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Applications of the duality method to generalizations of the Jordan canonical form

Olga Holtz

Department of Mathematics, University of Wisconsin, 480 Lincoln Drive, Madison, WI 53706, USA

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Abstract

The Jordan normal form for a matrix over an arbitrary field and the canonical form for a pair of matrices under contragredient equivalence are derived using Pták's duality method.

Keywords: Duality theory; Jordan normal form; Contragredient equivalence

1. Introduction

We show how Pták's duality method leads to short proofs of two extensions of the Jordan canonical form, viz. the normal form for a matrix over an arbitrary (not necessarily algebraically closed) field under similarity and the canonical form for a pair of matrices under contragredient equivalence.

The duality method is summarized in the following.

Lemma 1. *Let V be a finite-dimensional space over a field F, let $A : V \to V$ be a linear map, and $S \subset V$ be an A-invariant subspace of V. If $T \subset V^*$ is an A^*-invariant subspace of the dual V^* of V such that*

$$s \in S, \quad \langle s, t\rangle = 0 \ \forall t \in T \quad \Longrightarrow \quad s = 0, \tag{1}$$

$$t \in T, \quad \langle s, t\rangle = 0 \ \forall s \in S \quad \Longrightarrow \quad t = 0, \tag{2}$$

E-mail address: holtz@math.wisc.edu (O. Holtz).

PII: S 0 0 2 4 - 3 7 9 5 (0 0) 0 0 0 5 4 - 9

then $V = S \dot{+} \operatorname{ann}(T)$ *is an A-invariant direct sum decomposition of V, with* $\operatorname{ann}(T) := \{v \in V : \langle v, t\rangle = 0 \ \forall t \in T\}$ *the annihilator of T.*

We give a proof for the sake of completeness.

Proof. Condition (1) implies that the sum $S + \operatorname{ann}(T)$ is direct. If $\dim T \geqslant \dim S$ and $\{t_j\}_{j=1}^{\dim T}$ ($\{s_j\}_{j=1}^{\dim S}$) is a basis of T (S), then the matrix $G := (\langle s_i, t_j\rangle : i = 1, \ldots, \dim S,\ j = 1, \ldots, \dim T)$ has fewer rows than columns. Hence the equation $Gx = 0$ has a nontrivial solution, and so (2) fails. In other words, (2) implies that $\dim T \leqslant \dim S$. Hence $\dim \operatorname{ann}(T) \geqslant \dim V - \dim S$. Thus, $V = S \dot{+} \operatorname{ann}(T)$. Since T is A^*-invariant, $\operatorname{ann}(T)$ is A-invariant, which completes the proof. □

2. The analogue of the Jordan form for an arbitrary field

Theorem 1. *Let V be a finite-dimensional linear space over a field F and let* $A : V \to V$ *be a linear map. Then there exists a basis of V such that the representation of A with respect to that basis has the form*

$$\operatorname{diag}(A_1, \ldots, A_p), \tag{3}$$

where

$$A_i = \begin{pmatrix} C_i & 0 & \cdots & 0 & 0 \\ B_i & C_i & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & C_i & 0 \\ 0 & 0 & \cdots & B_i & C_i \end{pmatrix}, \quad B_i = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{d_i \times d_i},$$

$$C_i = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & a_{d_i} \\ 1 & 0 & 0 & \cdots & 0 & 0 & a_{d_i-1} \\ 0 & 1 & 0 & \cdots & 0 & 0 & a_{d_i-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & a_3 \\ 0 & 0 & 0 & \cdots & 1 & 0 & a_2 \\ 0 & 0 & 0 & \cdots & 0 & 1 & a_1 \end{pmatrix}_{d_i \times d_i},$$

$$x^{d_i} - a_1 x^{d_i - 1} - \cdots - a_{d_i} \quad \text{is a prime in } F[x].$$

This form is unique up to reordering of the blocks $A_1, \ldots, A_p$.

Proof. Since the space of all linear maps on V is finite-dimensional, there exists $k \in \mathbb{N}$ such that $A^k \in \operatorname{span}\{I, A, \ldots, A^{k-1}\}$, and hence some monic polynomial in $F[x]$ annihilates A.

Let $f \in F[x]$ be the monic polynomial of minimal degree such that $f(A) = 0$ and let $f = (f_1)^{k_1} \cdots (f_r)^{k_r}$ be its decomposition into powers of distinct (monic) primes f_i, $i = 1, \ldots, r$. Let $g_i := \prod_{j=1, j \neq i}^{r} (f_i)^{k_i}$. Since $F[x]$ is a Euclidean domain and $\gcd(g_1, \ldots, g_r) = 1$, it follows that $g_1 h_1 + \cdots + g_r h_r = 1$ for some $h_1, \ldots, h_r \in F[x]$. Hence $v = h_1(A) g_1(A) v + \cdots + h_r(A) g_r(A) v$ for any $v \in V$. But $h_i(A) g_i(A) V \subseteq V_i := \ker(f_i(A))^{k_i}$, so $V = V_1 + \cdots + V_r$. Suppose $v \in V_i \cap V_j$, $i \neq j$. As $(f_i)^{k_i}$ and $(f_j)^{k_j}$ are relatively prime, there exist $s_{i,j}, s_{j,i} \in F[x]$ such that $s_{i,j}(f_i)^{k_i} + s_{j,i}(f_j)^{k_j} = 1$, and hence $v = s_{i,j}(A)(f_i(A))^{k_i} v + s_{j,i}(A)(f_j(A))^{k_j} v = 0$, since $(f_i(A))^{k_i} v = (f_j(A))^{k_j} v = 0$. So, $V = V_1 \dot{+} \cdots \dot{+} V_r$ is a(n A-invariant) direct sum decomposition of V. The arguments given so far are standard.

Now show how to split the subspaces V_i. Let $\widetilde{V}$ stand for V_1, $\widetilde{A}$ for $A|_{V_1}$, $\widetilde{f}$ for f_1, k for k_1, d for $\deg f_1$. Since f is the minimal polynomial annihilating A, $\widetilde{f}^k$ is the minimal polynomial annihilating $\widetilde{A}$. So there exists $v \in \widetilde{V}$ such that $w := (\widetilde{f}(\widetilde{A}))^{k-1} \widetilde{A}^{d-1} v \neq 0$.

We claim that $w \notin \operatorname{span}\{(\widetilde{f}(\widetilde{A}))^{k-1} \widetilde{A}^j v : j = 0, \ldots, d-2\}$. Indeed, if w were in that span, it would imply $h(\widetilde{A})(\widetilde{f}(\widetilde{A}))^{k-1} v = 0$ for some polynomial h of degree $d-1$. But any polynomial of degree $d-1$ is coprime to f, and so there would exist a combination of h and f (with coefficients from $F[x]$) equal to 1, which would yield $(f(\widetilde{A}))^{k-1} v = 0$, contradicting $w \neq 0$. Hence the claim follows.

So, there exists $v' \in \widetilde{V}^*$ such that

$$\langle (f(\widetilde{A}))^{k-1} \widetilde{A}^j v, v' \rangle \begin{cases} = 0 & \text{if } j = 0, \ldots, d-2, \\ \neq 0 & \text{if } j = d-1. \end{cases}$$

Let

$$W_1 := \operatorname{span}\{(f(\widetilde{A}))^{i_1 - 1} \widetilde{A}^{i_2 - 1} v : i_1 = 1, \ldots, k,\ i_2 = 1, \ldots, d\},$$
$$W_1' := \operatorname{span}\{(f(\widetilde{A}^*))^{i_1 - 1} (\widetilde{A}^*)^{i_2 - 1} v' : i_1 = 1, \ldots, k,\ i_2 = 1, \ldots, d\}.$$

Notice that

$$g_{(i_1, i_2), (j_1, j_2)} := \langle (f(\widetilde{A}))^{i_1 - 1} \widetilde{A}^{d - i_2} v, (f(\widetilde{A}^*))^{k - j_1} (\widetilde{A}^*)^{j_2 - 1} v' \rangle \neq 0$$

only if $(i_1, i_2) \preceq (j_1, j_2)$ (in lexicographic order). So, the $kd \times kd$-matrix $(g_{(i_1, i_2), (j_1, j_2)} : i_1, j_1 = 1, \ldots, k,\ i_2, j_2 = 1, \ldots, d)$ is upper triangular with nonzero diagonal elements. Hence, by the lemma, $\widetilde{V} = W_1 \dot{+} \operatorname{ann}(W_1')$ is an $\widetilde{A}$-invariant direct sum decomposition of $\widetilde{V}$. The matrix representation of $\widetilde{A}|_{W_1}$ with respect to the basis $((f(\widetilde{A}))^{i_1 - 1} \widetilde{A}^{i_2 - 1} v : i_1 = 1, \ldots, k, i_2 = 1, \ldots, d)$ ordered lexicographically is one of the diagonal blocks in (3) with $d_i = d$ and $\widetilde{f}(x) = x^d - a_1 x^{d-1} - \cdots - a_d$.

Splitting the spaces $\operatorname{ann}(W_1')$, $V_2, \ldots, V_r$ in the same way as above, we obtain a direct sum $V = W_1 \dot{+} \cdots \dot{+} W_p$ of A-invariant indecomposable subspaces and a basis in each so that the matrix representation of A with respect to the concatenation of the bases of W_i's has the form (3).

Since the minimal polynomial f of A is unique, the (monic) prime factors f_i and the powers k_i with which they occur in f are determined uniquely. Let

$$n^i_j := \dim\ker(f_i(A))^j = \sum_{W_l \subseteq \ker(f_i(A))^{k_i}} \min(\dim W_l,\ j\ \deg f_i),$$

$$i = 1,\ldots,r, \quad j = 1,\ldots,k_i.$$

Then $\Delta n^i_j := n^i_{j+1} - n^i_j$ is the number of blocks for f_i of order greater than $j \cdot \deg f_i$ multiplied by $\deg f_i$. So the number of blocks of order $j \cdot \deg f_i$ equals $-\Delta^2 n^i_{j-1}/\deg f_i = (\Delta n^i_{j-1} - \Delta n^i_j)/\deg f_i$. Since the numbers n^i_j are uniquely determined by the map A, this completes the proof of the uniqueness of (3). □

Remarks.

1. The arguments in the two preceeding paragraphs are variations of those due to de Boor [1].
2. If F is algebraically closed, the polynomials f_i are of degree 1, and so (3) becomes the Jordan normal form of A.
3. In the proof above, all the factors of the minimal polynomial are treated in the same way in contrast to the proof in [7] where the canonical splitting is first given for the nilpotent part of A and then follows for all other parts by shifting A by an eigenvalue λ (for that completion of the proof in [7], see [1]).
4. Theorem 1 is classical and can be found, e.g., in [5, pp. 92–97]. In the sequel, we refer to a matrix in the form (3) as being in the *Jordan normal form for the field F*, and as the Jordan normal form of the operator A.

3. The canonical form under contragredient equivalence

Two pairs of matrices, (A, B) and (C, D), are called contragrediently equivalent if $A, C \in F^{m\times n}$, $B, D \in F^{n\times m}$, and $A = SCT^{-1}$, $B = TDS^{-1}$ for some invertible $S \in F^{m\times m}$, $T \in F^{n\times n}$.

The problem of classification of pairs of matrices under contragredient equivalence can be restated as follows. Given an n-dimensional linear space V and an m-dimensional linear space W and linear maps $A : V \to W$, $B : W \to V$, choose bases of V and W so that the pair (A, B) has a simple representation with respect to these bases.

Theorem 2. *Let V, W be finite-dimensional linear spaces over a field F and let $A : V \to W$, $B : W \to V$ be linear maps. Then there exist bases of V and W such that, with respect to those bases, the pair (A, B) has the representation*

$$\big(\mathrm{diag}(I, A_1, \ldots, A_p, 0), \quad \mathrm{diag}(J_{AB}, B_1, \ldots, B_p, 0)\big), \tag{4}$$

where J_{AB} is the nonsingular part of the Jordan form of AB, $A_i, B_i \in F^{m_i\times n_i}$, $|m_i - n_i| \leqslant 1$, and

$$(A_i, B_i) \in \left\{ \left(\begin{pmatrix} I_{m_i-1} & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ I_{m_i-1} \end{pmatrix} \right), \left(\begin{pmatrix} 0 \\ I_{m_i-1} \end{pmatrix}, \begin{pmatrix} I_{m_i-1} & 0 \end{pmatrix} \right), (I_{m_i}, J_{m_i}), (J_{m_i}, I_{m_i}) \right\},$$

where J_k denotes the $k \times k$-matrix with ones on the first subdiagonal and zeros elsewhere. The representation (4) *is unique up to reordering of the pairs of blocks (A_i, B_i), $i = 1, \ldots, p$. Two pairs (A, B) and (C, D) are contragrediently equivalent if and only if AB is similar to CD and*

$$\begin{aligned} &\operatorname{rank} A = \operatorname{rank} C, \quad \operatorname{rank} BA = \operatorname{rank} DC, \ldots, \operatorname{rank}(BA)^t = \operatorname{rank}(DC)^t, \\ &\operatorname{rank} B = \operatorname{rank} D, \quad \operatorname{rank} AB = \operatorname{rank} CD, \ldots, \operatorname{rank}(AB)^t = \operatorname{rank}(CD)^t, \\ &t := \min\{m, n\}. \end{aligned} \tag{5}$$

Proof. *Step* 1. By [7, Theorem 1] (whose proof holds over an arbitrary field), there exist V_1 (W_1) and V_2 (W_2) such that *BA* (*AB*) is invertible on V_1 (W_1) and nilpotent on V_2 (W_2) and $V = V_1 \dot{+} V_2$ $(W = W_1 \dot{+} W_2)$. Moreover, $V_1 = \operatorname{range}(BA)^r$, $V_2 = \ker(BA)^r$, $W_1 = (AB)^r$, and $W_2 = \ker(AB)^r$ for some $r \in \mathbb{N}$. If $x \in V_1$, then $x = (BA)^r y$ for some $y \in V$. Hence $(AB)^r Ay = Ax$, that is, $Ax \in W_1$. Analogously, $By \in V_1$ whenever $y \in W_1$. So, $V = V_1 \dot{+} V_2$, $W = W_1 \dot{+} W_2$, A maps V_i to W_i, B maps W_i to V_i for $i = 1, 2$.

If $x \in V_2$, then $(AB)^r Ax = 0$, and so $Ax \in W_2$. If $x \in V_1$ and $Ax = 0$, then $BAx = 0$, and therefore, $x = 0$ since *BA* is invertible on V_1. So, A induces a one–one map from V_1 to W_1. Likewise, B induces a one–one map from W_1 to V_1. So, V_1 and W_1 have the same dimension and the induced maps are also onto.

This step of the proof not only uses [7, Theorem 1], but also parallels it.

Now one can choose bases of V_1 and W_1 so that $A|_{V_1}$ is the identity matrix and $B|_{W_1}$ is in Jordan normal form (which is the nonsingular part of the Jordan normal form of *AB*).

Step 2. The spaces V_2 and W_2 are further split as follows. Let l be the length of the longest nonzero product of the form $\cdots ABA$ or $\cdots BAB$. Call such a product C and suppose it ends in A. Pick $x \in V_2$ so that $Cx \neq 0$ and form the sequence x, Ax, $BAx, \ldots, Cx$, whose elements are alternately in V_2 and W_2. Let V_3 (W_3) be the span of the elements of the sequence belonging to V_2 (W_2).

If l is even, then $\dim V_3 = \dim W_3 + 1 = 1 + l/2$. Pick $x' \in V_2^*$ so that $\langle Cx, x' \rangle \neq 0$. Form the sequence x', $B^*x', \ldots, A^*B^*x', \ldots, C^*x'$. Let V_4 (W_4) be the annihilator in V_2 (W_2) of the elements of the sequence that lie in V_2^* (W_2^*). The $(1 + l/2) \times (1 + l/2)$-matrix $(\langle (BA)^{i-1}x, (A^*B^*)^{1+l/2-j}x' \rangle : i, j = 1, \ldots, 1 + l/2)$ is upper triangular with nonzero diagonal entries. Hence, by Lemma 1, $V_2 = V_3 \dot{+} V_4$. This argument is exactly the same as the corresponding argument in [1].

Analogously, $W_2 = W_3 \dot{+} W_4$. Moreover, A maps V_i to W_i, B maps W_i to V_i, $i = 3, 4$, and the pair $(A|_{V_3}, B|_{W_3})$ has the form

$$\left(\begin{pmatrix} I_{l/2} & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ I_{l/2} \end{pmatrix}\right).$$

If l is odd, then $\dim V_3 = \dim W_3 = (1+l)/2$, and the above construction gives $V_2 = V_3 \dot{+} V_4$, $W_2 = W_3 \dot{+} W_4$ with A mapping V_i to W_i, B mapping W_i to V_i, $i = 3, 4$, the pair $(A|_{V_3}, B|_{W_3})$ having the form $(I_{(1+l)/2}, J_{(1+l)/2})$.

If C ends in B, then $(A|_{V_3}, B|_{W_3})$ has the form

$$\left(\begin{pmatrix} 0 \\ I_{l/2} \end{pmatrix}, \begin{pmatrix} I_{l/2} & 0 \end{pmatrix}\right) \quad \text{or} \quad (J_{(1+l)/2}, I_{(1+l)/2}).$$

This step of the proof parallels, with necessary modifications, [7, Theorem 2].

The problem is now reduced to splitting V_4 and W_4 in the same way. The splitting process ends at the jth stage if $A|_{V_{2j}} = 0$ and $B|_{W_{2j}} = 0$.

Thus, one obtains the canonical form (4). It is completely determined by the nonsingular part of the Jordan form of AB and the ranks $\operatorname{rank}(A)$, $\operatorname{rank}(BA)$, $\operatorname{rank}(ABA), \ldots, \operatorname{rank}(B)$, $\operatorname{rank}(AB)$, $\operatorname{rank}(BAB), \ldots$ Since the rank of any such product equals the size of J_{AB} if the length of the product exceeds $2\min\{m, n\}$, the infinite sequences above can be terminated at $(BA)^{\min\{m,n\}}$, $(AB)^{\min\{m,n\}}$. It follows that

1. the representation (4) is unique up to the order of the pairs of blocks and
2. two pairs (A, B) and (C, D) are contragrediently equivalent if and only if AB is similar to CD and (5) holds. □

Remarks.

1. Pták's duality method was rediscovered by Kaplansky [6], who also described how to derive the canonical form (4). The same form was first published by Dobrovol'skaya and Ponomarev [2]. Gelonch and Rubió i Diaz [3, Theorem 2] proved that the pair (A, B) can be represented as

   $$\left(\operatorname{diag}(A_1, \ldots, A_q), \quad \operatorname{diag}(B_1, \ldots, B_q)\right),$$

   where A_i and B_i^* are of the same size and

   $$(\dim\ker A_i, \dim\ker B_i) \in \{(0,1), (1,0)\} \quad \text{unless} \quad A_i = 0, \quad B_i = 0.$$

   Horn and Merino derived the canonical form (4) in [4, Theorem 5]. All the derivations (in [2–4,6]) were for the field $\mathbb{C}$.
2. Observe that the canonical form of the pair (I, A) under contragredient equivalence is (I, J_A), where J_A is the Jordan normal form of A. This and many other applications of the canonical form (4) are discussed in [4].

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