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Journal of Algebra 259 (2003) 313–322
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

The center of the generic division ring and twisted multiplicative group actions

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Received 17 August 2000

Communicated by M. Van den Bergh

Abstract

Let F be a field and let p be a prime. The problem we study is whether the center, C_p, of the division ring of $p \times p$ generic matrices is stably rational over F. Given a finite group G and a $\mathbb{Z}G$-lattice, we let $F(M)$ be the quotient field of the group algebra of the abelian group M. Procesi and Formanek [Linear Multilinear Algebra 7 (1979) 203–212] have shown that for all n there is a $\mathbb{Z}S_n$-lattice, G_n, such that C_n is stably isomorphic to the fixed field under the action of S_n of $F(G_n)$. Let H be a p-Sylow subgroup of S_p. Let A be the root lattice, and let $L = F(\mathbb{Z}S_p/H)$. We show that there exists an action of S_p on $L(\mathbb{Z}S_P \otimes_{\mathbb{Z}H} A)$, twisted by an element $\alpha \in \mathrm{Ext}^1_{S_p}(\mathbb{Z}S_p \otimes_{\mathbb{Z}H} A, L^*)$, such that $L_\alpha(\mathbb{Z}S_p \otimes_{\mathbb{Z}H} A)^{S_p}$ is stably isomorphic to C_p. The extension α corresponds to an element of the relative Brauer group of L over L^H. Since $\mathbb{Z}S_p \otimes_{\mathbb{Z}H} A$ and $\mathbb{Z}S_p/H$ are quasi-permutations, $L(\mathbb{Z}S_p \otimes_{\mathbb{Z}H} A)^{S_p}$ is stably rational over F. However, it is not known whether $L_\alpha(\mathbb{Z}S_p \otimes_{\mathbb{Z}H} A)^{S_p}$ is stably rational over F. Thus the result represents a reduction on the problem since $\mathbb{Z}S_p \otimes_{\mathbb{Z}H} A$ is quasi-permutation; however, the twist introduces a new level of complexity.

Keywords: Field invariants; Lattices; Generic matrices

Introduction

The problem we study is whether the center, C_n, of the division ring of $n \times n$ generic matrices is stably rational over the base field F. This is a major open question with

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0021-8693/03/$ – see front matter
PII: S0021-8693(02)00576-8

connections to important problems in other fields such as geometric invariant theory and Brauer groups.

Given a finite group G, a $\mathbb{Z}G$-lattice M, and a field F, we let $F(M)$ denote the quotient field of the group algebra $F[M]$ of the abelian group M written multiplicatively. The group G acts on $F(M)$ via its action on M. It was shown in [F] that C_n is stably isomorphic to $F(G_n)^{S_n}$, the fixed field under the action of S_n of $F(G_n)$, where G_n is a specific $\mathbb{Z}S_n$-lattice which we define below. If M and M' are G-faithful $\mathbb{Z}G$-lattices, their corresponding fields $F(M)$ and $F(M')$ are stably isomorphic if and only if M and M' are in the same flasque class. Thus C_n is stably equivalent to $F(M)^{S_n}$ for any $\mathbb{Z}S_n$-lattice in the flasque class of G_n; flasque classes of $\mathbb{Z}G$-lattices are defined in Section 1.

Let p be a prime and let N be the normalizer in S_p of a p-Sylow subgroup. In [B1] we have shown that G_p and $\mathbb{Z}S_p \otimes_{\mathbb{Z}N} G_p$ are in the same flasque class, which implies that C_p is stably isomorphic to $F(\mathbb{Z}S_p \otimes_{\mathbb{Z}N} G_p)^{S_p}$. In [B2] we show that the flasque class of G_p depends mostly on the structure of $\widehat{G}_p$ as a $\widehat{\mathbb{Z}}N$-lattice, where $\widehat{\mathbb{Z}}$ denotes the p-adic completion of $\mathbb{Z}$, and $\widehat{G}_p = G_p \otimes \widehat{\mathbb{Z}}$. These results, together with the decomposition of $\widehat{G}_p$ into indecomposable $\widehat{\mathbb{Z}}N$-modules from [B2], are used to find a family of $\mathbb{Z}S_p$-lattices whose corresponding fixed fields are stably isomorphic to C_p, the center of the division ring of $p \times p$ generic matrices, Theorem 1.5.

Let G be a finite group, let M be a $\mathbb{Z}G$-lattice, and let L be a field on which G acts. Given an element $\alpha \in \operatorname{Ext}^1_G(M, L^*)$, we have an action of G on $L(M)$ twisted by α. The field $L(M)$ with such an action will be denoted by $L_\alpha(M)$.

Let H be a p-Sylow subgroup of S_p, and let A be the root lattice. We find a field extension L of F, on which S_p acts faithfully as F-automorphism, and an element α in $\operatorname{Ext}^1_{S_p}(\mathbb{Z}S_p \otimes_{\mathbb{Z}H} A, L^*)$, such that $L_\alpha(\mathbb{Z}S_p \otimes_{\mathbb{Z}H} A)^{S_p}$ is stably isomorphic to the center of the division ring of $p \times p$ generic matrices over F. Moreover, L^{S_p} is stably rational over F. Theorem 2.1 asserts that if $L_\alpha(\mathbb{Z}S_p \otimes_{\mathbb{Z}H} A)^{S_p}$ is stably rational over F, then so is C_p. Since $\mathbb{Z}S_p \otimes_{\mathbb{Z}H} A$ is quasi-permutation, $L(\mathbb{Z}S_p \otimes_{\mathbb{Z}H} A)^{S_p}$ is rational over L^{S_p} by [B1, Theorem 2.1]; however, there are no known analogous results for $L_\alpha(\mathbb{Z}S_p \otimes_{\mathbb{Z}H} A)^{S_p}$. Thus this theorem represents a reduction on the problem since $\mathbb{Z}S_p \otimes_{\mathbb{Z}H} A$ is quasi-permutation; however, the twist introduces a new level of complexity.

1. Let G be a finite group. An equivalence relation is defined in the category $\mathcal{L}_G$ of $\mathbb{Z}G$-lattices as follows. The $\mathbb{Z}G$-lattices M and M' are said to be equivalent if there exist permutation modules P and P' such that $M \oplus P \cong M' \oplus P'$. The set of equivalence classes forms an abelian monoid under the direct sum. Lattices equivalent to 0 are said to be stably permutation. The equivalence class of a lattice M will be denoted by $[M]$.

For any integer n, $H^n(G, M)$ will denote the nth Tate cohomology group of G with coefficients in M. A $\mathbb{Z}G$-lattice M is flasque if $H^{-1}(H, M) = 0$ for all subgroups H of G. A flasque resolution of a $\mathbb{Z}G$-lattice M is an exact sequence

$$0 \to M \to P \to E \to 0$$

with permutation P and flasque E. It follows directly from [EM, Lemma 1.1] that any $\mathbb{Z}G$-lattice M has a flasque resolution. The flasque class of M is $[E]$, and will be denoted by $\phi(M)$. By [CTS, Lemma 5, Section 1], $\phi(M)$ is independent of the flasque resolution of M. Lattices whose flasque class is 0 are said to be quasi-permutation. For more on flasque classes, see [CTS, Section 1].

Definition. Let L and K be extension fields of a field F, and let G be a finite subgroup of their groups of F-automorphisms. Then L and K are said to be stably isomorphic if there exist G-trivial indeterminates $x_1, \ldots, x_r,\ y_1, \ldots, y_s$ such that $L(x_1, \ldots, x_r) \cong K(y_1, \ldots, y_s)$ as F-algebras, and the isomorphism respects their G-actions. If K is contained in L, we also say that L is stably rational over K.

We now define the $\mathbb{Z}S_n$-lattice G_n, mentioned in the introduction. Let U be the $\mathbb{Z}S_n$-lattice with $\mathbb{Z}$-basis $\{u_i\colon 1 \leqslant i \leqslant n\}$ and with S_n-action given by $gu_i = u_{g(i)}$, for all $g \in S_n$. Let A be the root lattice. A is defined by the exact sequence

$$\begin{aligned} 0 \to A \to U \ &\to\ \mathbb{Z} \to 0, \\ u_i \ &\mapsto\ 1 \end{aligned}$$

Then $G_n = A \otimes_{\mathbb{Z}} A$, and $F(G_n)^{S_n}$ is stably isomorphic to C_n [F, Theorem 3].

Henceforth we will adopt the following notation, unless otherwise specified:

- G denotes S_p, where p is a prime.
- H denotes p-Sylow subgroup of G. Thus H is cyclic of order p.
- $a \in \mathbb{Z}$ will denote a primitive $(p-1)$st root of 1 mod p.
- N is the normalizer of H in G. Thus $N = H \rtimes C$, is the semidirect product of H by a cyclic group C, of order $p-1$. H will be generated by h, C by c, and we have $chc^{-1} = h^a$.
- $\widehat{\mathbb{Z}}$ is the p-adic completion of $\mathbb{Z}$.
- For any finite group G and any $\mathbb{Z}G$-lattice M, $\widehat{M}$ will denote the p-adic completion of M, and for any prime q, M_q will denote the localization of M at q.
- The dual of a $\mathbb{Z}G$-lattice M, $\mathrm{Hom}(M, \mathbb{Z})$, will be denoted by M^*.

Since $\mathbb{Z}N/H \cong \mathbb{Z}C \cong \mathbb{Z}[x]/(x^{p-1} - 1)$ as $\mathbb{Z}N$-lattices, the decomposition of $\widehat{\mathbb{Z}}N/H$ into indecomposables is given by

$$\widehat{\mathbb{Z}}N/H \cong \bigoplus_{k=0}^{p-2} \mathbb{Z}[x]\big/\big(x - \vartheta^k\big) \cong \bigoplus_{k=0}^{p-2} \mathbb{Z}_k$$

where ϑ is a primitive $(p-1)$st root of 1 in $\widehat{\mathbb{Z}}$ which is congruent to a mod p, and $\mathbb{Z}_k$ is the $\widehat{\mathbb{Z}}N$-module of $\widehat{\mathbb{Z}}$-rank 1 on which H acts trivially, and such that $c1 = \vartheta^k$.

The restriction from G to N of U is isomorphic to $\mathbb{Z}H$, and the isomorphism is given by $u_i \mapsto h^i$, with $c.h = h^a$. $\widehat{U}$ is a $\widehat{\mathbb{Z}}N$-indecomposable module by [CR, Theorem 19.22].

For $k = 0, \ldots, p-2$, we set $U_k = \widehat{U} \otimes \mathbb{Z}_k$. Since $\widehat{\mathbb{Z}}N \cong \widehat{\mathbb{Z}}N \otimes_{\widehat{\mathbb{Z}}H} \widehat{\mathbb{Z}}H \cong \widehat{\mathbb{Z}}N/H \otimes \widehat{U}$, we have

$$\widehat{\mathbb{Z}}N = \bigoplus_{k=0}^{p-2} U_k.$$

For $k = 0, \ldots, p-2$, A_k will denote the $\widehat{\mathbb{Z}}N$-lattice $\widehat{\mathbb{Z}}H(h-1)^k$. Under this notation, $A_1 = \widehat{A}$ and $A_{p-1} = \widehat{A}^*$ by [B1, Theorem 3.2]. We also set $X_k = \mathbb{Z}_k/p\mathbb{Z}_k$.

Lemma 1.1. *There exists a $\mathbb{Z}N$-exact sequence*

$$0 \to U \to \mathbb{Z} \oplus A^* \to L \to 0,$$

where $L = \mathbb{Z}/p^r\mathbb{Z}$ for all integers $r \geqslant 1$.

Proof. Dualizing the defining sequence of the $\mathbb{Z}G$-lattice A, we obtain

$$0 \to \mathbb{Z} \to U \to A^* \to 0$$

since U is a permutation, and hence isomorphic to its dual. The map $U \to A^*$ is the composition of restriction $U^* \to A^*$ with the isomorphism from U to U^*. We denote it by Res. The map $U \to \mathbb{Z} \oplus A^*$ is given by $u_i \mapsto p^{r-1} + \operatorname{Res} u_i$. The result follows directly. □

Theorem 1.2. *There exists a $\widehat{\mathbb{Z}}N$-exact sequence*

$$0 \to \widehat{\mathbb{Z}}N \to \widehat{G}_p \oplus \widehat{A} \to \mathbb{Z}_1/p^r\mathbb{Z}_1 \to 0.$$

Proof. In [B2, Theorem 2.5] we show that the decomposition of $\widehat{G}_p$ into indecomposable $\widehat{\mathbb{Z}}N$-modules is

$$\widehat{G}_p \cong \bigoplus_{\substack{k=0\\k\neq 1}}^{p-2} U_k \oplus \mathbb{Z}_1.$$

By [B1, Theorem 3.2], $\widehat{A} \cong A_1 \cong \widehat{A}^* \otimes \mathbb{Z}_1$. Thus, tensoring the sequence of Lemma 1.1 by $\mathbb{Z}_1$, we obtain:

$$0 \to U_1 \to \mathbb{Z}_1 \oplus A_1 \to \mathbb{Z}_1/p^r\mathbb{Z}_1 \to 0.$$

Adding $\bigoplus_{k=0,\ k\neq 1}^{p-2} U_k$ to the first two terms of the sequence, we obtain:

$$0 \to \bigoplus_{\substack{k=0\\k\neq 1}}^{p-2} U_k \oplus U_1 \to \bigoplus_{\substack{k=0\\k\neq 1}}^{p-2} U_k \oplus \mathbb{Z}_1 \oplus A_1 \to \mathbb{Z}_1/p^r\mathbb{Z}_1 \to 0.$$

But $\widehat{\mathbb{Z}}N \cong \bigoplus_{k=0}^{p-2} U_k$, thus the first term of the sequence is isomorphic to $\widehat{\mathbb{Z}}N$, and the second term is isomorphic to $\widehat{G}_p \oplus A_1$. $\square$

Lemma 1.3. *Let $a \in \mathbb{Z}$ be a primitive $(p-1)$st root of* 1 mod p. *The map*

$$i : \mathbb{Z}C \to \mathbb{Z}C, \quad 1 \mapsto c - a$$

is an injection of $\mathbb{Z}N$-modules whose cokernel is $L_1 \oplus L_2$, where $L_1 = Z_1/p^r Z_1$ for some $r \geqslant 1$, and L_2 is a finite cohomologically trivial $\mathbb{Z}N$-module of order prime to p.

Proof. The map i is injective since $c - a$ is not a zero divisor, so its cokernel is finite. A computation shows that coker(i) is cyclic of order $a^{p-1} - 1$. Since a is a primitive $(p-1)$st root of 1 mod p, $a^{p-1} - 1$ is divisible by p, and the p-primary component of coker(i) is L_1. For primes $q \neq p$ we have

$$0 \to \mathbb{Z}_q C \xrightarrow{i} \mathbb{Z}_q C \to (L_2)_q \to 0.$$

Let C_q be any subgroup of N of q-power order. We may assume that C_q is contained in C. Thus $H^m(C_q, (L_2)_q) = 0$ for all integers m, which proves the claim. $\square$

Lemma 1.4. *Let G be a finite group, and $\mathbb{R}$ a Dedekind domain of characteristic* 0. *Suppose there exists $\mathbb{R}G$-exact sequences*

$$0 \to V \to E \to L \to 0, \qquad 0 \to V' \to E' \to L \to 0,$$

where E and E' are $\mathbb{R}G$-lattices, and V and V' are $\mathbb{R}G$-projectives. Then

$$E \oplus V' \cong E' \oplus V.$$

Furthermore, if $G = S_n$, then E and E' are in the same flasque class.

Proof. Consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \to & V & \to & E & \to & L & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & V & \to & M & \to & E' & \to & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & V' & \to & V' & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}.$$

Since projectives are injectives in the category of $\mathbb{R}G$-lattices, and since E and E' are $\mathbb{R}G$-lattices, the middle sequences split and we have

$$V \oplus E' \cong V' \oplus E.$$

Since $G = S_n$ and V and V' are $\mathbb{R}G$-projective, they are stably permutative by [EM, Theorem 3.3], therefore E and E' are in the same flasque class. □

Theorem 1.5. *Let p be a prime, let r be a positive integer, and let L be a finite G-module with the property that its p-primary component is isomorphic to $\mathbb{Z}G \otimes_{\mathbb{Z}N} (\mathbb{Z}_1/p^r\mathbb{Z}_1)$. Let*

$$0 \to \mathbb{Z}G \to E \to L \to 0$$

be any extension of L by $\mathbb{Z}G$ such that E is a $\mathbb{Z}G$-lattice. Then the center of the division ring of $p \times p$ generic matrices over an F is stably isomorphic to $F(E)^G$.

Proof. As above, let $G = S_p$, and let H be a p-Sylow subgroup of G. Let i_2 be the injection of $\mathbb{Z}G/H$ into $\mathbb{Z}G \cong \mathbb{Z}G \otimes_{\mathbb{Z}H} \mathbb{Z}H$, defined by $i_2(\bar{g}_i) = \sum_{j=1}^{p} g_i \otimes h^j$ where $\{g_i\}$ is a transversal for H in G. Let i_1 be any injective endomorphism of $\mathbb{Z}G/H$ whose cokernel satisfies the hypothesis of the theorem. Form the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathbb{Z}G/H & \xrightarrow{i_2} & \mathbb{Z}G & \longrightarrow & \mathbb{Z}G/H \otimes A & \longrightarrow & 0 \\
 & & {\scriptstyle i_1}\downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathbb{Z}G/H & \longrightarrow & E & \longrightarrow & \mathbb{Z}G/H \otimes A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \operatorname{coker}(i_1) & \longrightarrow & \operatorname{coker}(i) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \qquad (*)$$

Set $\operatorname{coker}(i_1) = S \oplus S'$, where $S = \mathbb{Z}G \otimes_{\mathbb{Z}N} (\mathbb{Z}_1/p^r\mathbb{Z}_1)$ and S' is order prime to p. The vertical middle sequence becomes

$$0 \to \mathbb{Z}G \to E \to S \oplus S' \to 0. \qquad (1)$$

Step 1. We show that $\widehat{\mathbb{Z}}G \otimes_{\widehat{\mathbb{Z}}N} \widehat{G}_p \oplus \widehat{\mathbb{Z}}G \otimes_{\widehat{\mathbb{Z}}N} \widehat{A} \cong \widehat{E}$. Tensoring the sequence

$$0 \to \widehat{\mathbb{Z}}N \to \widehat{G}_p \oplus A_1 \to \mathbb{Z}_1 \big/ p^r\mathbb{Z}_1 \to 0$$

of Theorem 1.2, by $\widehat{\mathbb{Z}}G$ over $\widehat{\mathbb{Z}}N$, we get

$$0 \to \widehat{\mathbb{Z}}G \to \widehat{\mathbb{Z}}G \otimes_{\widehat{\mathbb{Z}}N} \widehat{G}_p \oplus \widehat{\mathbb{Z}}G \otimes_{\widehat{\mathbb{Z}}N} \widehat{A} \to S \to 0. \qquad (2)$$

Tensoring sequence (1) by $\widehat{\mathbb{Z}}$, and applying Lemma 1.4 to the resulting sequence and to sequence (2) we get

$$\widehat{\mathbb{Z}}G \otimes_{\widehat{\mathbb{Z}}N} \widehat{G}_p \oplus \widehat{\mathbb{Z}}G \otimes_{\widehat{\mathbb{Z}}N} \widehat{A} \oplus \widehat{\mathbb{Z}}G \cong \widehat{E} \oplus \widehat{\mathbb{Z}}G.$$

By the Krull–Schmit–Azumaya theorem, we have

$$\widehat{\mathbb{Z}}G \otimes_{\widehat{\mathbb{Z}}N} \widehat{G}_p \oplus \widehat{\mathbb{Z}}G \otimes_{\widehat{\mathbb{Z}}N} \widehat{A} \cong \widehat{E}.$$

Step 2. We show that G_p and E are in the same flasque class. The defining sequence of the $\mathbb{Z}G$-lattice A is

$$0 \to A \to U \to \mathbb{Z} \to 0,$$
$$u_1 \mapsto 1.$$

For all primes $q \neq p$, this sequence splits, with splitting map $1 \to (1/p)\sum u_i$. Thus

$$U_q \cong A_q \oplus \mathbb{Z}_q \quad \text{and} \quad U_q \otimes A_q \cong A_q \otimes A_q \oplus A_q.$$

Since $G_p = A \otimes A$, we have

$$U_q \otimes A_q \cong (G_p)_q \oplus A_q.$$

As $\mathbb{Z}N$-modules, $U \cong \mathbb{Z}H \cong \mathbb{Z}N/C$, and $A \cong \mathbb{Z}H(h-1)$. We also have an isomorphism of $\mathbb{Z}C$-modules $A \cong \mathbb{Z}C$ given by $h^i - 1 \mapsto c^i$ for $i = 1, \ldots, p-1$. Therefore

$$U_q \otimes A_q \cong \mathbb{Z}_q N/C \otimes A_q \cong \mathbb{Z}_q N \otimes_{\mathbb{Z}_q C} \mathbb{Z}_q C \cong \mathbb{Z}_q N,$$

which implies

$$\mathbb{Z}_q N \cong (G_p)_q \oplus A_q \quad \text{and} \quad \mathbb{Z}_q G \cong \mathbb{Z}_q G \otimes_{\mathbb{Z}N} (G_p \oplus A).$$

On the other hand, since H is of order p, A_q is $\mathbb{Z}_q H$-projective for all primes $q \neq p$. Therefore the horizontal sequences in $(*)$, namely,

$$0 \to \mathbb{Z}G/H \to \mathbb{Z}G \to \mathbb{Z}G/H \otimes A \to 0, \qquad 0 \to \mathbb{Z}G/H \to E \to \mathbb{Z}GH/H \otimes A \to 0,$$

split when localized at a prime $q \neq p$, and so $E_q \cong \mathbb{Z}_q G$. Thus we have $E_q \cong \mathbb{Z}_q G \otimes_{\mathbb{Z}N} (G_p \oplus A)$ for all primes $q \neq p$. From Step 1, we have $\widehat{\mathbb{Z}}G \otimes_{\widehat{\mathbb{Z}}N} \widehat{G}_p \oplus \widehat{\mathbb{Z}}G \otimes_{\widehat{\mathbb{Z}}N} \widehat{A} \cong \widehat{E}$ which implies, by [CR, Proposition 30.17]:

$$E_p \cong \mathbb{Z}_p G \otimes_{\mathbb{Z}N} (G_p \oplus A).$$

Thus E and $\mathbb{Z}G \otimes_{\mathbb{Z}N} (G_p \oplus A)$ are of the same genus. By [BL, Proposition 2.2] they are in the same flasque class, since $G = S_p$. Since A is quasi-permutation, this implies that E

and $\mathbb{Z}G \otimes_{\mathbb{Z}N} G_p$ are in the same flasque class. By [B2, Corollary 1.2] G_p and $\mathbb{Z}G \otimes_{\mathbb{Z}N} G_p$ are in the same flasque class, thus so are E and G_p.

By [B2, Theorem 1.1] this implies that $F(G_p)^G$ and $F(E)^G$ are stably isomorphic. The result follows from [F, Theorem 3]. □

2. Given a finite group G, a $\mathbb{Z}G$-lattice M, and a field L on which G acts, we may form the field $L(M)$, and this field has a G-action via the action of G on M. However, there exist other G-actions on $L(M)$. These actions were found by Saltman [S], and called α-twisted actions. They are defined as follows.

Let $\alpha \in \mathrm{Ext}^1_G(M, L^*)$, where L^* is the multiplicative group of L. Let the equivalence class of

$$0 \to L^* \to M' \to M \to 0$$

in $\mathrm{Ext}^1_G(M, L^*)$ be α. Writing M and M' as multiplicative abelian groups, we have

$$M' = \{x.m\colon\ x \in L^*,\ m \in M\},$$

and the G-action on M' is given by $g^*x.m = g(x)\,\mathrm{d}_g(gm).gm$, where $\mathrm{d}: G \to \mathrm{Hom}_{\mathbb{Z}}(M, L^*)$ is the derivation corresponding to α. In particular, for $x = 1$, we have

$$g^*m = \mathrm{d}_g(gm).gm.$$

Thus we obtain an α-twisted action on $L(M)$. Denote by $L_\alpha(M)$ the field $L(M)$ with the corresponding G-action.

The following remark is needed in the proof of Theorem 2.1.

Remark. Recall that N is the normalizer of a p-Sylow subgroup H of G. Thus $N = H \rtimes C$ is the semidirect product of H by a cyclic group C, of order $p-1$. Let h and c generate H and C, respectively. Then $chc^{-1} = h^a$, where a is a primitive $(p-1)$st root of 1 mod p. Let $n_h = \sum_i h^i$ be the norm of H. The kernel of the $\mathbb{Z}H$-map $\mathbb{Z}H \to \mathbb{Z}H(h-1)$, multiplication by $h-1$, is $n_h\mathbb{Z}H$. Thus $A \cong \mathbb{Z}H(h-1) \cong \mathbb{Z}H/n_h\mathbb{Z}H$ as $\mathbb{Z}H$-modules.

Theorem 2.1. *Let $L = F(\mathbb{Z}G/H)$. There exists an α-twisted action of G on $L(\mathbb{Z}G/H \otimes A)$ such that $L_\alpha(\mathbb{Z}G/H \otimes A)^G$ is stably isomorphic to C_p. That is, if $L_\alpha(\mathbb{Z}G/H \otimes A)^G$ is stably rational over F, then C_p is stably rational over F. Furthermore, the extension α corresponds to an element of the relative Brauer group* $\mathrm{Br}(L/L^H)$.

Proof. Let i_1 be the map

$$\mathbb{Z}G/H \to \mathbb{Z}G/H, \quad \bar{1} \mapsto \bar{c} - \bar{a}.$$

Since $\mathbb{Z}G/H \cong \mathbb{Z}G \otimes_{\mathbb{Z}N} \mathbb{Z}C$, the map i_1 is the map i of Lemma 1.3 induced up to G, and thus it is injective. Consider the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathbb{Z}G/H & \xrightarrow{i_2} & \mathbb{Z}G & \longrightarrow & \mathbb{Z}G/H \otimes A & \longrightarrow & 0 \\
 & & {\scriptstyle i_1}\downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathbb{Z}G/H & \longrightarrow & M & \longrightarrow & \mathbb{Z}G/H \otimes A & \longrightarrow & 0\ . \\
 & & \downarrow & & \downarrow & & & & \\
 & & \operatorname{coker}(i_1) & \longrightarrow & \operatorname{coker}(i) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

It follows from Lemma 1.3 that $\operatorname{coker}(i_1)$ satisfies the hypothesis of Theorem 1.5, hence $F(M)^G$ is stably isomorphic to C_p.

Let $\{g_i\}$ be a transversal for H in G. Set $b_i = i_1(\bar{g}_i)$ and, as in Theorem 1.5, let $i_2(\bar{g}_i) = \sum_{j=1}^{p} g_i \otimes h^j$. Thus

$$M \cong \mathbb{Z}G/H \oplus \mathbb{Z}G \Big/ \left\{ \left(b_i - \sum_{j=1}^{p} g_i \otimes h^j \right) \colon i = 1, \ldots, (p-1)! \right\}.$$

From this isomorphism we obtain a G-surjection of rings

$$F[\mathbb{Z}G/H \oplus \mathbb{Z}G] \to F[M].$$

We let y_i and x_{ij} denote the elements of the $\mathbb{Z}$-basis of $\mathbb{Z}G/H \oplus \mathbb{Z}G$, corresponding to and $g_i \otimes h^j$, respectively, when $\mathbb{Z}G/H \oplus \mathbb{Z}G$ is viewed as a multiplicative abelian group. Thus the y_i and x_{ij} are independent commuting indeterminates over F. Let m_i be the monomial in the y_i corresponding to b_i. Then $F[\mathbb{Z}G/H \oplus \mathbb{Z}G] = F[y_i^{\pm 1}, j_{ij}^{\pm 1}]$, and the kernel of the above surjection by [P, Lemma 1.8] is:

$$I = \left\langle m_i \prod_{j=1}^{p} x_{ij}^{-1} - 1 \colon i = 1, \ldots, (p-1)! \right\rangle.$$

Thus $F[M] \cong F[y_i^{\pm 1}, x_{ij}^{\pm 1}]/I$. Let

$$\bar{y}_i = y_i \mod I, \qquad \bar{x}_{ij} = x_{ij} \mod I \quad \text{for } j = 1, \ldots, p-1.$$

Then $\bar{x}_{ip} = m_i \prod_{j=1}^{p} \bar{x}_{ij}^{-1}$ and $g\bar{y}_i = \overline{gy}_i$. The set $\{\bar{y}_i, \bar{x}_{ij} \colon i = 1, \ldots, (p-1)!, j = 1, \ldots, p-1\}$ is algebraically independent over F, since its cardinality, $p!$, is equal to the Krull dimension of $F[M]$. Thus $F(M) = F(\bar{y}_i, \bar{x}_{ij}\colon\ i = 1, \dots, (p-1)!,\ \ j = 1, \dots, p-1)$. We have a G-isomorphism

$$F[y_i] \to F[\bar{y}_i] \subseteq F[M], \quad y_i \mapsto \bar{y}_i.$$

Set $L = F(\bar{y}_i)$, then $L \cong F(\mathbb{Z}G/H)$ and $F(M) \cong L(\bar{x}_{ij})$. $\square$

Let M^* be the subgroup of $F(M)^*$ generated by L^* and M. By the remark preceding the theorem, $A \cong \mathbb{Z}H/n_h\mathbb{Z}H$ as a $\mathbb{Z}H$-module, hence $M^*/L^* \cong \mathbb{Z}G/H \otimes A$. We have a G-exact sequence

$$\alpha\colon \quad 0 \to L^* \to M^* \to \mathbb{Z}G/H \otimes A \to 0.$$

Clearly, $F(M) \cong F(M^*) = L_\alpha(\mathbb{Z}G/H \otimes A)$, where by $F(M^*)$ we mean the smallest subfield of $F(M)$ generated by F and M^*. Hence $F(M)^G \cong L_\alpha(\mathbb{Z}G/H \otimes A)^G$ and, by Theorem 1.5, $F(M)^G$ is stably isomorphic to C_p. This proves the first statement.

For the second statement, $\alpha \in \mathrm{Ext}^1_G(\mathbb{Z}G/H \otimes A, L^*) \cong \mathrm{Ext}^1_H(A, L^*)$ by Shapiro's Lemma. Taking the cohomology of the $\mathbb{Z}H$-sequence

$$0 \to A \to \mathbb{Z}H \to \mathbb{Z} \to 0,$$

we have $\mathrm{Ext}^1_H(A, L^*) \cong \mathrm{Ext}^2_H(\mathbb{Z}, L^*) \cong H^2(H, L^*) = \mathrm{Br}(L/L^H)$, the relative Brauer group of L over L^H.

References

[BL] C. Bessenrodt, L. Lebruyn, Stable rationality of certain PGL_n-quotients, Invent. Math. 104 (1991) 179–199.

[B1] E. Beneish, Induction theorems on the center of the ring of generic matrices, Trans. Amer. Math. Soc. 350 (9) (1998) 3571–3585.

[B2] E. Beneish, Monomial actions of the symmetric group, J. Algebra, to appear.

[CR] Curtis, Reiner, Methods of Representation Theory, Vol. 1, Wiley, New York, 1981.

[CTS] S.-L. Colliot-Thelene, J.-P. Sansuc, La R-equivalence sur les tores, Ann. Sci. École Norm. Sup. (4) 10 (1977) 175–230.

[EM] S. Endo, T. Miyata, On the projective class group of finite groups, Osaka J. Math. 13 (1976) 109–122.

[F] E. Formanek, The center of the ring of 3 × 3 generic matrices, Linear Multilinear Algebra 7 (1979) 203–212.

[P] D. Passman, The Algebraic Structure of Rings, Wiley, New York, 1977.

[S] D. Saltman, Twisted multiplicative invariants, Noether's problem and Galois extensions, J. Algebra 131 (2) (1990) 535–558.