Alternating Hamiltonian circuits in edge-coloured bipartite graphs

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Abstract


We show that if $G = K_{n,n}$ is edge-coloured with $r \geq 2$ colours so that the subgraph induced by the edges of each colour is regular of order $2n$, then $G$ has a Hamiltonian circuit in which adjacent edges have different colours. We also give a generalization of this result to the case when $G$ is a regular bipartite graph.

1. Introduction

An edge-colouring of a graph $G$ is a map $\phi : E(G) \to \mathcal{C}$, where $\mathcal{C}$ is a set of colours. A circuit in an edge-coloured graph is alternating if adjacent edges of the circuit have different colours. Various authors have found conditions which imply the existence of alternating circuits of various kinds in edge-coloured graphs (see for example Chen and Daykin [2], Daykin [3], Grossman and Häggkvist [5], Bollobás and Erdös [1]).

In this paper we prove the following theorems.

Theorem 1. Let $K_{n,n}$ be edge-coloured with $r \geq 2$ colours in such a way that the subgraph induced by each colour is regular of order $2n$ and degree at least $1$. Then $G$ contains an alternating Hamiltonian circuit.

Theorem 1.1 is a special case of Theorem 1.2.

Theorem 1.2. Let $G$ be a regular bipartite graph of order $2n$ and degree at least
\( \frac{1}{2}n + 1 \) which is edge-coloured by \( r \geq 2 \) colours \( c_1, \ldots, c_r \). If \( 1 \leq s < r \) and the subgraph induced by \( c_s + 1, \ldots, c_r \) is regular of degree at least \( \frac{1}{2}n \) and at most \( d(G) - 1 \), and has order \( 2n \), then \( G \) has an alternating Hamiltonian circuit.

2. Proof of Theorem 1.2

To prove Theorem 1.2, we use an approach via a corollary due to Nash-Williams [8] of a theorem of Ghouila-Houri [4] about directed graphs. Ghouila-Houri's theorem is in turn an easy consequence of an attractive theorem of Meyniel [7]. Let \( d^1(v) \) and \( d^0(v) \) denote the in- and out-degrees of a vertex \( v \) in a directed graph, respectively. The result of Nash-Williams is:

**Lemma 2.1.** Let \( D \) be a directed graph of order \( n \). If \( d^1(v) \geq \frac{1}{2}n \) and \( d^0(v) \geq \frac{1}{2}n \) for each \( v \in V(G) \), then \( G \) contains a directed Hamiltonian circuit.

We now prove Theorem 1.2.

Let \( G_1 \) be the subgraph of \( G \) induced by the edges coloured \( c_1, \ldots, c_r \), and let \( G_2 \) be the subgraph induced by the edges coloured \( c_{s+1}, \ldots, c_r \). Then \( G_1 \) and \( G_2 \) are both regular, and \( \frac{1}{2}n \leq d(G_2) < d(G) \), so that \( d(G_1) > 0 \). By a well-known theorem of König, \( G_1 \) is the union of \( d(G_1) \) edge-disjoint 1-factors. Let \( F_1 \) be such a 1-factor.

Let the vertex set of \( G \) be \( A \cup B \), where each edge joins a vertex of \( A \) to a vertex of \( B \). Let \( A = \{a_1, \ldots, a_n\} \) and \( B = \{b_1, \ldots, b_n\} \), and suppose that the edges of \( F_1 \) are \( a_1 b_1, \ldots, a_n b_n \). Then, for \( 1 \leq i \leq n \), \( a_i \) and \( b_i \) are nonadjacent in \( G_2 \).

Form a directed graph \( D \) on vertices \( d_1, \ldots, d_n \) as follows. For \( 1 \leq i \leq n \), \( i \neq j \), whenever \( a_i b_j \) is an edge of \( G_2 \), let \( d_i d_j \) be a directed edge of \( D \). Since \( d(G_2) \geq \frac{1}{2}n \) it follows that \( d^1(v) \geq \frac{1}{2}n \) and \( d^0(v) \geq \frac{1}{2}n \) for each \( v \in V(D) \). (Since \( G_2 \) is regular, it also follows that \( D \) is regular, although we do not use this fact.) By Lemma 2.1, \( D \) has a directed Hamiltonian circuit, which we may suppose consists of the edges \( d_1 d_2, d_2 d_3, \ldots, d_{n-1} d_n, d_n d_1 \).

It follows that \( G \) has a Hamiltonian circuit whose edges are, in order, \( b_1 a_1, a_1 b_2, b_2 a_2, a_2 b_3, \ldots, a_{n-1} b_n, b_n a_n, a_n b_1 \). The edges of \( F_1 \) are in \( G_1 \) and are interlaced with edges of \( G_2 \), and so the Hamiltonian circuit is alternating.

3. Comments

Theorem 1.2 is best possible, for if \( n \) is even and the subgraph induced by \( c_s + 1, \ldots, c_r \) is regular of degree \( \frac{1}{2}n - 1 \), then it does not follow that \( G \) has an alternating Hamiltonian circuit. To see this, suppose that \( s = 1 \), that the edges coloured \( c_1 \) form a 1-factor \( F \) of \( G \), and that \( G \) consists of two copies of \( K_{n/2, n/2} \). Then \( G \setminus F \) has degree \( \frac{1}{2}n - 1 \) and is edge-coloured with \( c_2, \ldots, c_r \). As \( G \) has no Hamiltonian circuit, it clearly does not have one that is alternating.
If, as in the proof of Theorem 1.2, $G_2$ is the subgraph of $G$ induced by the edges coloured $c_1, c_2, \ldots, c_r$, then, under the conditions of Theorem 1.2, it follows that $G$ has at least $\min(d(G) - d(G_2), d(G_2) - \frac{1}{2}n - 1)$ edge-disjoint Hamiltonian circuits.

If in Theorem 1.2 we were to stipulate that $G$ be connected, then it seems likely that the lower bound on $d(G)$ could be lowered considerably.

It is evident that the regularity condition could be weakened a great deal. All that is really needed is that, referring to the proof of Theorem 1.2, $G_1$ has a 1-factor $F_1$ and $D$ has a Hamiltonian circuit. Conditions for these are provided by Hall's theorem and Meyniel's theorem, respectively.

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References