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Locally *G*-homogeneous Busemann *G*-spaces

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ABSTRACT

We present short proofs of all known topological properties of general Busemann *G*-spaces (at present no other property is known for dimensions more than four). We prove that all small metric spheres in locally *G*-homogeneous Busemann *G*-spaces are homeomorphic and strongly topologically homogeneous. This is a key result in the context of the classical Busemann conjecture concerning the characterization of topological manifolds, which asserts that every *n*-dimensional Busemann *G*-space is a topological *n*-manifold. We also prove that every Busemann *G*-space which is uniformly locally *G*-homogeneous on an orbal subset must be finite-dimensional.

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1. Introduction

A metric space is said to be a *Busemann G-space* if it satisfies four basic axioms that, among other things, imply that the space is a complete geodesic space (a precise definition will be given later). This class of spaces was introduced in 1942 by Herbert Busemann [12,13,15] in an attempt to present Finsler manifolds in simple geometric terms. Subsequent investigations in *the geometry of geodesics* were summarized in [17,21]. Busemann and Phadke [20] introduced and studied an interesting generalization of Busemann *G*-spaces. Their survey [21] can be considered as a testament to researchers in the area of the geometry of geodesics.

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In the present paper we give short proofs of topological properties of general finite-dimensional Busemann G-spaces – no other property is known at present without specification of the dimension. A new result among them is that small spheres in every $(n \geqslant 3)$-dimensional Busemann G-spaces are simply connected.

At present, the answer to the Busemann question [15], which asks if every Busemann G-space must be finite-dimensional is still unknown. Heretofore, the best known result was that this is true for every Busemann G-space with small geodesically convex balls near some point [3]. The latter condition is satisfied at every point of a Busemann G-space X if X has nonpositive curvature in the Busemann sense [15,14], which means that in small triangles the length of the midsegment is no more than the half of the length of the corresponding side.

Note that nowadays many authors apply the term *Busemann space* to a geodesic space with a local or global condition of nonpositive curvature in the Busemann sense [35]. However, there exist metrically homogeneous Finsler 2-manifolds among the so-called *quasihyperbolic planes*, which are Busemann G-spaces with no geodesically convex balls of positive radius (the assertion was stated in [16] and proved in [25]). Weaker assertions have been proved in [19]. In this paper we shall generalize the result from [3] stated above to all Busemann G-spaces which are uniformly locally G-homogeneous on an orbal subset. It is unknown whether every Busemann G-space satisfies this property.

Busemann conjectured that for all $n < \infty$, every n-dimensional Busemann G-space is a topological n-manifold. The $(n \leqslant 4)$-dimensional Busemann G-spaces are known to be topological n-manifolds (cf. [15,32,38,39]). The *Busemann conjecture* is also known to be true in all dimensions under the additional hypothesis that the Aleksandrov curvature is bounded either *from below* or *from above*; such spaces are even Riemannian (hence Finsler) manifolds with continuous metric tensors [7,8]. There are other additional conditions which guarantee that a Busemann G-space is a topological manifold, or even a Finsler space with a continuous metric function [17,36]. We shall discuss these results more in details later in the paper. However, this classical problem, now over half a century old, has still not been solved in its complete generality. For more on the Busemann conjecture see the recent survey [29].

A finite-dimensional normed vector space $(V, \|\cdot\|)$ is a Busemann G-space if and only if its closed balls of positive radius are strongly convex in the affine sense, i.e. they are convex and their boundary spheres do not contain nontrivial affine segments. Under this condition, its shortest arcs are exactly affine segments. On the other hand, V has the Aleksandrov curvature bounded from above or below if and only if V is isometric to the Euclidean space [1]. Therefore there exist Busemann G-spaces with geodesically (strongly) convex balls which do not have the Aleksandrov curvature bounded from above or below. Note also that every normed vector space $(V, \|\cdot\|)$ is a *space with distinguished geodesics* in the sense of [20].

Let us observe that we shall, as did Busemann, assume that a *Finsler manifold* is a finite-dimensional C^1-differentiable manifold M with a continuous norm F on its tangent bundle TM. However, it should be noted that usually the Finsler geometry experts, including Finsler [24] himself, generally require additional conditions for the function F. There are no known examples of Busemann G-spaces which are topological manifolds but fail to be Finsler manifolds. However, every metrically homogeneous Busemann G-space is a homogeneous space of a (connected) Lie group by its compact subgroup, and hence a topological manifold [4,37]. It seems that every such space should be a Finsler manifold. This has actually been proved for dimensions 2 and 3 (cf. [5,6]), whereas every one-dimensional Busemann G-space is always a Riemannian (hence Finsler) manifold.

Pogorelov [36] proved that a Finsler manifold M with a "strictly convex" metric function F of the class $C^{1,1}$ is a Busemann G-space, and moreover, that this degree of regularity cannot be weakened. Namely, for any $\alpha < 1$ there exist Finsler manifolds with a strictly convex metric function of the class $C^{1,\alpha}$ which are not Busemann G-spaces. This result substantially improves upon an earlier result of Busemann and Mayer [18] – they proved the first statement above for C^3-functions F. Note that three versions of "strict convexity" were used in [36]. However, the discussion in the previous paragraph implies that there are Busemann–Finsler G-spaces V with metric function $F = \|\cdot\|$ which are not differentiable and not strictly convex for two of the three versions of this notion.

Pogorelov also proved in some sense the converse assertion: if in a Busemann G-space the intersecting shortest curves have a certain slope to each other which continuously depends on these shortest curves, then such a G-space is a Finsler space with a continuous metric function. Similar results were proved by Busemann [17]: if a G-space is "continuously differentiable and regular" at one point then it is a topological manifold (cf. (9) on p. 24 in [17]). Busemann stated that regularity condition can be avoided. In fact, it is more or less clear that if a G-space is continuously differentiable at every one of its points then it is isometric to a Finsler space with a continuous metric function.

The Busemann conjecture is a special case of another classical conjecture, the *Bing–Borsuk conjecture* [9]. A topological space X is said to be *topologically homogeneous* if for any two points $x_1, x_2 \in X$, there is a homeomorphism of X onto itself taking x_1 to x_2. It is a classical result that all connected manifolds without boundary are topologically homogeneous. The Bing–Borsuk conjecture states that all finite-dimensional topologically homogeneous ANR-spaces are manifolds.

It is well known that Busemann G-spaces are topologically homogeneous [39] (see also in the present paper) and locally contractible, so they are ANR-spaces if they are finite-dimensional [33]. Thus, even though it is hardly believable that the Busemann conjecture is not true, a counterexample to it would settle the Bing–Borsuk conjecture in the negative. On the other hand, a proof of the Busemann conjecture may shed some light on the Bing–Borsuk conjecture.

Implied from the basic geometric properties is that every small metric ball in a Busemann G-space is the cone from its center over its boundary. As a result of topological homogeneity and this cone structure, a Busemann G-space M is a manifold if and only if all small metric spheres in M are codimension one manifold factors. Thus the characterization of small metric spheres is of vital importance in addressing the question of whether high-dimensional Busemann G-spaces are

manifolds in general. Several geometric properties which imply that a given topological space is a codimension one manifold factor can be found in [26–28,30].

Demonstrating the topological homogeneity of small metric spheres is a key step to proving the general case of the Busemann conjecture [29]. In this paper we introduce a special type of homogeneity property, the so-called local G-homogeneity. Local G-homogeneity essentially requires that any sufficiently small metric ball can be represented as a cone with the vertex sufficiently close to the ball's center, the cone lines being geodesics. We shall demonstrate that in Busemann G-spaces the property of local G-homogeneity implies that all sufficiently small metric spheres are mutually homeomorphic and topologically homogeneous.

The following are main results of the present paper:

Theorem 1.1. *Suppose X is a locally G-homogeneous Busemann G-space. Then sufficiently small metric spheres in X are (strongly) topologically homogeneous.*

Theorem 1.2. *Suppose X is a Busemann G-space, uniformly locally G-homogeneous on an orbal subset. Then X is finite-dimensional.*

Theorem 1.3. *There exists a Busemann G-space X with the following properties*:

(1) *X is uniformly locally G-homogeneous on an orbal subset*;
(2) *X is locally G-homogeneous*; *and*
(3) *X has no convex metric balls of positive radius.*

In the Epilogue we shall collect some unsolved questions.

2. Preliminaries

Definition 2.1. Let (X, d) be a metric space. X is said to be a *Busemann G-space* provided it satisfies the following axioms of Busemann:

 (i) *Menger Convexity*: Given distinct points $x, y \in X$, there is a point $z \in X - \{x, y\}$ such that $d(x, z) + d(z, y) = d(x, y)$;
 (ii) *Finite Compactness*: Every d-bounded infinite set has an accumulation point;
(iii) *Local Extendibility*: For every point $w \in X$, there exists a radius $\rho_w > 0$, such that for any pair of distinct points x, y in the open ball $U(w, \rho_w)$, there is a point $z \in U(w, \rho_w) - \{x, y\}$ such that $d(x, y) + d(y, z) = d(x, z)$; and
(iv) *Uniqueness of Extension*: Given distinct points $x, y \in X$, if there are points $z_1, z_2 \in X$ for which both equalities

$$d(x, y) + d(y, z_i) = d(x, z_i) \quad \text{for } i = 1, 2,$$

and

$$d(y, z_1) = d(y, z_2)$$

hold, then $z_1 = z_2$.

Remark 2.2. From these basic properties, a rich structure on a Busemann G-space can be derived. If (X, d) is a Busemann G-space and $w \in X$ is any point, then (X, d) satisfies the following properties:

- *Complete Inner Metric*: (X, d) is a locally compact complete inner metric space;
- *Existence of Geodesics*: Any two points in X can be joined by a geodesic;
- *Local Uniqueness of Joins*: Any two points x, y in $U(w, \rho_w)$ can be joined by a unique shortest geodesic in X;
- *Local Cones*: The closed ball $B(w, r), 0 < r < \rho_w$, is homeomorphic to the cone over its boundary (cf. Proposition 3.3 below);
- *Topological Homogeneity*: Every Busemann G-space is topologically homogeneous. Moreover, topological homogeneity homeomorphism can be chosen to be isotopic to the identity (cf. Theorem 3.11 and Corollary 3.12 below).

Busemann [15] has proposed the following conjecture which still remains open in dimensions $n \geqslant 5$:

Conjecture 2.3 *(Busemann Conjecture). Every n-dimensional Busemann G-space, $n \in \mathbb{N}$, is a topological n-manifold.*

In this paper we shall show (cf. Theorem 1.1) that stably visible metric spheres in any Busemann G-space are strongly topologically homogeneous (cf. Definitions 4.3 and 4.5).

3. Topological properties of finite-dimensional Busemann G-spaces

Thurston [39] has shown that small metric spheres in any n-dimensional Busemann G-space are homology $(n-1)$-manifolds (throughout this paper we are working only with *singular* homology with $\mathbb{Z}$ coefficients).

In this section, using only old results, known from topological literature until 1963, we shall briefly prove all known topological properties, in particular the assertion above due to Thurston, for arbitrary finite-dimensional Busemann G-spaces.

For convenience we shall use the following notations. Let I denote the unit interval $[0, 1]$. $B(x, r)$ shall denote the closed ball of radius r centered on x and $U(x, r)$ shall denote the open ball of radius r centered on x.

Definition 3.1. If x, y, and z are distinct points in a Busemann G-space and

$$d(x, y) + d(y, z) = d(x, z)$$

we say that *y lies between x and z* and denote this by $x - y - z$.

Let (X, d) be any Busemann G-space. For a point $w \in X$ we denote by $\rho(w)$ the supremum of all numbers ρ_w which satisfy the condition (iii) from Definition 2.1.

The following statement is an easy consequence of definitions.

Lemma 3.2. *The function $\rho(w) = +\infty$ for all points $w \in X$ or*

$$\left|\rho(x) - \rho(y)\right| \leqslant d(x, y) \quad \text{for all } x, y \in X. \tag{1}$$

One can easily sequentially prove the assertions of the next proposition.

Proposition 3.3. *Suppose that $0 < r < \rho(x)$. Let $S := S(x, r)$ and $B := B(x, r)$. Then*

- *for every point y in the sphere S there is a unique shortest arc (segment) $\overline{xy}$, joining points x and y;*
- *segment $\overline{xy}$ continuously depends on point $y \in S$ in the sense that the real-valued function $\phi : S \times S \to \mathbb{R}$ where*

$$\phi(y_1, y_2) := d_H(\overline{xy_1}, \overline{xy_2})$$

  where d_H denotes the Hausdorff distance (between compact subsets), is continuous;
- *every point $z \in B - \{x\}$ lies on a unique segment $\overline{xy}$, $y = y(z) \in S$; and*
- *let $c : S \times I \to C(S)$ be the canonical map of S onto its cone, identifying all points $(y, 0) \in S \times I$ to the vertex v of the cone. Then the map $f : B \to C(S)$, defined by the formula*

$$f(z) = \begin{cases} c(y(z), \frac{d(x,z)}{d(x,y(z))}), & z \in B - \{x\}, \\ v, & z = x \end{cases}$$

  is a homeomorphism.

Remark 3.4. Note that the first two assertions of Proposition 3.3 remain true if we change x by any other point $x' \in B - S$. If for some point $x' \in B - S$, every segment $\overline{x'y}$, where $y \in S$, intersects S only at the point y (in other words, *the sphere S is visible from the point x'*), then the last statement of Proposition 3.3 is true after replacement x by x'. We shall say in this case that the above map f defines *the canonical structure of geodesic cone on B with the vertex x (or x') and the closed ball B is (geodesically) starlike with respect to the point x (or x').*

Lemma 3.5. *For any two numbers $r_1, r_2 \in (0, 1) \subset I$, there is an isotopy $h : I \times I \to I$ fixed on $\{0, 1\}$ such that $h(\cdot, 0) = id_I$ and $h(r_2, 1) = r_1$.*

Proof. We can suppose that $0 < r_1 < r_2 < 1$. Then there is a unique real number $\alpha > 1$ such that $r_1 = r_2^\alpha$. The map $h(r, t) = r^{1+t(\alpha-1)}$ is the required isotopy. $\quad\square$

Proposition 3.6. *Let $C := C(S)$ be a cone on a topological space S, $r_1, r_2 \in (0, 1)$, and $x = c(s_0, r_2)$, $y = c(s_0, r_1)$ for some $s_0 \in S$. Then there is an isotopy $H : C \times I \to C$ fixing the base and the vertex of the cone C such that $H(\cdot, 0) = id_C$ and $H(x, 1) = y$.*

Proof. The required isotopy is defined by the formula $H(c(s, r), t) = c(s, h(r, t))$, where $s \in S$, $r \in I$ and h is the isotopy from Lemma 3.5. $\quad\square$

Proposition 3.7. *Let $B := B(x_0, r)$, $x_0 \in X$, $0 < r < \rho(x_0)$, $S := S(x_0, r)$. Then for any two points w, z lying inside some segment $\overline{x_0 s_0}$, where $s_0 \in S$, there is an isotopy $H' : B \times I \to B$ fixing S and x_0 such that $H'(\cdot, 0) = id_B$ and $H'(w, 1) = z$.*
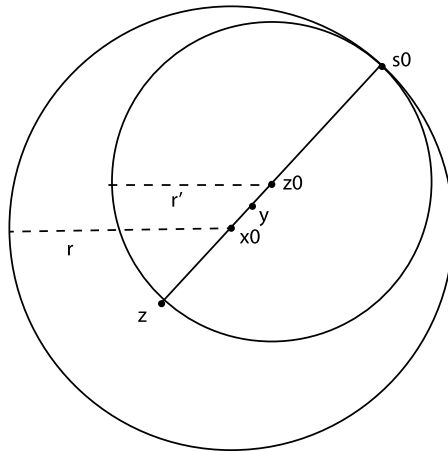
Fig. 1. Sketch for Lemma 3.8.

Proof. By Proposition 3.3, there is a homeomorphism $f : C = C(S) \to B$. Then there are numbers $r_1, r_2 \in (0, 1)$ such that $f(s_0, r_2) = w$, $f(s_0, r_1) = z$. We define the isotopy H' by the formula $H' = f \circ H \circ f^{-1}$, where H is the isotopy from Proposition 3.6. □

Lemma 3.8. *Let $B = B(x_0, r)$, $x_0 \in X$, $0 < r < \rho(x_0)$, $S = S(x_0, r)$. Then for every point $s_0 \in S$ there are a point y and an isotopy $H'' : B \times I \to B$ fixing S such that $x_0 - y - s_0$, $H''(\cdot, 0) = id_B$, and $H''(x_0, 1) = y$. (See Fig. 1.)*

Proof. There is a point $z \in U := U(x_0, r)$ such that $s_0 - x_0 - z$. Let z_0 be the midpoint of $\overline{s_0 z}$. Then $z - x_0 - z_0$ because $d(s_0, z) < 2r$. Thus there is a point y such that $z_0 - y - x_0$. Let $S' := S(z_0, r')$ and $B' := B(z_0, r')$, where $r' = d(z_0, s_0)$. Note that $z \in S'$. By the Triangle inequality, $B' \subset B$. Thus B' is a geodesic cone over S' with the vertex z_0. By Proposition 3.7, there is an isotopy $H : B' \times I \to B'$ fixing S' and z_0 such that $H(\cdot, 0) = id_{B'}$ and $H(x_0, 1) = y$. So we can extend the isotopy H to the required one H'' on B, fixing H outside of B'. □

Theorem 3.9. *Let $B = B(x_0, r)$, $x_0 \in X$, $0 < r < \rho(x_0)$. Then for every point $x \in U$ there is an isotopy $H : B \times I \to B$ fixing $S := S(x_0, r)$ such that $H(\cdot, 0) = id_B$ and $H(x_0, 1) = x$.*

Proof. We need only to consider the case when $x \neq x_0$. Then there is unique point $s_0 \in S$ such $x_0 - x - s_0$. By Lemma 3.8 there exist a point y and an isotopy $H'' : B \times I \to B$ fixing S such that $x_0 - y - s_0$, $H''(\cdot, 0) = id_B$, and $H''(x_0, 1) = y$. By Proposition 3.7, there is an isotopy $H' : B \times I \to B$ fixing S such that $H'(\cdot, 0) = id_B$ and $H'(y, 1) = x$. Now the "composition" H of isotopies H'' and H' gives us the required isotopy. □

Corollary 3.10. *For an arbitrary closed ball $B := B(x_0, r)$ in a Busemann G-space, of radius r, $0 < r < \rho(x_0)$, and any $x \in U(x_0, r)$ there is a homeomorphism $h : B \to B$, fixing the sphere $S(x_0, r)$, such that $h(x_0) = x$.*

Theorem 3.11. *Let x, y be any two points in a Busemann G-space, X. Then there is an isotopy $H : X \times I \to X$ such that $H(\cdot, 0) = id_X$ and $H(x, 1) = y$.*

Proof. We can suppose that $x \neq y$. Then there is a (shortest) segment $\overline{xy}$. By Lemma 3.2, the function $\rho(w)$ is infinite or continuous. In both cases there is a number $r > 0$ such that $r < \rho(w)$ for every point $w \in \overline{xy}$. Then there is a finite set $\{x_0 = x, \ x_1, \ldots, x_k = y\}$ of points in $\overline{xy}$ such that $d(x_i, x_{i+1}) < r$ for every $i = 0, 1, \ldots, k - 1$. Let $B_i := B(x_i, r)$ and $S_i := S(x_i, r)$. By Theorem 3.9, there is an isotopy $H_i : B_i \times I \to B_i$ fixing S_i such that $H_i(\cdot, 0) = id_{B_i}$ and $H_i(x_i, 1) = x_{i+1}$. We extend the isotopies to X requiring that H_i fixes all points outside B_i. Now the "composition" H of isotopies $H_0, \ldots, H_{k-1}$ gives us the required isotopy. □

Corollary 3.12. *Every Busemann G-space is topologically homogeneous.*

Proposition 3.13. *Any finite-dimensional Busemann G-space X is an absolute neighborhood retract (ANR).*

Proof. Clearly X is arcwise connected. Proposition 3.3 implies that X is locally contractible. Now the statement follows from [33]. □

Definition 3.14. (See [31,34].) X is a Kosiński r-space provided that:

(1) X is locally compact, metric, separable and finite-dimensional; and
(2) Each point of X has arbitrarily small closed neighborhoods U such that the boundary $Bd(U)$ is a strong deformation retract of $U - y$ for each interior point y of U.

Note that Kosiński [31] assumed that X is compact, but this condition can be replaced here by the local compactness. As an immediate consequence of Proposition 3.3 and Corollary 3.10 we get the following corollary (cf. also Theorem 3 on p. 16 in [17]).

Corollary 3.15. *Every finite-dimensional Busemann G-space is a Kosiński r-space.*

Theorem 3.16. *Every n-dimensional Busemann G-space X is a $\mathbb{Z}$-homology n-manifold, i.e., for every point $x \in X$, $H_k(X, X - \{x\}; \mathbb{Z}) \cong \mathbb{Z}$ if $k = n$ and $H_k(X, X - \{x\}; \mathbb{Z}) = 0$ if $k \neq n$.*

Proof. Alexandroff [2] proved that for any finite-dimensional (separable metric) space its cohomological dimension over the ring $\mathbb{Z}$ coincides with its topological (i.e. covering) dimension. The space X is arcwise connected, locally contractible and by Corollary 3.15 it is a Kosiński r-space. We can now simply apply results of Lee [34]. $\square$

Remark 3.17. Thurston [39] proved that any finite-dimensional Busemann G-space is an ANR $\mathbb{Z}$-homology manifold, using more recent results of Dydak and Walsh [22].

Proposition 3.18. *Let $S := S(x_0, r)$, where $0 < r < \rho(x_0)$, be any sphere in a Busemann G-space (X, d), let $x \in S$ be any point and denote by $x' \in S$ its (unique) "antipodal" point, i.e. $d(x, x') = 2r$. Then $\{x'\}$ is a strong deformation retract of $S' := S - \{x\}$.*

Proof. Let us first consider $S = S(x_0, r)$, where $0 < r < \frac{1}{2}\rho(x_0)$. We may assume that X is not 1-dimensional. Otherwise $S = \{x, x'\}$, and there is nothing to prove. By hypothesis, $2r < \rho(x_0)$. Then for every point z in $B(x_0, 2r) - \{x_0\}$ the (unique) segment $\overline{x_0 z}$ or its extension to a segment necessarily intersects the set S' in a unique point, which we shall denote by $f(z)$. By the condition on r, any two points $y, z \in S$ are joined in X by a unique segment $\overline{yz}$ (which does not necessarily lie in the closed ball $B(x_0, r)$) and these segments continuously depend on their ends. For every point $y \in S'$, the segment $\overline{yx'}$ does not go through x_0.

As a corollary of the Triangle inequality, every such segment $\overline{yx'}$ lies in the ball $B(x_0, 2r)$. For a point $y \in S'$ define $h_y(t)$, $t \in I$, as the point on the segment $\overline{yx'}$ such that $d(h_y(t), x') = (1 - t)d(y, x')$. Now because of all what was said before, the formula $H(y, t) = f(h_y(t))$ defines a homotopy $H : S' \times I \to S'$ such that $H(\cdot, 0) = id_{S'}$, $H(S', 1) = \{x'\}$ and $H(x', t) = x'$ for all $t \in I$. The proof for $0 < r < \rho(x_0)$ can be completed by Proposition 3.3. $\square$

See also pp. 17–18 in [17].

Theorem 3.19. *Let $S := S(x_0, r)$, where $0 < r < \rho(x_0)$, be any sphere in an n-dimensional Busemann G-space X. Then S is a $\mathbb{Z}$-homology $(n - 1)$-manifold and has the homology of the $(n - 1)$-sphere.*

Proof. We shall use the Eilenberg–Steenrod homology axioms [23]. The case $n = 1$ is trivial. So suppose that $n > 1$. Let $B := B(x_0, r)$ and $B' := B - \{x_0\}$. Evidently the closure of $X - B$ is contained in $X - \{x_0\}$. Then by the Excision axiom, $H_k(X, X - \{x_0\})$ is isomorphic to $H_k(B, B')$ and so by Theorem 3.16, the latter group is $\mathbb{Z}$ if $k = n$ and 0 if $k \neq n$. Moreover, as a corollary of Proposition 3.3, S is a strong deformation retract of B', while B is contractible. Hence by the Homotopy axiom, $H_k(B') \cong H_k(S)$ for all k; $H_k(B) = 0$ for $k \neq 0$ and $H_0(B) \cong \mathbb{Z}$.

Consider the following part of the exact homology sequence for the pair (B, B'):

$$\cdots \to H_{k+1}(B) \to H_{k+1}(B, B') \to H_k(B') \cong H_k(S) \to H_k(B) \to \cdots.$$

If $k > 0$ then the first and last terms are 0 and so

$$H_k(S) \cong H_{k+1}(B, B'),$$

which is nonzero only if $k + 1 = n$ or $k = n - 1$ and $H_{n-1}(S) \cong H_n(B, B') \cong \mathbb{Z}$. Also $H_0(S) \cong \mathbb{Z}$ because S is arcwise connected. This means that S has the homology of the $(n - 1)$-sphere.

For any point $x \in S$, consider the following part of the homology exact sequence for the pair $(S, S - \{x\}) = (S, S')$:

$$\cdots \to H_{k+1}(S') \to H_{k+1}(S) \to H_{k+1}(S, S') \to H_k(S') \to H_k(S) \to \cdots.$$

If $k > 0$ then $H_{k+1}(S') \cong H_k(S') = 0$ by Proposition 3.18. Then $H_{k+1}(S, S') \cong H_{k+1}(S)$ which is nonzero only if $k + 1 = n - 1$ and $H_{n-1}(S, S') \cong H_{n-1}(S) \cong \mathbb{Z}$ by the statements above. For $k > 0$ the latter two equalities make sense only if $n > 2$. If $k = 0$

then the last arrow is an isomorphism of groups, which are both isomorphic to $\mathbb{Z}$ because S and S' are arcwise connected by the argument from the proof of Proposition 3.18. Then $H_1(S, S') \cong H_1(S)$, which is zero if $n > 2$ and isomorphic to $\mathbb{Z}$ if $n = 2$. The arcwise connectivity of S and S' implies that $H_0(S, S') = 0$ by the definition of $H_0(S, S')$. We have thus proved that S is a homology $(n-1)$-manifold. $\square$

We get the following immediate corollary (an entirely different proof can be found in [17]):

Corollary 3.20. *In every finite-dimensional G-space X, every sphere $S(x, r)$ of radius $0 < r < \rho(x)$ is noncontractible.*

As a corollary of Theorem 3.16, finite-dimensional Busemann G-spaces also possess the *invariance of domain property*, which was first established for manifolds by Brouwer [10,11] and then generalized to homology manifolds by Wilder [41] (cf. Väisälä [40] for a short proof):

Theorem 3.21 *(Invariance of Domain Theorem). Let X be a finite-dimensional Busemann G-space, and $h : C \to D$ a homeomorphism of subsets in X. Then h maps $\mathrm{int}(C)$ onto $\mathrm{int}(D)$. Analogous assertion is true for every sphere $S(x, r)$ if $0 < r < \rho(x)$.*

We note that a very different argument for this theorem was given in [17].

Theorem 3.22. *Let X be an n-dimensional Busemann G-space where $n \geqslant 3$. Then every sphere $S = S(x, r)$, $0 < r < \rho(x)$, is simply connected.*

Proof. Since any two spheres $S = S(x, r)$, $S = S(x, r')$, $0 < r, r' < \rho(x)$, are homeomorphic by Proposition 3.3, we may suppose in the proof that $0 < r < \rho(x)/2$. It follows from Proposition 3.18 that it suffices to prove that every loop in S is homotopic to a loop whose image is a proper subset of S. Let $r_0 > 0$ be the minimal value of continuous function ρ (cf. Lemma 3.2) on compact ball $B(x_0, 2r)$ and $l : I \to S$ any loop in S. For the number $r_1 = \frac{1}{2} \min(r, r_0)$ there is $\delta > 0$ such that $d(l(s), l(s')) < r_1$ if $s, s' \in I$, $|s - s'| < \delta$. Take any numbers $s_0, s_1, \ldots, s_m$ such that $s_0 = 0 < s_1 < \cdots < s_m = 1$, $s_{j+1} - s_j < \delta$ for all $j = 0, 1, \ldots, m - 1$, and corresponding points $y_i = l(s_i)$, $i = 0, 1, \ldots, m$. Then $y_0 = y_m$. By Triangle inequality and choice of r_1 and δ, for every $j = 0, 1, \ldots, m - 1$, there is unique segment $\overline{y_j y_{j+1}}$, and this segment lies in $B(x, r + r_1/2)$. By the same reason, for all points $z \in \overline{y_j y_{j+1}}$ and $l(s)$, where $s \in \overline{s_j, s_{j+1}}$, there is unique segment $\overline{l(s)z}$, and this segment lies in $B(x, 2r)$. There is a loop $l_0 : I \to B(x, r + r_1/2)$ such that $l_0(s_i) = l(s_i)$ for all $i = 0, 1, \ldots, m$ and the restriction of l_0 to every segment $\overline{s_j, s_{j+1}}$; $j = 0, 1, \ldots, m - 1$, is a parametrization of the segment $\overline{y_j y_{j+1}}$. By arguments above, for every number $s \in I$ we can define unique path $h_s(t)$, $t \in I$, in $B(x, 2r)$ such that $h_s(t)$ is the point on unique segment $\overline{l(s)l_0(s)}$ with condition $d(h_s(t), l(s)) = td(l(s), l_0(s))$. Now we can define desired homotopy $H : I \times I \to S$ by formula $H(s, t) = f(h_s(t))$, where f is defined in the proof of Proposition 3.18. Indeed, it is clear that the mapping f homeomorphically sends every segment $\overline{yz}$ with ends $y, z \in S$, such that z is not antipodal to y with respect to x, onto its image in S. For this reason the image of loop $l_1(s) = H(s, 1)$ is no more than 1-dimensional because the image of the restriction of l_1 to every segment $\overline{s_j, s_{j+1}}$, $j = 0, 1, \ldots, m - 1$, is equal to $f(\overline{y_j y_{j+1}})$ which is homeomorphic to the segment $\overline{y_j y_{j+1}}$. Since S has topological dimension $\leqslant n$ and is a homology $(n-1)$-manifold by Theorem 3.19, its topological dimension is $(n-1) \geqslant 2$. Thus $l_1(I) \neq S$. $\square$

Remark 3.23. This theorem would also hold for *infinite-dimensional* Busemann G-spaces should they exist.

4. Local *G*-homogeneity

In this section we shall introduce some basic terminology and facts:

Definition 4.1. A set Z in a metric space X is said to be *starlike with respect to $x \in \mathrm{int}(Z)$* if x is joinable with every point in the boundary ∂Z by unique shortest geodesic (segment) and Z is the geodesic cone over ∂Z with cone point x. In particular, $Z = \bigcup \{\overline{xz} \mid z \in \partial Z\}$ and if $z, z' \in \partial Z$ are different, then $\overline{xz} \cap \overline{xz'} = \{x\}$.

It follows from Proposition 3.3 that in Busemann G-spaces, all metric balls $B(x, r)$, $0 < r < \rho(x)$, are starlike with respect to their centers.

Definition 4.2. A set Z in a metric space is said to be *stably starlike* at a point x if there is a $\delta > 0$ such that Z is starlike with respect to any point $y \in B(x, \delta)$.

Definition 4.3. A metric space X is said to be *locally G-homogeneous* if for every point $x \in X$, there is a radius $\varepsilon > 0$ such that $\varepsilon < \rho(x)$ and the ball $B(x, \varepsilon)$ is stably starlike at x (or, in other words, the sphere $S(x, \varepsilon)$ is stably visible at x).

Remark 4.4. The condition that the ball $B := B(x, \varepsilon)$ in a Busemann G-space X is metrically strongly convex, i.e. any two points $y, z \in B$ are joinable by a unique segment $\overline{yz}$ in X and this segment, except maybe for points y and z, is contained in $\text{int}(B)$, implies the assertion that B is starlike with respect to every point in $\text{int}(B)$. As a corollary, a Busemann G-space X, having for each point $x \in X$ a metrically strongly convex closed ball of positive radius with the center x, is locally G-homogeneous.

The terminology locally G-homogeneous was chosen to signify that an autohomeomorphism of X fixed outside of $B(x, \varepsilon)$ taking x to a nearby point y can be chosen to preserve cone lines in the sense that if $z \in S(x, \varepsilon)$, then $\overline{xz} \to \overline{yz}$. Although all Busemann G-spaces are topologically homogeneous, it is unknown whether all Busemann G-spaces are also locally G-homogeneous.

Definition 4.5. A space X is said to be *strongly topologically homogeneous* if for any two points $x, y \in X$ and path $\alpha : [0, 1] \to X$ such that $\alpha(0) = x$ and $\alpha(1) = y$, the map $h : \{x\} \times [0, 1] \to X$; $h(x, t) = \alpha(t)$ is an ambient isotopy, i.e. there is an isotopy $H : X \times [0, 1] \to X$ such that $H|_{\{x\} \times [0,1]} = h$.

5. Topological homogeneity

In this section we demonstrate the first version of our main result, the topological homogeneity of sufficiently small metric spheres in locally G-homogeneous Busemann G-spaces.

We now define two types of maps that are key to our proof and establish their continuity. We begin by citing the following well-known result:

Proposition 5.1. *Suppose that X is a Busemann G-space and $B(x, r) \subset X$, where $0 < r < \rho(x)$. Then for each point $z \in S(x, r')$, where $0 < r' \leqslant r$, there exists a unique point $z' \in S(x, r')$ such that $z - x - z'$.*

Proof. This follows directly from the uniqueness of extension property. □

The point z' from Proposition 5.1 is called the *antipode* of z in $S(x, r')$. We now define the antipodal map for $B(x, r)$.

Definition 5.2 (*Antipodal Map*). Suppose that X is a Busemann G-space and $B(x, r) \subset X$, where $0 < r < \rho(x)$. Then the *antipodal map* $\Phi : B(x, r) \to B(x, r)$ is defined so that $\Phi(x) = x$ and for each point $z \in S(x, r')$, where $0 < r' \leqslant r$, $\Phi(z) = z'$, where z' is the unique antipode of z in $S(x, r')$.

In the continuity arguments that follow, we shall use the following extensively, to show that a map is continuous: A map $f : X \to Y$ between compact metric spaces is continuous if and only if for every point $x \in X$ and every sequence $\{x_n\} \subset X$ such that $x_n \to x$ and $f(x_n) \to y^*$, one gets $y^* = f(x)$.

Proposition 5.3. *The antipodal map $\Phi : B(x, r) \to B(x, r)$ from Definition 5.2 is a homeomorphism.*

Proof. Suppose that $\{z_n\} \subset B(x, r)$ is a sequence such that $z_n \to z$ and $\Phi(z_n) \to z^*$. Let $r' = d(x, z)$ and $r_n = d(x, z_n)$. Then $r_n \to r'$. Note that

$$d(z_n, x) + d(x, \Phi(z_n)) = d(z_n, \Phi(z_n)) = 2r_n$$

which, by continuity of the distance function, implies that

$$d(z, x) + d(x, z^*) = d(z, z^*) = 2r'.$$

However, we also have

$$d(z, x) + d(x, \Phi(z)) = d(z, \Phi(z)) = 2r'.$$

Moreover,

$$d(x, z^*) = d(x, \Phi(z)) = r'.$$

By uniqueness of antipodes (cf. Proposition 5.1), $z^* = \Phi(z)$. Therefore, Φ is continuous.

Since $\Phi^{-1} = \Phi$, it follows that Φ is indeed a homeomorphism. □

The following projection map will also be key in defining our homogeneity homeomorphism.

Definition 5.4 *(Projection Map).* Suppose that X is a Busemann G-space and $T, Z \subset X$ are compact starlike sets with respect to $x \in \text{int}(T) \cap \text{int}(Z)$. Define the *projection map* $\psi : Z - \{x\} \to \partial T$ such that for each point $z \in Z - \{x\}$, $\psi(z) = t$, where t is the unique point of ∂T so that one of $x - t - z$, $t = z$, or $x - z - t$ holds. We say that ψ is *centered at x*.

The existence and uniqueness of t in Definition 5.4 easily follows from the definition of starlike set and inclusion $x \in \text{int}(T) \cap \text{int}(Z)$. The cases $x - t - z$, $t = z$, and $x - z - t$ correspond the cases $d(x, t) < d(x, z)$, $d(x, t) = d(x, z)$, and $d(x, t) > d(x, z)$, respectively.

Proposition 5.5. *The projection map $\psi : Z - \{x\} \to \partial T$ from Definition 5.4 is continuous. Moreover, the restriction map $\psi|_{\partial Z} : \partial Z \to \partial T$ is a homeomorphism.*

Proof. By hypothesis, T, Z are starlike sets with respect to x. For any $z \in \partial Z - \{x\}$, $\psi(z)$ is the unique point $t \in \partial T$ so that one of $x - t - z$, $t = z$, or $x - z - t$ holds.

We shall now show the continuity of ψ. In particular, we shall show that ψ is continuous on the restriction to any compact set $Z_\delta = \overline{Z - B(x, \delta)}$, where $B(x, \delta) \subset \text{int}(Z)$ and $\delta > 0$. Suppose that there is a sequence $\{z_n\} \subset Z_\delta$ such that $z_n \to z$ and $\psi(z_n) = t_n \to t^*$. By the compactness of T, and hence ∂T, $t^* \in \partial T$.

For each n, one of $x - t_n - z_n$ or $x - z_n - t_n$ is the case. Let

$$L = \{z_n \mid x - z_n - t_n\}$$

and

$$M = \{z_n \mid x - t_n - z_n\}.$$

If L is finite, it follows from the continuity of the distance function and the relation $d(x, t_n) + d(t_n, z_n) = d(x, z_n)$ for large n, that $x - t^* - z$ or $t^* = z$. If M is finite, then $d(x, z_n) + d(z_n, t_n) = d(x, t_n)$ for large n, so that $x - z - t^*$ or $z = t^*$. If neither L nor M is finite, then $z = t^*$ must be the case. However, t is the unique point of ∂T so that one of $x - t - z$, $t = z$, or $x - z - t$ holds. Hence $t^* = t$. Therefore ψ is continuous.

Note that both $\psi|_{\partial Z}$ and $\psi|_{\partial Z}^{-1}$ are well-defined and 1–1 by our choice of Z and T. Since ψ is continuous, $\psi|_{\partial Z}$ is also continuous. The map $\psi|_{\partial Z}^{-1}$ is the restriction of the projection map $\phi : T - x \to \partial Z$ to ∂T. Thus the continuity of $\psi|_{\partial Z}^{-1}$ also follows from a similar argument as above. Therefore $\psi|_{\partial Z}$ is a homeomorphism. $\square$

Theorem 5.6. *In a locally G-homogeneous Busemann G-space X, every two spheres $S(x, r(x))$ and $S(y, r(y))$, of radii $0 < r(x) < \rho(x)$ and $0 < r(y) < \rho(y)$, respectively are homeomorphic.*

Proof. It suffices to show that for every $x \in X$ there is a $\delta > 0$ such that the result is true for each $y \in B(x, \delta)$. Let $B(x, \epsilon)$ be the ball promised by the definition of locally G-homogeneous and $\delta > 0$ ($\delta < \epsilon$) be the value promised by the definition of stably starlike. Then for any $y \in B(x, \delta)$ we have the desired homeomorphism to be the composition of homeomorphisms

$$S\big(x, r(x)\big) \xrightarrow{\psi_1} S(x, \epsilon) \xrightarrow{\psi_2} S\big(y, r(y)\big)$$

where ψ_1 is the projection map centered at x and ψ_2 is the projection map centered at y. $\square$

We are now ready to prove a weaker version of our first main theorem.

Theorem 5.7. *Suppose that X is a locally G-homogeneous Busemann G-space. Then any metric sphere $S(x, r)$, $0 < r < \rho(x)$, is topologically homogeneous.*

Proof. It suffices to show that $S(x, \epsilon)$ is homogeneous for sufficiently close points. Without loss of generality, choose y sufficiently close to z so that $B(x, \epsilon)$ is starlike with respect to the midpoint m of $\overline{y'z}$. Let $\gamma = \frac{1}{2}d(y, y')$. The desired map sequence provides a homeomorphism taking y to z, where Φ_i are antipodal maps and ψ is a projection map centered at m,

$$S(x, \epsilon) \xrightarrow{\Phi_1} S(x, \epsilon) \xrightarrow{\psi} S_\gamma(m) \xrightarrow{\Phi_2} S_\gamma(m) \xrightarrow{\psi^{-1}} S(x, \epsilon). \quad \square$$

6. Strong homogeneity

In this section we shall show that small metric spheres in locally G-homogeneous Busemann G-spaces X are in fact, strongly homogeneous. We shall call a sphere $S(x, r) \subset X$ *sufficiently small* if $r < \rho(x)$ (cf. Lemma 3.2).

Definition 6.1. Let X be a Busemann G-space. Then $\Omega \subset X$ is said to be a *fundamental region in X* provided that:

(1) Ω is an open region with compact closure; and
(2) For any closed metric ball $B(x, r) \subset \Omega$, $r < \rho(x)$.

Each point in a Busemann G-space X is contained in a fundamental region. For example, one can easily prove that every open ball $U(x, r) \subset X$, where $0 < r < \rho(x)$ is a fundamental region.

We shall begin with some continuity theorems.

Theorem 6.2. *Suppose that X is a Busemann G-space and $\Omega \subset X$ is a fundamental region. Let*

$$\hat{\Omega} = \left\{ (a, r, x) \in X \times \mathbb{R} \times X \mid x \in B(a, r) \subset \Omega \right\}$$

and $\Phi[a, r] : B(a, r) \to B(a, r)$ is an antipodal map. Then the map

$$f : \hat{\Omega} \to X; \qquad f(a, r, x) := \Phi[a, r](x)$$

is continuous.

Proof. Let Ω^* be a compact subset of Ω. Let

$$\hat{\Omega}^* = \left\{ (a, r, x) \in X \times \mathbb{R} \times X \mid B(a, r) \subset \Omega^* \right\}.$$

It suffices to show that f is continuous on $\hat{\Omega}^*$.

Suppose there is a sequence $(a_n, r_n, x_n) \to (a, r, x)$ in $\hat{\Omega}^*$ and $f(a_n, r_n, x_n) \to x^*$. Without loss of generality we may choose r'_n so that $r_n \leqslant r'_n$ and $B(a, r) \subset B(a_n, r'_n) \subset \Omega$ (this can be accomplished if the sequence $\{a_n\}$ is modified to contain only points very close to a). Define $x'_n = f(a_n, r'_n, x_n)$. Note that

$$d(x'_n, a_n) + d(a_n, x) = d(x'_n, x_n) \quad \text{and} \quad d(x'_n, a_n) = d(a_n, x_n).$$

By continuity of the distance function

$$d(x^*, a) + d(a, x) = d(x^*, x) \quad \text{and} \quad d(x^*, a) = d(a, x).$$

However x', the antipode of x in $B(a, r)$, satisfies these same relations in place of x^*. It then follows from uniqueness of the antipode that $x^* = x'$. Therefore f is continuous. $\square$

Theorem 6.3. *Suppose that X is a Busemann G-space, $\Omega \subset X$ is a fundamental region, $B(x, \rho) \subset \Omega$ is stably starlike at x and $\delta > 0$ is a radius promised in the definition of stably starlike (cf. Definition 4.2). Let*

$$\hat{\Omega} = \left\{ (a, r, y) \mid a \in B(x, \delta) \subset U(a, r) \subset B(a, r) \subset \Omega, \ y \in \overline{B(x, \rho) - B(x, \delta)} \right\}.$$

Then the map $g : \hat{\Omega} \to X; g(a, r, y) = \psi[a, r](y)$, where $\psi[a, r] : \overline{B(x, \rho) - B(x, \delta)} \to S(a, r)$ is the projection map centered at a, is continuous.

Proof. Let Ω^* be a compact subset of Ω. Define

$$\hat{\Omega}^* = \left\{ (a, r, y) \in \Omega \mid y \in B(a, r) \subset \Omega^* \right\}.$$

It suffices to show that g is continuous on $\hat{\Omega}^*$.

Suppose there is a sequence $(a_n, r_n, y_n) \in \Omega^*$ such that $(a_n, r_n, y_n) \to (a, r, y)$ and $g(a_n, r_n, y_n) \to y^*$. Let $y'_n = g(a_n, r_n, y_n)$. Note that $y^* \in S(a, r)$. Also, precisely one of $a_n - y'_n - y_n$, $y'_n = y_n$, or $a_n - y_n - y'_n$ holds for each n. By continuity of the distance function, one of $a - y^* - y$, $y^* = y$, or $a - y - y^*$ holds. However, $y' = g(a, r, y)$ is a point of $S(a, r)$ that also satisfies this condition when replaced with y^*. Thus $y^* = y'$. Therefore g is indeed continuous. $\square$

Theorem 6.4. *Suppose that X is a Busemann G-space, $\Omega \subset X$ is a fundamental region and $B(x, \epsilon) \subset \Omega$. Let y be a fixed point in $S(x, \epsilon)$ and let y' denote the antipode of y in $S(x, \epsilon)$. Then the midpoint map*

$$\Gamma : S(x, \epsilon) \to X$$

such that $\Gamma(z) = m$, where m is the midpoint of $\overline{zy'}$, is continuous.

Proof. Suppose $\{z_n\} \subset S(x, \epsilon)$ is a sequence such that $z_n \to z$ and $\Gamma(z_n) = m_n \to m^*$. Then $z_n - m_n - y'$ and $d(z_n, m_n) = d(m_n, y')$. By the continuity of the distance function, $z - m^* - y'$ and $d(z, m^*) = d(m^*, y')$. However, m also satisfies these relations in place of m^*. By uniqueness of joins, $m = m^*$. Thus Γ is indeed continuous. $\square$

We are now ready to prove the strong form of the first main theorem.

Proof of Theorem 1.1. Let $y, z \in S(x, \epsilon)$ and $\alpha : I \to S(x, \epsilon)$ be a path from y to z. Let y' be the antipode of y, $m(t)$ the midpoint of $\overline{\alpha(t)y'}$, and $\gamma(t) = \frac{1}{2}d(\alpha(t), y')$. The isotopy $H : S_\epsilon(x) \times I \to S_\epsilon(x)$ is given by H_t which is the composition of homeomorphisms

$$S_\epsilon(x) \xrightarrow{\Phi[x,\epsilon]} S_\epsilon(x) \xrightarrow{\psi[m(t),\gamma(t)]} S_{\gamma(t)}\big(m(t)\big) \xrightarrow{\Phi[m(t),\gamma(t)]} S_{\gamma(t)}\big(m(t)\big) \xrightarrow{\psi[m(t),\gamma(t)]^{-1}} S_\epsilon(x).$$

Continuity of H_t in the variable t follows from the propositions above. Thus H is the desired isotopy. $\quad\square$

7. Uniformly locally G-homogeneous Busemann G-spaces

Definition 7.1. We say that a metric space (X, d) is *uniformly locally G-homogeneous on a set* $C \subset X$ if there are numbers δ, ε_1, ε_2 such that

- $0 < \delta \leqslant \varepsilon_1 < \varepsilon_2$.
- For every $c \in C$ and every $\varepsilon \in (\varepsilon_1, \varepsilon_2)$ the closed ball $B(c, \varepsilon)$ is starlike with respect to every point x in open ball $U(c, \delta)$.

We say that a metric space (X, d) is *uniformly locally G-homogeneous on an orbal subset* $C \subset X$ if additionally

- C contains a ball $B(c_0, r)$ such that $\varepsilon_2 < r$.

The following is our second main theorem (cf. Theorem 1.2 from Section 1).

Theorem 7.2. *If a Busemann G-space is uniformly locally G-homogeneous on an orbal subset C, then it has finite topological dimension.*

Proof. Assume the setup given by Definition 7.1. Then we can find numbers ε_1', ε_2' such that

$$\varepsilon_1 < \varepsilon_1' < \varepsilon_2' < \varepsilon_2 \quad \text{and} \quad \frac{\varepsilon_2' - \varepsilon_1'}{2} + \varepsilon_2' < \varepsilon_2. \tag{2}$$

Let us choose numbers

$$r_1 = \frac{1}{2}\min(\varepsilon_2' - \varepsilon_1', \delta) \quad \text{and arbitrary} \quad \varepsilon_0 : \quad 0 < \varepsilon_0 < \min(\delta, \varepsilon_2 - \varepsilon_2', \varepsilon_1' - \varepsilon_1). \tag{3}$$

Consider the set

$$D = \big\{(x, z) \in B(c_0, r_1) \times B(c_0, r) \,\big|\, d(x, z) = \varepsilon_2'\big\}$$

and define a metric d_1 on D by the formula

$$d_1\big((x, z), (x', z')\big) = \max\big(d(x, x'), d(z, z')\big). \tag{4}$$

Evidently the metric space (D, d_1) is compact. Thus there is a finite ε_0-net

$$\big\{(x_1, z_1), \ldots, (x_m, z_m)\big\}$$

in (D, d_1). Define a (continuous) map $f : B(c_0, r_1) \to \mathbb{R}^m$ by the formula

$$f(y) = \big(d(y, z_1), \ldots, d(y, z_m)\big).$$

We state that f is a topological embedding. For this it is enough to show that $f(x) \neq f(y)$ if $x, y \in B(c_0, r_1)$ and $x \neq y$. Indeed, it follows from the Triangle inequality and the formula (3) that $d(x, y) \leqslant \varepsilon_2' - \varepsilon_1' < \varepsilon_2'$. Using once more Definition 7.1 and Eqs. (2) and (3), we see that there is unique extension of the segment $\overline{xy}$ to a segment $\overline{xz}$ of length ε_2', and this segment lies in $B(c_0, r)$. Clearly, $(x, z) \in D$. By construction, there is an index $i \in \{1, \ldots, m\}$ such that $d_1((x_i, z_i), (x, z)) < \varepsilon_0$. We claim that $d(x, z_i) \neq d(y, z_i)$ (and so $f(x) \neq f(y)$). Otherwise, using the Triangle inequality and Eqs. (2), (3), and (4), we see that

$$\varepsilon := d(z_i, x) = d(z_i, y) \in (\varepsilon_1, \varepsilon_2). \tag{5}$$

On the other hand, $z - y - x$ and $d(z_i, z) < \delta$, and Eq. (5) contradicts the statement that the closed ball $B(z_i, \varepsilon)$ must be starlike with respect to the point z. So f is one-to-one on $B(c_0, r_1)$ and the topological dimension of $B(c_0, r_1)$ is less than or equal to m. Now Corollary 3.12 implies that the topological dimension of (X, d) is less than or equal to m. $\quad\square$

Remark 7.3. As we said in the Introduction, it was proved in [3] that a Busemann G-space X is finite-dimensional if X has small metrically convex balls near some of its points. It is known that this implies that X also has small metrically strongly convex balls near the same points [15]. This fact together with Remark 4.4 implies that a Busemann G-space, which has small convex balls near some point, is also uniformly locally G-homogeneous on an orbal subset. So Theorem 7.2 generalizes the result from [3] mentioned above.

8. Example

In this section we prove Theorem 1.3, i.e. we present an example of a Busemann G-space that is uniformly locally G-homogeneous on an orbal subset and locally G-homogeneous, but has no convex metric ball of positive radius.

In 1999 Gribanova [25] found all inner metrics on the upper half plane which are invariant under the action of the group

$$\Gamma: \quad x' = \alpha x + \beta, \quad y' = \alpha y, \quad \alpha > 0, \ -\infty < \beta < +\infty,$$

as well as their geodesics. It follows from [5] that every such metric must be Finslerian. Thus it is easy to see that the corresponding line element must have a form $ds = y^{-1} F(dx, dy)$ with a fixed norm F. Gribanova completely classified all quasihyperbolic geometries determined by the above line element (i.e. Busemann G-spaces) depending on the properties of F.

In particular, the following theorem was proven:

Theorem 8.1. *The line element of a quasihyperbolic plane can be written in the form*

$$ds = y^{-1} F(dx, dy);$$

moreover, the function $F(u_1, u_2)$ is defined for all u_1 and u_2 and satisfies the following conditions

(1) $F(u_1, u_2) > 0$ *for* $(u_1, u_2) \neq (0, 0)$;
(2) $F(ku_1, ku_2) = |k| F(u_1, u_2)$ *for every real k*;
(3) *F is convex*;
(4) *F is differentiable everywhere except for* $(0, 0)$;
(5) *The tangents of the curve $F(u_1, u_2) = 1$, parallel to the straight line $u_2 = 0$, touch this curve at a unique point.*

Conversely, each line element of the form

$$ds = y^{-1} F(dx, dy)$$

with F possessing these properties determines a quasihyperbolic geometry.
Define the function $F^*(x, y)$ by

$$F^* = \max_{F(u_1, u_2) \leqslant 1} (xu_2 - yu_1).$$

Geodesics of the quasihyperbolic plane with the line element $ds = y^{-1} F(dx, dy)$ are the intersections of the half-plane $y > 0$ with the curves $F^*(x - a, y) = k$, $k > 0$, $-\infty < a < \infty$, and with the tangents of these curves at their intersection points with the x-axis. For two distinct points of a quasihyperbolic plane, there is exactly one geodesic passing through them.

Remark 8.2. Geometric meaning of geodesics is that they are solutions of isoperimetric problem for two-dimensional normed vector space with the norm F [16]. Here, as well as below, a prescription is given how to construct them. However, Busemann also said in [17] (cf. p. 82) that spaces, defined in this manner by two norms F_1 and F_2, are isometric if and only if there is a linear transformation l of $\mathbb{R}^2$ such that $F_2 = F_1 \circ l$. This statement is not true because the usual Euclidean norm $F(u, v) = \sqrt{u^2 + v^2}$ gives a hyperbolic plane of curvature -1, while the norm kF, $k > 0$, gives a hyperbolic plane of curvature $\frac{-1}{k^2}$.

Gribanova also proved a theorem which can equivalently be stated as follows.

Theorem 8.3. *Suppose also that both tangent lines to the curve (so-called indicatrix) $C = \{(x, y) \in \mathbb{R}^2 \mid F(x, y) = 1\}$ at intersection points of this curve with x-axis have nontrivial joint segments with the curve C. Then no closed ball $B(p, r)$ of the quasihyperbolic plane with the line element $ds = y^{-1} F(dx, dy)$ is geodesically convex for any $r > 0$.*

8.1. Stadium space norm

Let us consider a quasihyperbolic plane X which we shall call the "Stadium space". It is defined by the set "Stadium" which consists of squares with side length 2 together with semidisks on the top and the bottom with radius 1, as pictured in Fig. 2.

The Stadium defines a norm $F = \| \cdot \|$ on $\mathbb{R}^2$, if we assume that its boundary curve C is a unit circle. It is clear that the norm F satisfies all hypotheses of Theorems 8.1 and 8.3. The norm of a vector $\mathbf{v}$ in the direction with the angle ψ measured from the positive x-axis is equal to
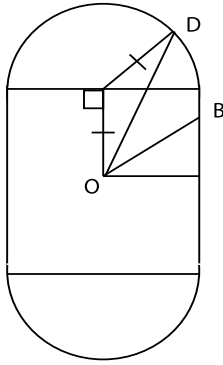
$$\|\mathbf{v}\| = l(\psi) |\mathbf{v}|,$$
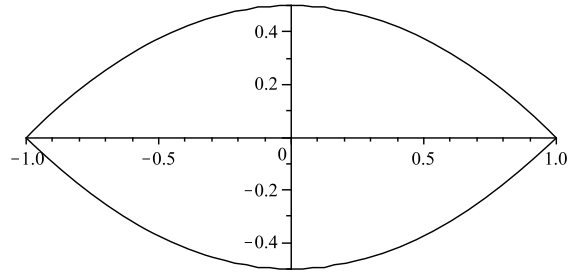
Fig. 2. The Stadium.
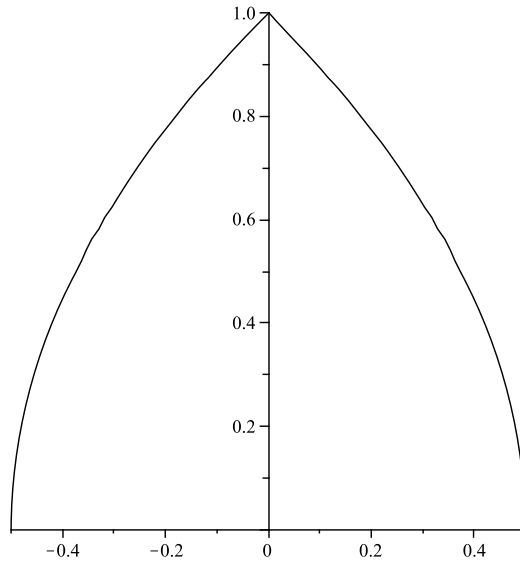

Fig. 3. The dual curve C_1.

Fig. 4. The standard geodesic C_2.

where $|\mathbf{v}|$ is the usual Euclidean norm and

$$l(\psi) = \begin{cases} |\cos \psi|, & \text{if } -\frac{\pi}{4} \leqslant \psi \leqslant \frac{\pi}{4} \text{ or } \frac{3\pi}{4} \leqslant \psi \leqslant \frac{5\pi}{4}, \\ \frac{1}{2|\sin \psi|}, & \text{if } \frac{\pi}{4} \leqslant \psi \leqslant \frac{3\pi}{4} \text{ or } -\frac{3\pi}{4} \leqslant \psi \leqslant -\frac{\pi}{4}. \end{cases} \tag{6}$$

To see this, observe in Fig. 2 that $l(\psi)$ is $\frac{1}{OB}$ in the first case and $\frac{1}{OD}$ in the second case.

8.2. Geodesics

Geodesics in the metric geometry are determined by the dual curve to the Stadium space (cf. Fig. 3),

$$C_1 = \left\{ (x, y) \in \mathbb{R}^2 \,\Big|\, \max_{(u_1, u_2) \in \mathcal{C}} (xu_1 + yu_2) = 1 \right\}.$$

In particular, if C_1 is rotated by $\frac{\pi}{2}$ to obtain C_2 (cf. Fig. 4), then the geodesics are the portions of the vertical lines and curves of the form

$$C_2(\lambda, x_0) := \lambda C_2 + (x_0, 0); \quad \lambda > 0, \ x_0 \in \mathbb{R}$$

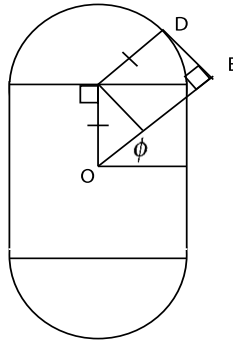contained in $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 \colon y > 0\}$.

Fig. 5. Defining the dual curve.

The dual curve C_1 to the boundary of the Stadium space is defined in polar coordinates, $[\tau, \phi]$ by the following function $\tau_1(\phi)$,

$$\tau_1(\phi) = \frac{1}{OE} = \frac{1}{1 + |\sin\phi|},$$

where E is the orthogonal projection of the point O onto the tangent line to the curve C at a point $D \in C$ (cf. Fig. 5).

Rotating by $\frac{\pi}{2}$ we get the defining function τ_2 for C_2 as

$$\tau_2 = \tau_1\left(\phi - \frac{\pi}{2}\right) = \frac{1}{1 + |\sin(\phi - \frac{\pi}{2})|} = \frac{1}{1 + |\cos\phi|}.$$

We shall omit calculations for the left part of the curve C_2, using later the symmetry of curves C and C_2 relative to y-axis. Also we consider only the upper halves of curves $C_2(\lambda, x)$. So we get the equation

$$\tau_2 = \frac{1}{1 + \cos\phi}, \quad 0 < \phi \leqslant \pi/2. \tag{7}$$

It is known that this is part of parabola. Setting $\phi = 0$ and $\phi = \pi/2$, we see that the right side of C_2 has equation

$$x = \frac{1 - y^2}{2}.$$

Hence, the right side of λC_2 has equation

$$x = \frac{\lambda^2 - y^2}{2\lambda}.$$

So the entire curve λC_2 is

$$x = \pm\frac{\lambda^2 - y^2}{2\lambda}, \quad 0 \leqslant |y| \leqslant \lambda. \tag{8}$$

Since the metric space, which we consider, is homogeneous, we can study only the circles of this metric with the center at the point $(0, 1)$. On the curve λC_2, if $y = 1$ we get $x_1 = \frac{\lambda^2 - 1}{2\lambda}$. So the shifted curve $\lambda C_2 + (x_0(\lambda), 0)$ passing through $(0, 1)$ has shifting term

$$x_0(\lambda) = -x_1 = \frac{1}{2}\left(\frac{1}{\lambda} - \lambda\right). \tag{9}$$

So we get the equation of the right side of $\lambda C_2 + x_0(\lambda)$ to be

$$x = \frac{\lambda^2 - y^2}{2\lambda} + x_0(\lambda) = \frac{1 - y^2}{2\lambda}, \quad 0 < y \leqslant \lambda, \ 1 \leqslant \lambda. \tag{10}$$

8.3. Distance formulas

The tangent vector for the right side of $\lambda C_2 + x_0(\lambda)$ (see Fig. 6) has direction vector

$$\left\langle \frac{dx}{dy}, 1 \right\rangle = \left\langle -\frac{y}{\lambda}, 1 \right\rangle, \quad \text{where } 0 < \frac{y}{\lambda} \leqslant 1.$$
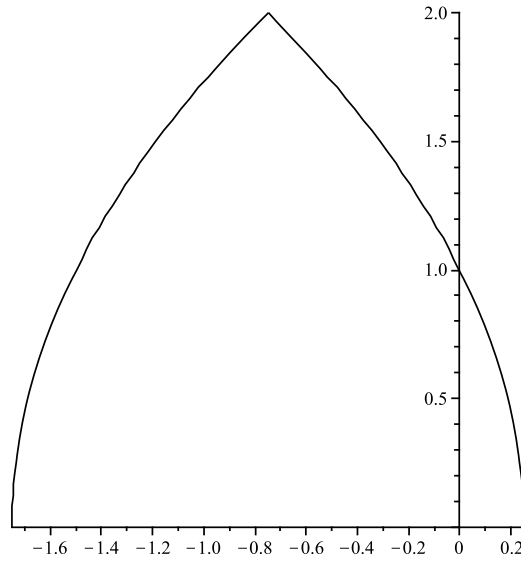
Fig. 6. Shifted geodesic $\lambda C_2 + (x_0(\lambda), 0)$.

Then the angle ψ of this direction changes between $\pi/2$ and $3\pi/4$. So we need to use only the second formula in (6). Here

$$\frac{1}{2|\sin \psi|} = \frac{\sqrt{1 + (\frac{y}{\lambda})^2}}{2}.$$

The line element on $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2: y > 0\}$ is $ds = \frac{1}{y}\|\cdot\|$. Then the length of a geodesic between two points (x_1, y_1) and (x_2, y_2) on the right side of $\lambda C_2 + x_0(\lambda)$ where $0 < y_1 < y_2 \leqslant \lambda$ is

$$l(y_1, y_2) = \int_{y_1}^{y_2} \frac{\sqrt{1 + (\frac{y}{\lambda})^2}}{2} \frac{\sqrt{1 + (\frac{y}{\lambda})^2}}{y} \, dy = \frac{1}{2} \int_{y_1}^{y_2} \frac{1 + (\frac{y}{\lambda})^2}{\frac{y}{\lambda}} d\left(\frac{y}{\lambda}\right)$$

$$= \frac{1}{2} \int_{\frac{y_1}{\lambda}}^{\frac{y_2}{\lambda}} \frac{1 + z^2}{z} \, dz = \frac{1}{2} \int_{\frac{y_1}{\lambda}}^{\frac{y_2}{\lambda}} \left(z + \frac{1}{z}\right) dz = \frac{1}{2} \left(\ln z + \frac{z^2}{2}\right)\Big|_{\frac{y_1}{\lambda}}^{\frac{y_2}{\lambda}} = \frac{1}{2} \ln \frac{y_2}{y_1} + \frac{1}{4\lambda^2}(y_2^2 - y_1^2).$$

In the subcases $y_2 = 1 > y_1 = y$ or $y = y_2 > y_1 = 1$ we get respectively

$$l(y) = \frac{1}{4}\left(\frac{1 - y^2}{\lambda^2} - \ln y^2\right) \quad \text{for } y < 1 \text{ (on the right side of the curve)} \tag{11}$$

or

$$l(y) = \frac{1}{4}\left(\frac{y^2 - 1}{\lambda^2} + \ln y^2\right) \quad \text{for } y > 1 \text{ (on the right side of the curve).} \tag{12}$$

Applying the last formula to the point of maximal height $(x_0(\lambda), \lambda)$, where $\lambda > 1$, we get

$$l(\lambda) = \frac{1}{4}\left(\frac{\lambda^2 - 1}{\lambda^2} + \ln \lambda^2\right) = \frac{1}{4}\left(1 - \frac{1}{\lambda^2} + \ln \lambda^2\right),$$

so

$$l(\lambda) = \frac{1}{4}\left(1 - \frac{1}{\lambda^2} + \ln \lambda^2\right). \tag{13}$$

8.4. Metric spheres

Now we shall find a form of the sphere $S_K := S((0, 1), K)$ with radius $K > 0$ and center $(0, 1)$, using only the right side of curves $\lambda C_2 + x_0(\lambda)$. Then we can apply the symmetry of the geodesic relative to the line $x = x_0$.
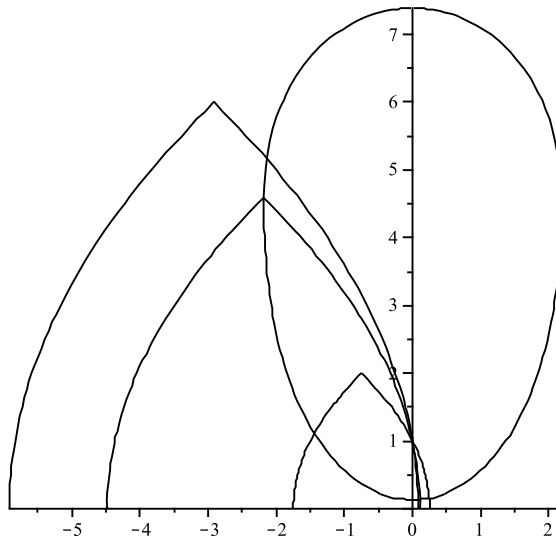
Fig. 7. A unit sphere in the Stadium space with sample geodesic radii.

We shall have three cases when the geodesic radii to the point on S_K is nonvertical (cf. Fig. 7).

(1) If $(x, y) \in S_K$, $x > x_0$, $y < 1$, then we have by formula (11) that

$$K = l(y) = \frac{1}{4} \left(\frac{1 - y^2}{\lambda^2} - \ln y^2 \right).$$

(2) If $(x, y) \in S_K$, $x \geqslant x_0$, $y > 1$, then $K \leqslant l(\lambda)$ and we have by formula (12) that

$$K = l(y) = \frac{1}{4} \left(\frac{y^2 - 1}{\lambda^2} + \ln y^2 \right).$$

(3) Consider now the case when $(x, y) \in S_K$, $x < x_0$. Note that in this case $K > l(\lambda)$. Using the symmetry of the geodesic with respect to the line $x = x_0$ we have

$$K = l(\lambda) + d\big(\big(x_0(\lambda), \lambda\big), (x, y)\big) = \frac{1}{4} \left(1 - \frac{1}{\lambda^2} + \ln \lambda^2 \right) + \frac{1}{2} \ln \frac{\lambda}{y} + \frac{\lambda^2 - y^2}{4\lambda^2},$$

which is equivalent to the equation

$$\frac{y^2 + 1}{4\lambda^2} + \frac{1}{2} \ln y - \ln \lambda = \frac{1}{2} - K. \tag{14}$$

In this case

$$x = -\frac{\lambda^2 - y^2}{2\lambda} + x_0(\lambda) = -\frac{\lambda^2 - y^2}{2\lambda} + \frac{1}{2} \left(\frac{1}{\lambda} - \lambda \right) = -\lambda + \frac{y^2 + 1}{2\lambda}.$$

Hence

$$x = -\lambda + \frac{y^2 + 1}{2\lambda}. \tag{15}$$

In the case of a vertical geodesic radii, the points on S_K are easily evaluated from solving the integral equation

$$K = \left| \int_{1}^{y_0} \frac{1}{2y} \, dy \right|$$

for y_0. It follows that the boundary points along $x = 0$ are $(0, e^{\pm 2K})$. Note that we can also get this by taking limits as $\lambda \to +\infty$ in formulas (11) and (12) where $l(y) = K$.
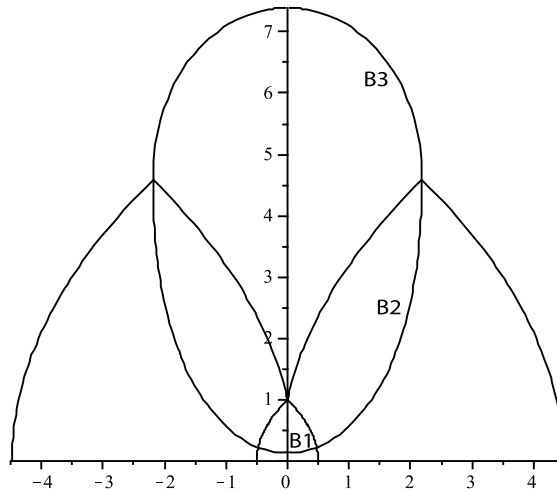
Fig. 8. The boundary curves of a unit ball.

8.5. Tangents to spheres

Next we shall find tangents to the sphere S_K, using the equations above and considering only its right part. This part in turn, consists of 3 curves: the bottom right curve B_1, the side right curve B_2, and the top right curve B_3 (cf. Fig. 8). The joint point of curves B_1 and B_2 is defined by the equality $\lambda = 1$, while the joint point of curves B_2 and B_3 is defined by the equality $y_0 = \lambda_0$, where $l(\lambda_0) = K$ (cf. Eq. (13)).

Equations of the bottom right curve B_1 are

$$x = \frac{1 - y^2}{2\lambda},$$
$$\frac{1 - y^2}{4\lambda^2} - \frac{1}{2}\ln y = K.$$

Differentiating these equations and using once more the first one of them, we get the following system

$$2\lambda y \frac{dy}{dx} + \left(1 - y^2\right)\frac{d\lambda}{dx} = -2\lambda^2,$$
$$\lambda\left(y^2 + \lambda^2\right)\frac{dy}{dx} + y\left(1 - y^2\right)\frac{d\lambda}{dx} = 0.$$

Solving for $\frac{dy}{dx}$ and $\frac{d\lambda}{dx}$ we get by the Cramer rule

$$\frac{dy}{dx} = \frac{2y\lambda}{\lambda^2 - y^2} > 0, \qquad \frac{d\lambda}{dx} = \frac{2\lambda^2(y^2 + \lambda^2)}{(1 - y^2)(y^2 - \lambda^2)} < 0. \tag{16}$$

In the other two cases we need to multiply Eqs. (15) and (10) by -1. Equations of the side right curve B_2 are

$$x = \lambda - \frac{1 + y^2}{2\lambda},$$
$$\frac{y^2 + 1}{4\lambda^2} + \frac{1}{2}\ln y - \ln \lambda = \frac{1}{2} - K.$$

We differentiate these equations to get

$$-2\lambda y \frac{dy}{dx} + \left(2\lambda^2 + 1 + y^2\right)\frac{d\lambda}{dx} = 2\lambda^2,$$
$$\lambda\left(y^2 + \lambda^2\right)\frac{dy}{dx} - y\left(2\lambda^2 + 1 + y^2\right)\frac{d\lambda}{dx} = 0.$$

Solving for $\frac{dy}{dx}$ and $\frac{d\lambda}{dx}$, using the Cramer rule, we get the following

$$\frac{dy}{dx} = \frac{2y\lambda}{\lambda^2 - y^2} > 0, \qquad \frac{d\lambda}{dx} = \frac{2\lambda^2(y^2 + \lambda^2)}{(2\lambda^2 + 1 + y^2)(\lambda^2 - y^2)} > 0. \tag{17}$$

Equations of the top right curve B_3 are

$$x = \frac{y^2 - 1}{2\lambda},$$

$$\frac{y^2 - 1}{4\lambda^2} + \frac{1}{2}\ln y = K.$$

By differentiating these equations we get

$$2\lambda y \frac{dy}{dx} + (1 - y^2)\frac{d\lambda}{dx} = 2\lambda^2,$$

$$\lambda(y^2 + \lambda^2)\frac{dy}{dx} + y(1 - y^2)\frac{d\lambda}{dx} = 0.$$

Solving for $\frac{dy}{dx}$ and $\frac{d\lambda}{dx}$ using the Cramer rule, we get the following

$$\frac{dy}{dx} = \frac{2y\lambda}{y^2 - \lambda^2} < 0, \qquad \frac{d\lambda}{dx} = \frac{2\lambda^2(y^2 + \lambda^2)}{(1 - y^2)(\lambda^2 - y^2)} < 0. \tag{18}$$

We see from Eqs. (16) and (17) that $\frac{dy}{dx}$ is continuous on $B_1 \cup B_2$ except at the top point of B_2, where B_2 meets B_3 and $y = \lambda$, so the slopes of both curves B_2 and B_3 approach infinity and have vertical tangents at their joint point. Also $\frac{dy}{dx} \to 0$ as $\lambda \to \pm\infty$ (or $x \to 0$). All these assertions imply that the curve S_K is smooth.

8.6. Convexity properties

To see that the ball $B((0, 1), K)$ is not convex, it is enough to take a geodesic $\lambda C_2 \cap \mathbb{R}^2_+$ with a number λ, which is a little more than e^{2K}.

Using Eqs. (16), (17), and (18), we can show after a little tedious calculations that $\frac{d^2y}{dx^2} > 0$ at interior points of B_1 and B_2 and $\frac{d^2y}{dx^2} < 0$ at interior points of B_3. This implies that the curve S_K is strongly convex in affine sense.

8.7. Uniform local G-homogeneity

In order to get this result, we look carefully at the geometry of the Stadium space.

Right-sided tangent vectors to every geodesic may have only directions with angles in intervals $(\frac{\pi}{4}, \frac{\pi}{2})$ or $(-\frac{\pi}{2}, -\frac{\pi}{4})$, with directions $\pm\frac{\pi}{4}$ only at its top point, where $y = \lambda$. This implies that a geodesic with origin inside $B((0, 1), K)$ can intersect $B_1 \cup B_2$ at most once. Thus geodesics with origin inside $B((0, 1), K)$ and parameters $\lambda, 1 \leqslant \lambda \leqslant \lambda_0$ can intersect the right side of S_K at most once.

Further we shall consider without any mention only geodesics which intersect the set $U((0, 1), K) \cap \{(x, y) \in \mathbb{R}^2 \mid y = 1\}$. It is clear that the width of S_K is equal to $2|x(\lambda_0)| = \lambda_0 - \frac{1}{\lambda_0}$. Now one can easily see that

$$2|x(\lambda)| = \lambda - \frac{1}{\lambda} \geqslant 2\left(\lambda_0 - \frac{1}{\lambda_0}\right) \quad \text{if } \lambda \geqslant 2\lambda_0$$

and a geodesic with parameter $\lambda \geqslant 2\lambda_0$ can intersect the right side of S_K at most once. So we need to consider only geodesics with parameters $\lambda, \lambda_0 < \lambda < 2\lambda_0$. It follows from Eq. (8) that right-side derivatives on any geodesic with such parameter at any point (x, y), where $y_0 = \lambda_0 \leqslant y \leqslant \lambda$, satisfy condition

$$\frac{dy}{dx} = \pm\frac{\lambda}{y} \geqslant -\frac{\lambda}{y} > -\frac{2\lambda_0}{\lambda_0} = -2.$$

This implies that geodesics with such parameters can intersect the right side of S_K at least twice only at points with $y > y_1$, where $\frac{dy}{dx}(y_1) = -2$ for derivative along S_K. We can deduce from Eqs. (18) that $y_1 = \frac{\sqrt{5}-1}{2}\lambda_1$, where

$$\frac{y_1^2 - 1}{4\lambda_1^2} + \frac{1}{2}\ln y_1 = K.$$

So this is possible only if $\lambda_1 < \lambda < 2\lambda_0$, where $\lambda_1 > \lambda_0$. The top point of every geodesic with such parameter λ, going through $(0, 1)$ and intersecting B_3, is $(|x(\lambda)|, \lambda)$ and $|x(\lambda)| > |x(\lambda_1)|$, while the most right point of S_K is $(|x(\lambda_0)|, \lambda_0)$. Thus the shift of this geodesic to the left of size less than

$$\nu = |x(\lambda_1)| - |x(\lambda_0)| = \frac{1}{2}\left(\lambda_1 - \lambda_0 + \frac{1}{\lambda_0} - \frac{1}{\lambda_1}\right)$$

will have the top point to the right of S_K.

Let $\eta = \max\{x \mid (x, 1) \in B((0, 1), K)\}$ and $\xi = \min(\nu, \eta)$. Using previous considerations, one can check that the set $P = D \cap U((0, 1), K)$, where D is the set bounded above by curves

$$\lambda_0 C_2 + \left(-\left(x(\lambda_0) + \xi\right), 0\right), \qquad \lambda_0 C_2 + \left(x(\lambda_0) + \xi, 0\right)$$

and below by curves

$$\lambda_0 C_2 + \left(x(\lambda_0) - \xi, 0\right), \qquad \lambda_0 C_2 + \left(\xi - x(\lambda_0), 0\right),$$

(cf. formula (9)) has the following properties:

(1) $(0, 1) \in \text{int}(P)$;
(2) For every point $Y \in \text{int}(P)$, every geodesic with parameter $\lambda \geqslant \lambda_0$ going through Y intersects the set $\{(x, 1) \mid x \in (-\xi, \xi)\}$; and
(3) For every point $Y \in \text{int}(P)$ every geodesic, going through Y, intersects the sphere S_K exactly at two (mutually antipodal) points.

Since numbers ξ and λ_0 continuously depend on K, and the Stadium space is (metrically) homogeneous, we get the following theorem.

Theorem 8.4. *The Stadium space X is uniformly locally G-homogeneous on X. As a corollary, it is locally G-homogeneous and uniformly locally G-homogeneous on an orbal subset. On the other hand, X has no convex ball of positive radius.*

9. Epilogue

The Busemann conjecture remains an important problem in the characterization of manifolds. Proposition 3.3 and Corollary 3.12 imply that it is equivalent to the statement that sufficiently small metric spheres in a finite-dimensional Busemann G-space are codimension one manifold factors. We conclude the paper by some questions.

Question 9.1. Is every Busemann G-space X necessarily locally G-homogeneous or uniformly locally G-homogeneous on an orbal subset?

Question 9.2. Is every sufficiently small sphere in n-dimensional Busemann G-space homotopy equivalent to the $(n-1)$-sphere?

Question 9.3. Are there finite-dimensional locally G-homogeneous Busemann G-spaces with nonmanifold arbitrary small metric spheres?

A positive answer to the last question would provide an example of a compact topologically homogeneous finite-dimensional nonmanifold ANR which is a homology sphere having the property that the complement of every one of its points is contractible.

Acknowledgements

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