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Smooth Global Lagrangian Flow for the 2D Euler and Second-Grade Fluid Equations

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Abstract—We present a very simple proof of the global existence of a C^∞ Lagrangian flow map for the 2D Euler and second-grade fluid equations (on a compact Riemannian manifold with boundary) which has C^∞ dependence on initial data u_0 in the class of H^s divergence-free vector fields for $s > 2$

Keywords—Euler equations, Second-grade fluids, Lagrangian flow, Global regularity, Diffeomorphism groups

1. INCOMPRESSIBLE EULER EQUATIONS

Let (M, g) be a C^∞ compact oriented Riemannian 2-manifold with smooth boundary ∂M, let ∇ denote the Levi-Civita covariant derivative, and let μ denote the Riemannian volume form The incompressible Euler equations are given by

$$\begin{gathered}\partial_t u + \nabla_u u = -\operatorname{grad} p, \\ \operatorname{div} u = 0, \quad u(0) = u_0, \quad g(u, n) = 0, \qquad \text{on } \partial M,\end{gathered} \tag{1 1}$$

where $p(t, x)$ is the pressure function, determined (modulo constants) by solving the Neumann problem $-\triangle p = \operatorname{div} \nabla_u u$ with boundary condition $g(\operatorname{grad} p, n) = S_n(u)$, S_n denoting the second fundamental form of ∂M

The now standard global existence result for two-dimensional classical solutions states that for initial data $u_0 \in \chi^s = \{v \in H^s(TM) \mid \operatorname{div} u = 0,\ g(u, n) = 0\}$, $s > 2$, the solution u is in $C^0(\mathbb{R}, \chi^s)$ and has C^0 dependence on u_0 (see, for example, [1]) Equation (1 1) gives the Eulerian or spatial representation of the dynamics of the fluid The Lagrangian representation which is in terms of the volume-preserving fluid particle motion or flow map $\eta(t, x)$ is obtained by solving

$$\begin{aligned}\partial_t \eta(t, x) &= u(t, \eta(t, x)), \\ \eta(0, x) &= x\end{aligned} \tag{1 2}$$

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This is an ordinary differential equation on the infinite-dimensional volume-preserving diffeomorphism group $\mathcal{D}^s_\mu$, the set of H^s class bijective maps of M into itself with H^s inverses which leave ∂M invariant Ebin and Marsden [2] proved that $\mathcal{D}^s_\mu$ is a C^∞ manifold whenever $s > 2$ They also showed that for an interval I, whenever $u \in C^0(I, \chi^s)$ and $s > 3$, there exists a unique solution $\eta \in C^1(I, \mathcal{D}^s_\mu)$ to (1 2) Thus, for $s > 3$ the existence of a global C^1 flow map immediately follows from the fact that u remains bounded in H^s for all time It is often essential, however, for the Euler flow to depend smoothly on the initial data, in the case of vortex methods, for example, Hald in Assumption 3 of [3] requires this as a necessary condition to establish convergence

THEOREM 1 1 *For $u_0 \in \chi^s$, $s > 2$, there exists a unique global solution to (1 3) which is in $C^\infty(\mathbb{R}, T\mathcal{D}^s_\mu)$ and has C^∞ dependence on u_0*

PROOF The smoothness of the flow map follows by considering the Lagrangian version of (1 1) given by

$$\begin{gathered} \frac{D}{dt}\partial_t\eta(t,x) = -\operatorname{grad} p(t,\eta(t,x)), \qquad \det T\eta(t,x) = 1, \\ \partial_t\eta(0,x) = u_0(x), \\ \eta(0,x) = x, \end{gathered} \tag{1 3}$$

where $T\eta(t,x)$ denotes the tangent map of η (which in local coordinates is given by the 2×2 matrix of partial derivatives $\frac{\partial \eta^i}{\partial x^j}$), and where $\frac{D}{dt}$ is the covariant derivative along the curve $t \mapsto \eta(t,x)$ (which in Euclidean space is the usual partial time derivative) Since

$$\operatorname{grad} p \circ \eta = \operatorname{grad} \Delta^{-1} \left[\operatorname{Tr}(\nabla u \ \ \nabla u) + \operatorname{Ric}(u,u)\right] \circ \eta,$$

where Ric is the Ricci curvature of M, and since S_n is C^∞ and $H^{s-1}(TM)$ forms a multiplicative algebra whenever $s > 2$, we see that the linear operator $u \mapsto \operatorname{grad}\Delta^{-1}[\operatorname{Tr}(\nabla u \ \ \nabla u) + \operatorname{Ric}(u,u)]$ maps H^s back into H^s Denote by $f \ \ T\mathcal{D}^s_\mu \to TT\mathcal{D}^s_\mu$ the vector field

$$(\eta, \partial_t\eta) \mapsto \operatorname{grad}\Delta^{-1}\left[\operatorname{Tr}(\nabla u \ \ \nabla u) + \operatorname{Ric}(u,u)\right] \circ \eta$$

Then,

$$f(\eta, \partial_t\eta) = \operatorname{grad}_\eta \Delta_\eta^{-1}\left[\operatorname{Tr}(\nabla_\eta\partial_t\eta \ \ \nabla_\eta\partial_t\eta) + \operatorname{Ric}_\eta(\partial_t\eta, \partial_t\eta)\right],$$

where $\operatorname{grad}_\eta g = [\operatorname{grad}(g \circ \eta^{-1})] \circ \eta$ for all $g \in H^s(M)$, $\operatorname{div}_\eta X_\eta = [\operatorname{div}(X_\eta \circ \eta^{-1})] \circ \eta$, and $\nabla_\eta(X_\eta) = [\nabla(X_\eta \circ \eta^{-1})] \circ \eta$ for all $X_\eta \in T_\eta\mathcal{D}^s_\mu$, $\Delta_\eta = \operatorname{div}_\eta \circ \operatorname{grad}_\eta$, and $\operatorname{Ric}_\eta = \operatorname{Ric} \circ \eta$ It follows from Lemmas 4–6 in [4] and Appendix A in [2] that f is a C^∞ vector field Thus, (1 3) is an ordinary differential equation on the tangent bundle $T\mathcal{D}^s_\mu$ governed by a C^∞ vector field on $T\mathcal{D}^s_\mu$, it immediately follows from the fundamental theorem of ordinary differential equations on Hilbert manifolds, that (1 3) has a unique C^∞ solution on *finite* time intervals which depends smoothly on the initial velocity field u_0, i e , there exists a unique solution $\partial_t\eta \in C^\infty((-T,T), T\mathcal{D}^s_\mu)$ with C^∞ dependence on initial data u_0, where T depends only on $\|u_0\|_{H^s}$

When $s > 3$, this interval can be extended globally to $\mathbb{R}$ by virtue of η remaining in $\mathcal{D}^s_\mu$ Unfortunately, the global existence and uniquess of a C^∞ flow map $\eta(t,x)$ does not follow for initial data $u_0 \in \chi^s$ for $s \in (2,3]$, so we provide a simple argument to fill this gap We must show that η can be continued in $\mathcal{D}^s_\mu$ It suffices to prove that $T\eta$ and $T\eta^{-1}$ are both bounded in H^{s-1} This is easily achieved using energy estimates We have that

$$\frac{D}{dt}T\eta = \nabla\partial_t\eta = \nabla u \ \ T\eta$$

and

$$\frac{D}{dt}T\eta^{-1} = -T\eta^{-1} \ \ \nabla\partial_t\eta \ \ T\eta^{-1} = -T\eta^{-1} \ \ \nabla u$$

Computing the H^{s-1} norm of $T\eta$ and $T\eta^{-1}$, respectively, we obtain

$$\frac{1}{2}\frac{d}{dt}\|T\eta\|_{H^{s-1}} = \left\langle D^{s-1}\left(\nabla u \cdot T\eta\right), D^{s-1}T\eta\right\rangle_{L^2}$$

and

$$\frac{1}{2}\frac{d}{dt}\left\|T\eta^{-1}\right\|_{H^{s-1}} = \left\langle D^{s-1}\left(T\eta^{-1} \cdot \nabla u\right), D^{s-1}T\eta^{-1}\right\rangle_{L^2}$$

It is easy to estimate

$$\begin{aligned}\left\langle D^{s-1}\left(\nabla u \cdot T\eta\right), D^{s-1}T\eta\right\rangle_{L^2} &\le C\left(\|\nabla u\|_{L^\infty}\|T\eta\|^2_{H^{s-1}} + \|\nabla u\|_{H^{s-1}}\|T\eta\|_{L^\infty}\|T\eta\|_{H^{s-1}}\right) \\ &\le C\left(\|\nabla u\|_{L^\infty}\|T\eta\|^2_{H^{s-1}} + \|u\|_{H^s}\|T\eta\|^2_{H^{s-1}}\right),\end{aligned}$$

where the first inequality is due to Cauchy-Schwartz and Moser's inequalities and the second is the Sobolev embedding theorem. Similarly,

$$\left\langle D^{s-1}\left(-T\eta^{-1} \cdot \nabla u\right), D^{s-1}T\eta^{-1}\right\rangle_{L^2} \le C\left(\|\nabla u\|_{L^\infty}\left\|T\eta^{-1}\right\|^2_{H^{s-1}} + \|u\|_{H^s}\left\|T\eta^{-1}\right\|^2_{H^{s-1}}\right)$$

Since the solution u to (1.1) is in χ^s for all t, we have that $\|u\|_{H^s}$ is bounded for all t. Because the vorticity $\omega = \operatorname{curl} u$ is in L^∞, we have by Lemma 2.4 in [1, Chapter 17] that $\|\nabla u\|_{L^\infty} \le C(1 + \log\|u\|_{H^s})$, hence, $\|\nabla u\|_{L^\infty}$ is bounded for t. It then follows that η and η^{-1} are in $\mathcal{D}^s_\mu$ for all time.

2. SECOND-GRADE FLUID EQUATIONS

In this section, we establish the global existence of a C^∞ Lagrangian flow map for the second-grade fluids equations, also known as the Lagrangian averaged Euler or Euler-α equations when $\nu = 0$, which has C^∞ dependence on initial data. These equations are given on (M, g) by

$$\begin{gathered}\partial_t\left(1 - \alpha\Delta_r\right)u - \nu\Delta_r u + \nabla_u\left(1 - \alpha\Delta_r\right)u - \alpha\left(\nabla u\right)^t \cdot \Delta_r u = -\operatorname{grad} p, \\ \operatorname{div} u = 0, \quad u(0) = u_0, \quad u = 0, \qquad \text{on } \partial M, \\ \alpha > 0, \quad \nu \ge 0, \qquad \Delta_r = -\left(d\delta + \delta d\right) + 2\operatorname{Ric},\end{gathered} \tag{2.1}$$

(see [4]), and were first derived in 1955 by Rivlin and Ericksenn [5] in Euclidean space ($\operatorname{Ric} = 0$) as a first-order correction to the Navier-Stokes equations. In Euclidean space, the operator Δ_r is just the component-wise Laplacian, and the equation may be written as

$$\partial_t\left(1 - \alpha\Delta\right)u - \nu\Delta u + \operatorname{curl}\left(1 - \alpha\Delta\right)u \times u = -\operatorname{grad} p$$

For convenience, we set $\alpha = 1$. We define the unbounded, self-adjoint operator $(1 - \mathcal{L}) = (1 - 2\operatorname{Def}^*\operatorname{Def})$ on $L^2(TM)$ with domain $H^2(TM) \cap H^1_0(TM)$. The operator Def^* is the formal adjoint of Def with respect to L^2, $2\operatorname{Def}^*\operatorname{Def} u = -(\Delta + \operatorname{grad}\operatorname{div} + 2\operatorname{Ric})u$ so that $2\operatorname{Def}^*\operatorname{Def} u = -(\Delta + 2\operatorname{Ric})u$ if $\operatorname{div} u = 0$. We let $\mathcal{D}^s_{\mu,D}$ denote the subgroup of $\mathcal{D}^s_\mu$ whose elements restrict to the identity on the boundary ∂M. $\mathcal{D}^s_{\mu,D}$ is a C^∞ manifold (see [2,4]). Define $\chi^s_D = \{u \in \chi^s \mid u = 0, \text{ on } \partial M\}$.

The following is Proposition 5 in [4].

PROPOSITION 2.1. *For $s > 2$, let $\eta(t)$ be a curve in $\mathcal{D}^s_{\mu,D}$, and set $u(t) = \partial_t\eta \circ \eta(t)^{-1}$. Then u is a solution of the initial-boundary value problem (2.1) with Dirichlet boundary conditions $u = 0$ on ∂M if and only if*

$$\begin{gathered}\overline{\mathcal{P}}_\eta \circ \left[\frac{\nabla\eta}{dt} + \left[-\nu(1 - \mathcal{L})^{-1}\Delta_r u + \mathcal{U}(u) + \mathcal{R}(u)\right] \circ \eta\right] = 0, \qquad \operatorname{Det} T\eta(t, x) = 1, \\ \partial_t\eta(0, x) = u_0(x), \\ \eta(0, x) = x,\end{gathered} \tag{2.2}$$

where

$$\mathcal{U}(u) = (1-\mathcal{L})^{-1}\left\{\operatorname{div}\left[\nabla u \quad \nabla u^t + \nabla u \quad \nabla u - \nabla u^t \quad \nabla u\right] + \operatorname{grad}\operatorname{Tr}(\nabla u \quad \nabla u)\right\},$$
$$\mathcal{R}(u) = (1-\mathcal{L})^{-1}\left\{\operatorname{Tr}\left[\nabla\left(R(u,\ )u\right) + R(u,\ )\nabla u + R(\nabla u,\ )u\right] + \operatorname{grad}\operatorname{Ric}(u,u) - (\nabla_u \operatorname{Ric}) \quad u + \nabla u^t \quad \operatorname{Ric}(u)\right\},$$

and $\overline{\mathcal{P}}_\eta \quad T_\eta\mathcal{D}^s_D \to T_\eta\mathcal{D}^s_{\mu,D}$ *is the Stokes projector defined by*

$$\overline{\mathcal{P}}_\eta \quad T_\eta\mathcal{D}^s_{\mu,D} \to T_\eta\mathcal{D}^s_{\mu,D},$$
$$\overline{\mathcal{P}}_\eta(X_\eta) = \left[\mathcal{P}_e\left(X_\eta \circ \eta^{-1}\right)\right] \circ \eta,$$

and where $\mathcal{P}_e(F) = v$, v *being the unique solution of the Stokes problem*

$$(1-\mathcal{L})\,v + \operatorname{grad} p = (1-\mathcal{L})\,F,$$
$$\operatorname{div} v = 0,$$
$$v = 0, \qquad \text{on } \partial M$$

Equation (2 2) is an ordinary differential equation for the Lagrangian flow Notice again that H^{s-1}, $s > 2$, forms a multiplicative algebra, so that both $\mathcal{U}$ and $\mathcal{R}$ map H^s into H^s

THEOREM 2 1 *For* $u_0 \in \chi^s_D$, $s > 2$, *and* $\nu \geq 0$, *there exists a unique global solution to (2 2) which is in* $C^\infty(\mathbb{R}, T\mathcal{D}^s_\mu)$ *and has* C^∞ *dependence on* u_0

We note that one cannot prove the statement of this theorem from an analysis of (2 1) alone (see [6,7], and references therein)

PROOF The ordinary differential equation (2 2) can be written as $\partial_{tt}\eta = S(\eta, \partial_t\eta)$ (see in [4, p 23]) Remarkably, $S \quad T\mathcal{D}^s_{\mu,D} \to TT\mathcal{D}^s_{\mu,D}$ is a C^∞ vector field, and [4, Theorem 2] provides the existence of a unique short-time solution to (2 2) in $C^\infty((-T,T), T\mathcal{D}^s_{\mu,D})$ which depends smoothly on u_0, and where T only depends on $\|u_0\|_{H^s}$

Thus, it suffices to prove that the solution curve η does not leave $\mathcal{D}^s_{\mu,D}$ Following the proof of Theorem 1 1, and using the fact that the solution $u(t,x)$ to (2 1) remains in H^s for all time [6,7], it suffices to prove that ∇u is bounded in L^∞

Letting $q = \operatorname{curl}(1 - \alpha\triangle_r)u$ denote the potential vorticity, and computing the curl of (2 1), we obtain the 2D vorticity form as

$$\partial_t q + g\,(\operatorname{grad} q, u) = \nu \operatorname{curl} u$$

It follows that for all $\nu \geq 0$, $q(t,x)$ is bounded in L^2 (conserved when $\nu = 0$), and therefore, by standard elliptic estimates $\nabla u(t,x)$ is bounded in H^2, and hence, in L^∞ ∎

As a consequence of Theorem 2 1 being independent of viscosity, we immediately obtain the following

COROLLARY 2 1 *Let* $\eta^\nu(t,x)$ *denote the Lagrangian flow solving (2 2) for* $\nu > 0$, *so that* $u^\nu = \partial_t\eta^\nu \circ \eta^{\nu-1}$ *solves (2 1) Then for* $u_0 \in \chi^s_D$, $s > 2$, *the viscous solution* $\eta^\nu \in C^\infty(\mathbb{R}, T\mathcal{D}^s_\mu)$ *converges regularly (in* H^s*) to the inviscid solution* $\eta^0 \in C^\infty(\mathbb{R}, T\mathcal{D}^s_\mu)$ *Consequently,* $u^\nu \to u^0$ *in* H^s *on infinite-time intervals*

This gives an improvement of Busuioc's result in [8] in two ways

(1) we are able to prove the regular limit of zero viscosity on manifolds with boundary, and
(2) in the Lagrangian framework, we are able to get C^∞ in time solutions

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