is a closed subspace of $W^{1,p}(D)$, and $u_i \to u$ in $W^{1,p}(D)$, we must also have $u \in W_0^{1,p}(D)$. Letting $w=\psi^{-1}(u)$, we obtain w as a nonnegative solution of (2.1) with $\underline{w} \leqslant w \leqslant \bar{w}$.

With the addition of hypothesis (H3) and the assumption that the lower solution $\underline{w}$ is positive in D we now deduce a uniqueness result.

THEOREM 2.1. *Assume all the hypotheses in Lemma* 2.1. *In addition, suppose* that (H3) *is valid and that* $\underline{w} > 0$ *in D. Then there exists a unique positive solution* w^* *of* (2.1) *satisfying*

$$0 < \underline{w} \leqslant w^* \leqslant \bar{w} \qquad \text{in } D.$$

Proof. Let w be the solution of (2.1) obtained from the monotonic sequence in Lemma 2.1. Now $w > 0$ in D, since $w \geqslant \underline{w} > 0$ in D. Let z be any positive solution of (2.1) with $\underline{w} \leqslant z \leqslant \bar{w}$ in D. Then, $u = \psi(w)$, $v = \psi(z)$ are two positive solutions of (2.2) in $W^{2,p}(D) \cap W_0^{1,p}(D)$ with $0 < \underline{u} \leqslant u \leqslant \bar{u}$, $0 < \underline{u} \leqslant v \leqslant \bar{u}$ in D. By applying the same argument as that used in the proof of Lemma 2.1, we obtain $\underline{u} \leqslant v \leqslant u_i \leqslant \bar{u}$ in D for each $i = 0, \ldots$. Hence, we have the inequality

$$0 \leqslant \underline{u} \leqslant v \leqslant u \leqslant \bar{u} \qquad \text{in } D. \tag{2.16}$$

It remains to show that $v = u$ in $\bar{D}$. Since both u, v are in $W_0^{1,p}(D)$, there are two sequences $\{u_n\}$, $\{v_n\}$ in $C_0^\infty(D)$ which converge to u, v, respectively, in $W^{1,p}(D)$. Since u, v are solutions of (2.2) in $W^{2,p}(D)$, and $\{u_n\}$, $\{v_n\}$ have compact support in D, we use the definition of the weak derivative to obtain

$$\int_D u\, \Delta v_n\, dx + \int_D f(x, \psi^{-1}(u)) v_n\, dx = 0 \tag{2.17a}$$

$$\int_D v\, \Delta u_n\, dx + \int_D f(x, \psi^{-1}(v)) u_n\, dx = 0 \tag{2.17b}$$

for $n = 1, 2, \ldots$. Subtracting the two previous equations, we obtain

$$\int_D [v\, \Delta u_n - u\, \Delta v_n]\, dx = \int_D [f(x, \psi^{-1}(u)) v_n - f(x, \psi^{-1}(v)) u_n]\, dx. \tag{2.18}$$

It follows from the definition of the weak derivative and $u \in W^{1,p}(D)$, u_n, $v_n \in C_0^\infty(D)$ that we have

$$\int_D u\, \Delta v_n\, dx = -\int_D \nabla u\, \nabla v_n\, dx$$

$$\int_D v\, \Delta u_n\, dx = -\int_D \nabla v\, \nabla u_n\, dx.$$

Hence, the left side of (2.18) becomes

$$\int_D [v\,\Delta u_n - u\,\Delta v_n]\,dx = \int_D [\nabla u\,\nabla v_n - \nabla v\,\nabla u_n]\,dx$$

$$= \int_D \nabla u(\nabla v_n - \nabla v)\,dx - \int_D \nabla v(\nabla u_n - \nabla u)\,dx. \qquad (2.19)$$

From the Schwarz inequality, we have

$$\int_D |\nabla u(\nabla v_n - \nabla v)|\,dx \leqslant \|\nabla u\|_{L^2(D)}\,\|\nabla(v_n - v)\|_{L^2(D)}.$$

Since $v_n \to v$ in $W^{1,p}(D)$, $p > N \geqslant 2$, it follows that

$$\int_D \nabla u(\nabla v_n - \nabla v)\,dx \to 0, \qquad \text{as} \quad n \to \infty.$$

Similarly, one also has

$$\int_D \nabla v(\nabla u_n - \nabla u)\,dx \to 0, \qquad \text{as} \quad n \to \infty.$$

Consequently, it follows from (2.19) that

$$\int_D [v\,\Delta u_n - u\,\Delta v_n]\,dx \to 0, \qquad \text{as} \quad n \to \infty. \qquad (2.20)$$

Equations (2.20) and (2.18) lead to the property that

$$\int_D [f(x, \psi^{-1}(u))v_n - f(x, \psi^{-1}(v))u_n]\,dx \to 0, \qquad \text{as} \quad n \to \infty. \quad (2.21)$$

On the other hand, from Sobolev's Imbedding Theorem, u_n and v_n are uniformly bounded in $\bar{D}$, so the Lebesgue Convergence Theorem leads to

$$\int_D [f(x, \psi^{-1}(u))v_n - f(x, \psi^{-1}(v))u_n]\,dx$$

$$\to \int_D [f(x, \psi^{-1}(u))v - f(x, \psi^{-1}(v))u]\,dx, \qquad \text{as} \quad n \to \infty. \quad (2.22)$$

From (2.21) and (2.22), we deduce that

$$\int_D [f(x, \psi^{-1}(u))v - f(x, \psi^{-1}(v))u]\,dx = 0. \qquad (2.23)$$

Suppose that $v \not\equiv u$ in $\bar{D}$. The set

$$D_1 \stackrel{\text{def}}{=} \{x \in D \mid v(x) < u(x)\}$$

then has measure greater than zero. From assumption (H3), we have

$$\begin{aligned} &f(x, \psi^{-1}(u))v - f(x, \psi^{-1}(v))u \\ &\quad = uv\left[\frac{f(x, \psi^{-1}(u))}{u} - \frac{f(x, \psi^{-1}(v))}{v}\right] > 0 \text{ (or } < 0) \qquad \text{a.e. in } D_1 \end{aligned} \tag{2.24}$$

(recall that $v(x) \geqslant \psi(\underline{w}(x)) > 0$ for all x in D). This leads to

$$\int_{D_1} \left[f(x, \psi^{-1}(u))v - f(x, \psi^{-1}(v))u \right] dx > 0 \text{ (or } < 0), \tag{2.25}$$

which contradicts Eq. (2.23), since

$$\begin{aligned} 0 &= \int_D [f(x, \psi^{-1}(u))v - f(x, \psi^{-1}(v))u]\, dx \\ &= \int_{D_1} [f(x, \psi^{-1}(u))v - f(x, \psi^{-1}(v))u]\, dx \neq 0. \end{aligned} \tag{2.26}$$

This completes the proof of the theorem.

3. Systems

In this section, we study the existence of positive solutions for elliptic systems of the type

$$\begin{aligned} \Delta\psi(w_1) + f_1(x, w_1, w_2) &= 0 \qquad \text{a.e. in } D, \\ \Delta\psi(w_2) + f_2(x, w_1, w_2) &= 0 \qquad \text{a.e. in } D, \\ w_1 = w_2 &= 0 \qquad \text{on } \partial D, \end{aligned} \tag{3.1}$$

where the derivatives are taken in the weak sense; D is a bounded domain in R^N $(N \geqslant 2)$ with boundary $\partial D \in C^2$; and ψ: $[0, \infty) \to [0, \infty)$, f_i: $D \times [0, \infty) \times [0, \infty) \to R^1$ are functions satisfying the following assumptions:

($\tilde{\text{H}}$1) $\psi \in C^1[0, \infty)$, $\psi(0) = 0$, and $\psi'(s) > 0$ for $s > 0$.

($\tilde{\text{H}}$2) There are two positive constants b_1, b_2 such that

(i) $f_i \in L^\infty(D \times [0, b_1] \times [0, b_2])$ for $i = 1, 2$;

(ii) for any fixed $x \in D$ a.e. the functions $f_i(x, y_1, y_2)$ are continuous in (y_1, y_2) for all $(y_1, y_2) \in [0, b_1] \times [0, b_2]$, $i = 1, 2$;

(iii) there is a constant $M>0$ such that

$$f_1(x, \xi, y_2)-f_1(x, \eta, y_2) \geqslant -M(\psi(\xi)-\psi(\eta))$$
$$\text{for} \quad x \in D \quad \text{a.e.}, \; y_2 \in [0, b_2], \quad 0 \leqslant \eta \leqslant \xi \leqslant b_1.$$
$$f_2(x, y_1, \xi)-f_2(x, y_1, \eta) \geqslant -M(\psi(\xi)-\psi(\eta))$$
$$\text{for} \quad x \in D \quad \text{a.e.}, \; y_1 \in [0, b_1], \quad 0 \leqslant \eta \leqslant \xi \leqslant b_2.$$

DEFINITION 3.1. A pair of continuous functions (w_1, w_2) in $C(\bar{D})$ is called a positive solution of (3.1) if $\psi(w_i) \in W^{2,p}(D) \cap W_0^{1,p}(D)$ $(p>N)$ and (3.1) holds.

THEOREM 3.1. *Assume hypotheses* ($\tilde{\mathrm{H}}1$) *and* ($\tilde{\mathrm{H}}2$). *Suppose that there are functions* $\underline{w}_i(x)$, $\bar{w}_i(x)$ $(i=1,2)$ *in* $C(\bar{D})$ *with* $\psi(\underline{w}_i)$, $\psi(\bar{w}_i)$ $(i=1,2)$ *in* $W^{1,p}(D)$ $(p>N)$ *satisfying the inequalities*

$$-\int_D \nabla\psi(\underline{w}_1)\nabla\phi\, dx+\int_D f_1(x, \underline{w}_1, w_2)\phi\, dx \geqslant 0 \qquad \textit{for} \quad \underline{w}_2 \leqslant w_2 \leqslant \bar{w}_2 \quad (3.2a)$$

$$-\int_D \nabla\psi(\bar{w}_1)\nabla\phi\, dx+\int_D f_1(x, \bar{w}_1, w_2)\phi\, dx \leqslant 0 \qquad \textit{for} \quad \underline{w}_2 \leqslant w_2 \leqslant \bar{w}_2 \quad (3.2b)$$

$$-\int_D \nabla\psi(\underline{w}_2)\nabla\phi\, dx+\int_D f_2(x, w_1, \underline{w}_2)\phi\, dx \geqslant 0 \qquad \textit{for} \quad \underline{w}_1 \leqslant w_1 \leqslant \bar{w}_1 \quad (3.2c)$$

$$-\int_D \nabla\psi(\bar{w}_2)\nabla\phi\, dx+\int_D f_2(x, w_1, \bar{w}_2)\phi\, dx \leqslant 0 \qquad \textit{for} \quad \underline{w}_1 \leqslant w_1 \leqslant \bar{w}_1 \quad (3.2d)$$

for all $\phi \in C_0^1(D)$, $\phi \geqslant 0$. *Here* $w_i=w_i(x)$ *are assumed to be continuous in* $\bar{D}$, *and* $0 \leqslant \underline{w}_i \leqslant w_i \leqslant \bar{w}_i \leqslant b_i$ *in* $\bar{D}$, $\underline{w}_i>0$ *in* D, *and* $\underline{w}_i=0$ *on* ∂D. *Then there exists at least one positive solution* (w_1^*, w_2^*) *of* (3.1) *satisfying* $\underline{w}_i \leqslant w_i^* \leqslant \bar{w}_i$ *in* $\bar{D}$.

Proof. Let $\underline{u}_i=\psi(\underline{w}_i)$, $\bar{u}_i=\psi(\bar{w}_i)$, $X_i=\{u \in C(\bar{D}),\ \underline{u}_i \leqslant u_i \leqslant \bar{u}_i \text{ in } \bar{D}\}$, $i=1,2$, and let M be described as in ($\tilde{\mathrm{H}}2$, iii). The set $X_1 \times X_2$ is a bounded closed convex set in $C(\bar{D}) \times C(\bar{D})$. We define the map $T\colon X_1 \times X_2 \to X_1 \times X_2$ as

$$T(u_1, u_2)=(v_1, v_2) \qquad \text{for} \quad (u_1, u_2) \in X_1 \times X_2,$$

where $v_1, v_2 \in W^{2,p}(D) \cap W_0^{1,p}(D) \subset C(\bar{D})$ $(p>N)$ and (v_1, v_2) is uniquely determined as the solution of the (decoupled) system

$$\Delta v_i-Mv_i+f_i(x, \psi^{-1}(u_1), \psi^{-1}(u_2))+Mu_i=0 \qquad \text{in } D, \quad i=1,2. \; (3.3)$$

(Here the derivatives are meant in the weak sense.)

We first show that $(v_1, v_2) \in X_1 \times X_2$. From Eq. (3.3), and hypotheses ($\tilde{\text{H}}$1), ($\tilde{\text{H}}$2, iii) and (3.2b), we have, for any $\phi \in C_0^1(D)$, $\phi \geqslant 0$,

$$-\int_D \nabla(\bar{u}_1 - v_1)\nabla\phi\, dx - M\int_D (\bar{u}_1 - v_1)\phi\, dx \leqslant -\int_D [f_1(x, \psi^{-1}(\bar{u}_1), \psi^{-1}(u_2))$$

$$-f_1(x, \psi^{-1}(u_1), \psi^{-1}(u_2))]\phi\, dx - M\int_D (\bar{u}_1 - u_1)\phi \leqslant 0. \tag{3.4}$$

Hence the weak maximum principle implies that $\bar{u}_1 \geqslant v_1$. Analogously, since

$$-\int_D \nabla(\underline{u}_1 - v_1)\nabla\phi\, dx - M\int_D (\underline{u}_1 - v_1)\phi\, dx \geqslant -\int_D [f_1(x, \psi^{-1}(\underline{u}_1), \psi^{-1}(u_2))$$

$$-f_1(x, \psi^{-1}(u_1), \psi^{-1}(u_2))]\phi\, dx - M\int_D (\underline{u}_1 - u)\phi \geqslant 0 \tag{3.5}$$

for any $\phi \in C_0^1(D)$, $\phi \geqslant 0$, we deduce that $\underline{u}_1 \leqslant v_1$. We apply the same procedure to prove that $v_2 \in X_2$.

We next show that T is a continuous operator from $X_1 \times X_2$ into itself. Let $(u_1^{(n)}, u_2^{(n)})$ be a sequence in $X_1 \times X_2$, which converges to (u_1, u_2) in $X_1 \times X_2$. Define $(v_1^{(n)}, v_2^{(n)}) = T(u_1^{(n)}, u_2^{(n)})$, and $(v_1, v_2) = T(u_1, u_2)$ as in (3.1). By the classical L^p-estimate for the linear problem (3.3), we have

$$\|v_i^{(n)}\|_{2,p} \leqslant \bar{C}_i \, \|f_i(x, \psi^{-1}(u_1^{(n)}), \psi^{-1}(u_2^{(n)})) + Mu_i^{(n)}\|_p \tag{3.6}$$

with $\underline{u}_i \leqslant v_i^{(n)} \leqslant \bar{u}_i$ for $n = 1, 2, ...$, where $\bar{C}_i$ are positive constants. By ($\tilde{\text{H}}$2, i), there exist constants $M_i > 0$ such that

$$|f_i(x, y_1, y_2)| \leqslant M_i \qquad \text{for almost all } (x, y_1, y_2) \in D \times [0, b_1] \times [0, b_2]. \tag{3.7}$$

Since D is a bounded domain in R^N, (3, 7) implies that $\{f_i(x, \psi^{-1}(u_1^{(n)}), \psi^{-1}(u_2^{(n)})\}$ are bounded sequences in $L^p(D)$. It follows from (3.6) that $\{v_i^{(n)}\}$ is a bounded sequence in $W^{2,p}(D) \cap W_0^{1,p}(D)$ $(p > N)$. Applying Sobolev's theorem, we can select a subsequence $\{v_i^{(n_k)}\}$ from $\{v_i^{(n)}\}$ such that $\{v_i^{(n_k)}\}$ converges in $C(\bar{D})$ to, say, v_i^*. To see whether $\{v_i^{(n_k}\}$ actually converges to v_i^* in $W^{2,p}(D)$, we first deduce from ($\tilde{\text{H}}$2, ii) that

$$f_i(x, \psi^{-1}(u_1^{(n_k)}), \psi^{-1}(u_2^{(n_k)})) \to f_i(x, \psi^{-1}(u_1), \psi^{-1}(u_2)) \tag{3.8}$$

pointwise in D. Since D is bounded, the Lebesgue Convergence Theorem implies that the convergence in (3.8) is true in the $L^p(D)$ norm

($N<p<\infty$). The estimate (3.6) hence implies that $\{v_i^{(n_k)}\}$ converges to v_i^* in $W^{2,p}(D)$. By the definition of $\{v_i^{(n_k)}\}$, we have, for $k=1,2,\ldots$, that

$$\Delta v_i^{(n_k)} - M v_i^{(n_k)} + f_i(x, \psi^{-1}(u_1^{(n_k)}), \psi^{-1}(u_2^{(n_k)})) + M u_i^{(n_k)} = 0 \quad \text{in } D, \quad i=1,2. \tag{3.9}$$

Passing to the limit in (3.9), we obtain

$$\Delta v_i^* - M v_i^* + f_i(x, \psi^{-1}(u_1), \psi^{-1}(u_2)) + M u_i = 0 \quad \text{in } D, \quad i=1,2. \tag{3.10}$$

From (3.3) and (3.10), we see that both (v_1^*, v_2^*) and (v_1, v_2) are positive solutions of the same linear problem. We conclude by uniqueness of the positive solution of the linear problem (3.3) that $(v_1^*, v_2^*) = (v_1, v_2)$. Hence we have $\{v_i^{(n_k)}\} \to v_i$ in $C(\bar{D})$. Finally, we claim that the full sequence $\{v_i^{(n)}\} \to v_i$ in $C(\bar{D})$ as $i \to \infty$. Suppose not; then there exist a subsequence $\{v_i^{(n_j)}\}$ and a constant $\varepsilon_0 > 0$ such that

$$\|v_i^{(n_j)} - v_i\| \geqslant \varepsilon_0 \qquad \text{for} \quad j=1,2,\ldots. \tag{3.11}$$

Here the norm is taken in $C(\bar{D})$. Using the same argument as that used above, by replacing $\{v_i^{(n)}\}$ with $\{v_i^{(n_j)}\}$, we can select a subsequence of $\{v_i^{(n_j)}\}$ which converges to v_i in $C(\bar{D})$. This contradicts the inequality (3.11). Consequently, $\{v_i^{(n)}\}$ converges to v_i in $C(\bar{D})$ as $i \to \infty$. This leads to the conclusion that T is a continuous operator from $X_1 \times X_2$ into itself.

We finally show that T is a compact operator. From (3.6), T maps a bounded set in $X_1 \times X_2$ to a bounded set in $W_0^{1,p}(D) \times W_0^{1,p}(D)$. By the Sobolev Compact Imbedding Theorem, the identity map from $W_0^{1,p}(D)$ to $C(\bar{D})$ is compact. Hence, we can view T as a composition of a bounded map from $X_1 \times X_2$ to $W_0^{1,p}(D) \times W_0^{1,p}(D)$ followed by a compact identity map from $W_0^{1,p}(D) \times W_0^{1,p}(D)$ to $X_1 \times X_2$; and we conclude that T is a compact operator from $X_1 \times X_2$ into itself. Schauder's fixed point theorem asserts that T has a fixed point (u_1^*, u_2^*) in $X_1 \times X_2$. It follows from (3.3) that

$$\begin{aligned} \Delta u_1^* + f_1(x, \psi^{-1}(u_1^*), \psi^{-1}(u_2^*)) &= 0 \qquad \text{a.e. in } D, \\ \Delta u_2^* + f_2(x, \psi^{-1}(u_1^*), \psi^{-1}(u_2^*)) &= 0 \qquad \text{a.e. in } D, \\ u_1^* = u_2^* &= 0 \qquad \text{on } \partial D. \end{aligned} \tag{3.12}$$

The fact that (u_1^*, u_2^*) is in $X_1 \times X_2$ implies that (u_1^*, u_2^*) is in $W^{2,p}(D) \cap W_0^{1,p}(D)$ and that $\underline{u}_i \leqslant u_i^* \leqslant \bar{u}_i$ is in $\bar{D}$ for $i=1,2$. Consequently, $(w_1^*, w_2^*) = (\psi^{-1}(u_1^*), \psi^{-1}(u_2^*))$ is a positive solution of (3.1) with $\underline{w}_i \leqslant w_i^* \leqslant \bar{w}_i$ in $\bar{D}$.

The following corollary is sometimes more readily applicable than Theorem 3.1.

COROLLARY 3.1. *Assume hypotheses* ($\tilde{\text{H}}1$) *and* ($\tilde{\text{H}}2$). *Suppose that there are functions* $\underline{w}_i(x)$, $\bar{w}_i(x)$ $(i=1,2)$ *in* $C(\bar{D})$ *with* $\psi(\underline{w}_i)$, $\psi(\bar{w}_i)$ $(i=1,2)$ *in* $W^{2,p}(D)$ $(p>N)$ *satisfying the inequalities*

$$\Delta\psi(\underline{w}_1)+f_1(x,\underline{w}_1,w_2)\geqslant 0 \quad \text{a.e. in } D \quad \text{for} \quad \underline{w}_2\leqslant w_2\leqslant \bar{w}_2 \tag{3.13a}$$

$$\Delta\psi(\bar{w}_1)+f_1(x,\bar{w}_1,w_2)\leqslant 0 \quad \text{a.e. in } D \quad \text{for} \quad \underline{w}_2\leqslant w_2\leqslant \bar{w}_2 \tag{3.13b}$$

$$\Delta\psi(\underline{w}_2)+f_2(x,w_1,\underline{w}_2)\geqslant 0 \quad \text{a.e. in } D \quad \text{for} \quad \underline{w}_1\leqslant w_1\leqslant \bar{w}_1 \tag{3.13c}$$

$$\Delta\psi(\bar{w}_2)+f_2(x,w_1,\bar{w}_2)\leqslant 0 \quad \text{a.e. in } D \quad \text{for} \quad \underline{w}_1\leqslant w_1\leqslant \bar{w}_1, \tag{3.13d}$$

where the derivatives are taken in a weak sense. Here $w_i=w_i(x)$ *are assumed to be continuous in* $\bar{D}$, *and* $0\leqslant \underline{w}_i\leqslant w_i\leqslant \bar{w}_i\leqslant b_i$ *in* $\bar{D}$, $\underline{w}_i>0$ *in* D, *and* $\underline{w}_i=0$ *on* ∂D. *Then there exists at least one positive solution* (w_1^*, w_2^*) *of* (3.1) *satisfying* $\underline{w}_i\leqslant w_i^*\leqslant \bar{w}_i$ *in* $\bar{D}$.

Proof. This is an immediate result of Theorem 3.1 since (3.13) implies (3.2). To see this, we let $\phi\in C_0^1(D)$, $\phi\geqslant 0$, and multiply (3.13a) by ϕ. We integrate over D to find

$$\int_D \Delta\psi(\underline{w}_1)\phi\,dx+\int_D f_1(x,\underline{w}_1,w_2)\phi\,dx\geqslant 0 \qquad \text{for} \quad \underline{w}_2\leqslant w_2\leqslant \bar{w}_2. \tag{3.14}$$

It follows from the definition of the weak derivative that

$$-\int_D \nabla\psi(\underline{w}_1)\nabla\phi\,dx+\int_D f_1(x,\underline{w}_1,w_2)\phi\,dx\geqslant 0 \qquad \text{for} \quad \underline{w}_2\leqslant w_2\leqslant \bar{w}_2. \tag{3.15}$$

Similarly, we can verify the rest of the inequalities in (3.2). By application of Theorem 3.1, the proof is completed.

4. APPLICATIONS TO ECOLOGICAL MODELS

In the first part of this section, we apply the results in Section 3 to a prey–predator ecological model with degenerate density-dependent diffusion

$$\Delta u^m = u(a(x)-bu^k-cv)=0 \qquad \text{in } D, \tag{4.1a}$$

$$\Delta v^m = v(e(x)+fu-gv^k)=0 \qquad \text{in } D, \tag{4.1b}$$

$$u=v=0 \qquad \text{on } \partial D. \tag{4.1c}$$

Here D is a bounded connected domain in R^N $(N \geqslant 2)$ with boundary $\partial D \in C^2$, and m, k, b, c, f, g are positive constants with $1+k > m > 1$. We assume that $a(x)$, $e(x)$ are two positive functions in $L^\infty(D)$ with

$$\underline{a} \stackrel{\text{def}}{=} \operatorname*{ess\,inf}_{x \in D} a(x) > 0 \qquad \text{and} \qquad \underline{e} \stackrel{\text{def}}{=} \operatorname*{ess\,inf}_{x \in D} e(x) > 0. \tag{4.2}$$

For convenience, we denote $\bar{a} =^{\text{def}} \operatorname{ess\,sup}_{x \in D} a(x) > 0$ and $\bar{e} =^{\text{def}} \operatorname{ess\,sup}_{x \in D} e(x) > 0$. The following theorem gives sufficient conditions for the coexistence of the two species. If one compares them with the results in [10] for the nondegenerate case, we see that the conditions here are much more readily satisfied. For example, there is no need for the intrinsic growth rates $a(x)$ and $e(x)$ to be larger than the principal eigenvalue for the domain D. Other related references can be found in the Introduction and in [8, 11, 12, 13, 14, and 16].

THEOREM 4.1. *Assume* $1+k > m > 1$, *hypothesis* (4.2), *and*

$$g(\underline{a}/c)^k > \bar{e} + f(\bar{a}/b)^{1/k}. \tag{4.3}$$

Then there exists a positive solution (u, v) *of* (4.1) *with* $u, v \in C(\bar{D})$ *and* $u, v \in W^{2,p}(D) \cap W_0^{1,p}(D)$ $(p > N)$. *Moreover, the solution satisfies*

$$0 < u \leqslant (\bar{a}/b)^{1/k}, \qquad 0 < v \leqslant [g^{-1}(\bar{e} + f(\bar{a}/b)^{1/k}]^{1/k} \qquad \text{in } D$$

Proof. We will apply Corollary 3.1. Let $b_1 = (\bar{a}/b)^{1/k}$, $b_2 = [g^{-1}(\bar{e} + f(\bar{a}/b)^{1/k}]^{1/k}$. Define

$$\psi(s) = s^m \qquad \text{for} \quad s \geqslant 0$$

and

$$f_1(x, y_1, y_2) = y_1(a(x) - by_1^k - cy_2),$$
$$f_2(x, y_1, y_2) = y_2(e(x) + fy_1 - gy_2{}^k) \quad \text{for } (x, y_1, y_2) \in D \times [0, \infty) \times [0, \infty).$$

Then one can immediately verify that $(\tilde{\text{H}}1)$ and $(\tilde{\text{H}}2, i\text{–}\tilde{\text{H}}2, \text{ii})$ are satisfied. Since

$$\begin{aligned} f_1(x, \xi, y_2) - f_1(x, \eta, y_2) &= \xi(a(x) - b\xi^k - cy_2) - \eta(a(x) - b\eta^k - cy_2) \\ &= (a(x) - cy_2)(\xi - \eta) - b(\xi^{k+1} - \eta^{k+1}) \\ &\geqslant (\underline{a} - cb_2)(\xi - \eta) - b(\xi^{k+1} - \eta^{k+1}) \\ &\quad \text{for } \ x \in D \text{ a.e., } y_2 \in [0, b_2], \quad 0 \leqslant \eta \leqslant \xi \leqslant b_1, \end{aligned}$$

we can verify the first part of ($\tilde{\text{H}}2$, iii) by showing that there is a constant $M > 0$ such that

$$(\underline{a} - cb_2)(\xi - \eta) - b(\xi^{k+1} - \eta^{k+1}) > -M(\xi^m - \eta^m), \qquad 0 \leqslant \eta \leqslant \xi \leqslant b_1. \quad (4.4)$$

From hypothesis (4.3), we have $\underline{a} - cb_2 > 0$; thus (4.4) is satisfied if

$$M(\xi^m - \eta^m) \geqslant b(\xi^{k+1} - \eta^{k+1}) \qquad \text{for} \quad 0 \leqslant \eta \leqslant \xi \leqslant b_1. \qquad (4.5)$$

However, (4.5) can be readily verified if we note that the function $h(\xi) = M\xi^m - b\xi^{k+1}$ is increasing in $[0, b_1]$ by choosing $M > (b/m)(k+1)b_1^{k+1-m}$. Similarly, we verify the second part of ($\tilde{\text{H}}2$, iii),

$$\begin{aligned} f_2(x, y_1, \xi) - f_2(x, y_1, \eta) &= \xi(e(x) + fy_1 - g\xi^k) - \eta(e(x) + fy_1 - g\eta^k) \\ &\geqslant \underline{e}(\xi - \eta) - g(\xi^{k+1} - \eta^{k+1}) \\ &\geqslant -M(\psi(\xi) - \psi(\eta)), \end{aligned} \qquad (4.6)$$

for $x \in D$ a.e., $y_1 \in [0, b_1]$, $0 \leqslant \eta \leqslant \xi \leqslant b_2$ if $M > (g/m)(k+1)b_2^{k+1-m}$.

To construct upper and lower solutions $(\underline{u}, \underline{v})$ and $(\bar{u}, \bar{v})$, we let $\lambda_1 > 0$ be the principal eigenvalue for the problem

$$\begin{aligned} \Delta w + \lambda w &= 0 \qquad \text{in } D, \\ w &= 0 \qquad \text{on } \partial D, \end{aligned}$$

and $\phi(x)$ be the principal eigenfunction. Then we have $\phi(x) > 0$ in D and $\phi(x) = 0$ on ∂D. We define $\underline{u} = \underline{v} = (\delta\phi)^{1/m}$ in D for a small $\delta > 0$ to be determined. Thus they satisfy $\underline{u} = \underline{v} = 0$ on ∂D, $\underline{u} = \underline{v} > 0$ in D. Also, we define $\bar{u} = b_1$, $\bar{v} = b_2$. We verify that

$$\begin{aligned} &\Delta\bar{u}^m + \bar{u}(a(x) - b\bar{u}^k - cv) \leqslant \bar{u}(a(x) - b\bar{u}^k) \\ &\qquad = (\bar{a}/b)^{1/k}(a(x) - b(\bar{a}/b)) \leqslant 0 \qquad \text{a.e. in } D, \qquad \text{for all} \quad \underline{v} \leqslant v \leqslant \bar{v}, \quad (4.7) \\ &\Delta\bar{v}^m + \bar{v}(e(x) + fu - g\bar{v}^k) \leqslant \bar{v}(\bar{e} + f\bar{u} - g\bar{v}^k) = 0 \qquad \text{for all} \quad \underline{u} \leqslant u \leqslant \bar{u}. \quad (4.8) \end{aligned}$$

Moreover, we have

$$\begin{aligned} \Delta\underline{u}^m + \underline{u}(a(x) - b\underline{u}^k - cv) &= -\lambda_1(\delta\phi) + (\delta\phi)^{1/m}(a(x) - b(\delta\phi)^{k/m} - cv) \\ &\geqslant (\delta\phi)^{1/m}(-\lambda_1(\delta\phi)^{1-1/m} + \underline{a} - b(\delta\phi)^{k/m} - c\bar{v}) \geqslant 0 \end{aligned} \qquad (4.9)$$

for x a.e. in D and all $\underline{v} \leqslant v \leqslant \bar{v}$, when δ is sufficiently small, since $\underline{a} - c\bar{v} > 0$ by assumption (4.3). Finally

$$\Delta \underline{v}^m + \underline{v}(e(x) + fu - g\underline{v}^k) = -\lambda_1(\delta\phi) + (\delta\phi)^{1/m}(e(x) + fu - g(\delta\phi)^{k/m}) \geqslant (\delta\phi)^{1/m}(-\lambda_1(\delta\phi)^{1-1/m} + \underline{e} + f\underline{u} - g(\delta\phi)^{k/m}) \geqslant 0 \tag{4.10}$$

for x a.e. in D all $\underline{u} \leqslant u \leqslant \bar{u}$, when δ is sufficiently small, since $\underline{e} + f\underline{u} > 0$. From Corollary 3.1, the four inequalities above imply that there is a positive solution (u, v) of (4.1) with u, v in $C(\bar{D})$; u^m, $v^m \in W^{2,p}(D) \cap W_0^{1,p}(D)$; and

$$0 < u \leqslant (\bar{a}/b)^{1/k}, \qquad 0 < v \leqslant [g^{-1}(\bar{e} + f(\bar{a}/b)^{1/k}]^{1/k} \qquad \text{in } D.$$

In the second part of this section, we consider a population model with possibly *discontinuous growth rate* of the following type:

$$\begin{aligned} \Delta w + w(a(x) - bw) &= 0 \qquad \text{a.e. in } D, \\ w &= 0 \qquad \text{on } \partial D, \end{aligned} \tag{4.11}$$

where b is a positive constant, and $a(x)$ is a function in $L^\infty(D)$, which is possibly *discontinuous.* Here D is a bounded connected domain in R^N $(N \geqslant 2)$ with boundary $\partial D \in C^2$. Suppose that

$$a(x) > \lambda_1 \qquad \text{a.e. in } D. \tag{4.12}$$

We can construct $\bar{w} = \bar{a}/b$ as an upper solution of (4.11), where $\bar{a} =^{\text{def}} \operatorname{ess\,sup}_{x \in D} a(x) > 0$. Moreover, $\underline{w} = \delta\phi$ is a lower solution, where δ is sufficiently small and $\phi(x) > 0$ is as defined in Theorem 4.1. Then, we apply Theorem 2.1 to obtain a unique positive solution of (4.11) which satisfies $0 < w \leqslant \bar{a}/b$ in D. (Note that λ_1 and $\bar{a}$ are used with the same meanings as those given in the first example of this section.) However, in a highly spatially heterogeneous habitat in ecological problems, hypothesis (4.12) is not commonly satisfied. Consequently, we assume that $a(x)$ is relatively large in a subdomain D_s of D, and may be small or even negative outside D_s. Let λ_s be the principal eigenvalue for D_s, i.e., the first eigenvalue for $\Delta u + \lambda u = 0$ in D_s, $u = 0$ on ∂D_s. More specifically, we write $a(x)$ in the form

$$a(x) = \begin{cases} a_1(x), & \text{in } D_s \\ a_2(x), & \text{in } D \setminus D_s \end{cases} \tag{4.13}$$

and assume that D_s has boundary $\partial D_s \in C^2$; moreover, we impose the hypothesis

$$a_1(x) > \lambda_s \qquad \text{a.e. for} \quad x \in D_s. \tag{4.14}$$

Again, we let

$$\bar{a} = \operatorname*{ess\,sup}_{x \in D} a(x). \tag{4.15}$$

THEOREM 4.2. *Assume that* $a(x) \in L^\infty(D)$ *and that hypothesis* (4.14) *holds. Then the Dirichlet problem* (4.11) *has one and only one nonnegative nontrivial solution* w (*in the sense of Definition* 2.3 *with* $\psi(w) = w$) *satisfying* $0 \leqslant w \leqslant \bar{a}/b$ *in D. Moreover,* $w > 0$ *in D.*

Proof. Let $\psi(s) = s$ for $s \geqslant 0$ and $f(x, w) = w(a(x) - bw)$. One readily verifies that (H1), (H2, i–H2, iii) are satisfied. We will apply Lemma 2.1 to prove the existence of the solution. Let $\underline{w}$, $\bar{w}$ be defined as

$$\bar{w} = \bar{a}/b \quad \text{in } \bar{D}, \qquad \underline{w} = \begin{cases} \delta\theta(x), & x \in D_s \\ 0, & x \in \bar{D} \setminus D_s, \end{cases} \tag{4.16}$$

where $\theta(x)$ is a positive principal eigenfunction associated with the principal eigenvalue λ_s of the domain D_s, and $\delta > 0$ is to be determined. For $\delta > 0$ sufficiently small, we clearly have $0 \leqslant \underline{w} \leqslant \bar{w}$ in $\bar{D}$. The constant function $\bar{w}$ is in $W^{1,p}(D)$, and we now verify that $\underline{w} \in W^{1,p}(D)$. By the definition of $\underline{w}$, we have, for $|\alpha| = 1$, $\phi \in C_0^1(D)$,

$$-\int_D \underline{w} D^\alpha \phi \, dx = -\int_{D_s} \delta\theta(x) \, D^\alpha \phi \, dx. \tag{4.17}$$

Integrating by parts, we obtain

$$-\int_D \underline{w} D^\alpha \phi \, dx = \int_{D_s} \delta D^\alpha \theta(x) \phi \, dx, \tag{4.18}$$

since $\theta(x) = 0$ on ∂D_s. Hence the αth weak derivative of $\underline{w}$ is

$$D^\alpha \underline{w}(x) = \begin{cases} \delta D^\alpha \theta(x), & x \in D_s \\ 0, & x \in D \setminus D_s. \end{cases} \tag{4.19}$$

Since $D^\alpha \underline{w} \in L^p(D)$, we obtain $\underline{w} \in W^{1,p}(D)$. To see whether (2.3a) holds, we calculate

$$-\int_D \nabla \bar{w} \, \nabla \phi \, dx + \int_D \bar{w}(a(x) - b\bar{w}) \phi \, dx \leqslant \int_D \frac{\bar{a}}{b}\left(a(x) - b\,\frac{\bar{a}}{b}\right) \phi \, dx \leqslant 0 \tag{4.20}$$

for all $\phi \in C_0^1(D)$, $\phi \geqslant 0$. To verify (2.3b), one has

$$-\int_D \nabla \underline{w} \, \nabla\phi \, dx + \int_D \underline{w}(a(x) - b\underline{w})\phi \, dx$$

$$= -\int_{D_s} \delta\nabla \, \theta\nabla\phi \, dx + \int_{D_s} \delta\theta(x)(a_1 - b\,\delta\theta(x))\phi \, dx$$

$$= \int_{D_s} \delta\Delta \, \theta\varphi \, dx - \int_{\partial D_s} \delta \frac{\partial\theta}{\partial n} \phi \, ds + \int_{D_s} \delta\theta(x)(a_1 - b\,\delta\theta(x))\phi \, dx$$

$$= \int_{D_s} [-\lambda_s + a_1 - b\,\delta\theta(x)] \, \delta\theta \, \phi \, dx - \int_{\partial D_s} \delta \frac{\partial\theta}{\partial n} \phi \, ds, \qquad (4.21)$$

which is positive for $\delta > 0$ sufficiently small, by hypothesis (4.14) and the fact that $\partial\theta/\partial n \leqslant 0$ on ∂D_s. Applying Lemma 2.1, we conclude that (4.11) has a nonnegative solution w in $W^{2,p}(D) \cap W_0^{1,p}(D)$ with $0 \leqslant w \leqslant \bar{a}/b$ in D and $w > 0$ in D_s.

To prove that $w > 0$ in D, let $u(t, x) = e^{ct}w(x)$ for $(t, x) \in [0, +\infty) \times \bar{D}$, where c is a positive constant such that $c \geqslant \operatorname{ess\,sup}_{x \in D}(bw(x) - a(x))$. Thus, since w satisfies (4.11) (in the sense of distributions), we also have $\Delta u + u(a(x) - bw(x)) = 0$ in $[0, +\infty) \times D$; hence

$$u_t = cu = cu + \Delta u + u(a(x) - bw(x))$$

$$= \Delta u + u(c + a(x) - bw(x)) \geqslant \Delta u \qquad \text{in} \quad (0, +\infty) \times D,$$

by the choise of c. Thus u is an upper solution (in the sense of distributions) to the problem

$$\begin{aligned} v_t &= \Delta v && \text{in} \quad (0, +\infty) \times D \\ v &= 0 && \text{on} \quad (0, +\infty) \times \partial D \qquad (4.22) \\ v(0, x) &= w(x) && \text{in } D. \end{aligned}$$

Thus, if v is the solution to (4.22), we have $u(t, x) \geqslant v(t, x) > 0$ in $(0, +\infty) \times D$ (the last inequality follows from the maximum principle). Thus $w(x) > 0$ in D, by the definition of u (here use has been made of comparison results for upper solutions in the sense of distributions; this result can be found in [2]).

Finally, we prove that such a w is unique. Let w^* be the solution of (4.11) obtained from the monotonic convergence sequence as in Lemma 2.1, using $w_0 = \bar{a}/b$ as the first iterate and defining $w_{j+1} = S(w_j)$, (recall $\psi(s) = s$). Using the fact that $w \leqslant w_0$, we can prove that $w \leqslant w_j$ in D by using the maximum principle as in (2.14) with u_{j+1} and u_j respectively replaced by w and w_j. This leads to the fact that $0 < w \leqslant w^*$ in D. Let z be any nonnegative (nontrivial) solution of (4.11) with $z \leqslant \bar{a}/b$ in D. As above, we have $z \leqslant w^*$ in D. Let $f(x, w) = w(a(x) - bw)$. We follow the proof of Theorem 2.1, with the role of u, v respectively replaced by w^*, z until (2.23). Then (2.23) implies that

$$0 = \int_D [f(x, w^*)z - f(x, z)w^*]\,dx = \int_{\Omega_1} [f(x, w^*)z - f(x, z)w^*]\,dx$$

$$= \int_{\Omega_2} [f(x, w^*)z - f(x, z)w^*]\,dx, \tag{4.23}$$

where $\Omega_1 = \{x \in D \mid z(x) < w^*(x)\}$ and $\Omega_2 = \{x \in D \mid 0 < z(x) < w^*(x)\}$. The last equality follows from the fact that $f(x, 0) = 0$ for x a.e. in D. However,

$$f(x, w^*)z - f(x, z)w^* < 0 \qquad \text{in } \Omega_2. \tag{4.24}$$

We therefore conclude that $w^* = z$ in the set $\Omega = \{x \in D \mid 0 < z(x)\}$. We observe that the set Ω is open in D. Moreover, the set Ω is also closed in D for the following reason: Let $x_n \in \Omega$, $x_n \to x \in D$. Then

$$z(x) = \lim_{n \to \infty} z(x_n) = \lim_{n \to \infty} w^*(x_n) = w^*(x); \tag{4.25}$$

however, $w^*(x) > 0$ in D, therefore $z(x) > 0$, and $x \in \Omega$. Consequently, we must have $\Omega = D$. In Ω, we have concluded that $w^* = z$. Thus every nonnegative nontrivial solution bounded above by $\bar{a}/b$ must be identically equal to the same w^* in D. This completes the proof.

Remark 4.1. If the function $a(x)$ is in $C^\alpha(D)$, $0 < \alpha < 1$, and we have $a(x) < \lambda_1$ in D, then the only nonnegative solution of (4.11) in D is the trivial solution. (Here λ_1 is the principal eigenvalue for the domain D.)

Remark 4.2. If the function $a(x)$ is continuous, more general results for the degenerate case can be found in [16] and other works.

Remark 4.3. In this entire article, D has been assumed connected. However, if D is not connected, we do not have $\phi > 0$ in D but rather $\phi \geqslant 0$, $\phi \not\equiv 0$ in D. Thus, in this case have the following corollary for Theorem 4.2.

COROLLARY 4.1. *Suppose that D is not connected and that D_s is connected. Assume that $a(x) \in L^\infty(D)$ and that hypothesis* (4.14) *holds. Then the Dirichlet problem* (4.11) *has at least one nonnegative solution w* (*in the sense of Definition* 2.3 *with $\psi(w) = w$*) *satisfying $0 \leqslant w \leqslant \bar{a}/b$ in D and $w > 0$ in the component of D which contains D_s. Furthermore, if there is a positive solution v of* (4.11), *with $0 < v \leqslant \bar{a}/b$ in D, then it is the unique nonnegative* (*nontrivial*) *solution of* (4.11) *satisfying $0 < w \leqslant \bar{a}/b$ in D.*

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