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A remark on the geometry of uniformly rotating stars ☆

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ABSTRACT

In this paper we classify the free boundary associated to equilibrium configurations of compressible, self-gravitating fluid masses, rotating with constant angular velocity. The equilibrium configurations are all critical points of an associated functional and not necessarily minimizers. Our methods also apply to alternative models in the literature where the angular momentum per unit mass is prescribed. The typical physical model our results apply to is that of uniformly rotating white dwarf stars.

1. Introduction

In this paper we study the free boundary associated to rotating star models of white dwarf stars with prescribed constant angular velocity. Thus we are considering figures of equilibrium for compressible, self-gravitating fluid masses.

There has been a tremendous amount of work on incompressible, self-gravitating fluid masses rotating with prescribed constant angular velocity since the primary investigations by Newton. Various mathematicians like MacLaurin, Jacobi, Dirichlet, Riemann, Poincaré, H. Cartan and Chandrasekhar made significant contributions to the field, studying bifurcation sequences and analyzing the stability of various equilibrium shapes. A historical account and details of these investigations may be found in Chandrasekhar's treatise [10] and Tassoul's book [18].

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In the compressible case for the model with prescribed constant angular velocity $\omega > 0$ (cf. [15]), we consider the functional

$$J(\rho) = \int_{\mathbb{R}^3} A(\rho)\,d\xi - \frac{1}{2}\int_{\mathbb{R}^3} \omega^2 r^2 \rho\,d\xi - \frac{1}{2}\int_{\mathbb{R}^3} \rho B\rho\,d\xi, \tag{1.1}$$

where $\xi = (\xi_1, \xi_2, \xi_3)$, $r = \sqrt{\xi_1^2 + \xi_2^2}$, $\rho \geqslant 0$, $\rho \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and

$$(B\rho)(\xi) = \int_{\mathbb{R}^3} \frac{\rho(\eta)}{|\xi - \eta|}\,d\eta.$$

Moreover we impose the constraint

$$\int_{\mathbb{R}^3} \rho = 1. \tag{1.2}$$

As $\rho(\xi)$ represents the density of the stellar material, (1.2) means that the mass of the star is prescribed. Later in our paper we will assume in addition that the density $\rho(\xi)$ is axisymmetric, i.e.

$$\rho(\xi) = \rho(r, z) \quad \text{where } z(\xi) = \xi_3.$$

This assumption means that the star is rotating about the ξ_3-axis. The function $A(\rho)$ is the pressure and thus represents the equation of state of the stellar material. We assume that $A(\rho) \in C^1([0, +\infty))$ with A strictly convex so that $A'(\rho)$ is invertible. Further conditions on $A(\rho)$ will be stipulated below. The first term in (1.1) represents then the *internal energy* of the star, the second term the *rotational kinetic energy* and the last term the *gravitational potential energy*.

In [15] the existence of minimizers of J under the constraint (1.2) has been obtained. Ref. [14] contains further results for this model of prescribed angular velocity. In [11] support estimates for critical points of (1.1) under the constraint (1.2) have been shown. In particular, [11, Theorem 1] states that for $\omega \geqslant \omega_0 > 0$, the support of ρ is contained in a ball $B_\sigma(0, 0, \xi_3)$ for some ξ_3, where $\sigma = \sigma(\omega_0)$. It follows that

$$0 \leqslant B\rho \leqslant C \quad \text{in } \mathbb{R}^3.$$

Furthermore, [11, Theorem 2] shows that the number of connected components of the set $\{\rho > 0\}$ is finite for any critical point ρ.

Critical points of J with the constraint (1.2) are according to [11, (0.6)] characterized by the problem

$$\rho \text{ is continuous and nonnegative} \quad \text{and} \quad A'(\rho) - \frac{1}{2}\omega^2 r^2 - B\rho = \lambda(\omega) \quad \text{in } \{\rho > 0\}, \tag{1.3}$$

where $\lambda(\omega)$ is a Lagrange multiplier arising from the constraint (1.2). The focus in this paper is to study the free boundary $\partial\{\rho > 0\}$ arising from (1.3).

There is another model of rotating stars which has been studied in the literature, where the angular momentum per unit mass is prescribed. Existence of minimizers for this alternative model has been obtained in [5], and the study of critical points has been carried out in [16]. Caffarelli and Friedman investigated in [7] the free boundary of minimizers for this alternative model. As Caffarelli and Friedman deal with minimizers, they are able to apply rearrangement methods to their functional to obtain solutions that are increasing in one direction which simplifies the analysis as well as the result. Unfortunately this technique does not work for critical points in either model and creates a difficulty for our analysis. Let us remark that the proofs presented in this paper for critical points of J with the constraint (1.2) work equally well for the study of the free boundary of critical points in the model in [7].

The principal difficulty we encounter in our classification of singularities of the free boundary is that the nonlinearity is *not* an increasing function of the solution, so that various methods stemming from the well-known obstacle problem do not apply. Neither does the monotonicity formula derived in [1]. Let us also mention that our problem cannot be transformed into the type of problems studied in [8], so we cannot use those results either. Another difficulty is that our equation is inhomogeneous. In particular, the leading order term on the right-hand side is not of the form $f(u)$. This—together with a higher order degeneracy—distinguishes the present problem also from the recently researched "unstable obstacle problem" (see [17,2–4]).

Last, let us point out that—due to the fact that the free boundary $\partial\{\rho > 0\}$ does not necessarily coincide completely with the free boundary of the PDE problem obtained by transformation—we obtain in our classification of singularities several cases later called "pseudo-cases." We suggest that in the case of minimizers, rearrangement techniques similar to those used in [7] may be used to show that solutions are decreasing in a certain direction, thus ruling out the pseudo-cases.

Setting $u = \frac{1}{2}\omega^2 r^2 + B\rho + \lambda(\omega)$, we obtain in the set $\overline{\{\rho > 0\}}$ that $u = A'(\rho) = \Phi^{-1}(\rho)$, where $\Phi : [0, +\infty) \to \mathbb{R}$ is an increasing function satisfying according to the asymptotics

$$A(\rho) = c_1\rho^{\frac{5}{3}} + o\big(\rho^{\frac{5}{3}}\big) \quad \text{as } \rho \to 0, \qquad A(\rho) = c_2\rho^{\frac{4}{3}} + o\big(\rho^{\frac{4}{3}}\big) \quad \text{as } \rho \to +\infty$$

(where c_1, c_2 are positive constants) from Chandrasekhar's book [9, Chapter 10] and [11, (0.2)] the asymptotic relations

$$A'(\rho) = \frac{5}{3}c_1\rho^{\frac{2}{3}} + o\big(\rho^{\frac{2}{3}}\big) \quad \text{as } \rho \to 0, \qquad A'(\rho) = \frac{4}{3}c_2\rho^{\frac{1}{3}} + o\big(\rho^{\frac{1}{3}}\big) \quad \text{as } \rho \to +\infty, \tag{1.4}$$

$$\lim_{z\to 0+} z^{-3/2}\Phi(z) = \tilde{c}_1 \in (0, +\infty), \qquad \lim_{z\to +\infty} z^{-3}\Phi(z) = \tilde{c}_2 \in (0, +\infty). \tag{1.5}$$

It follows (cf. [11, (3.3)]) that

$$\Delta u = 3\omega^2 - 4\pi\rho = 3\omega^2 - 4\pi\Phi(u) \quad \text{in } \{\rho > 0\}$$

and

$$\Delta u = 3\omega^2 \quad \text{in } \{\rho = 0\}.$$

Note that as $u = A'(\rho) = \Phi^{-1}(\rho)$ is only valid in the set $\overline{\{\rho > 0\}}$, we obtain $\{\rho > 0\} \subset \{u > 0\}$ but not necessarily the opposite inclusion. It is however true that $\partial\{\rho > 0\} \subset \partial\{u > 0\}$ and that if $\rho(x^0) > 0$ then the connected component of $\{\rho > 0\}$ containing x^0 coincides with the connected component of $\{u > 0\}$ containing x^0.

Normalizing the equation as well as Φ we obtain the free boundary problem

$$\Delta u = 1 - \Phi(u)\chi_\Omega \quad \text{with an open set } \Omega \text{ satisfying } u > 0 \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega, \tag{1.6}$$

where $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^3)$, (1.6) is to be satisfied in the sense of distributions, and $\Phi \in C^1([0, +\infty))$ such that $\int_\Omega \Phi(u) \in (0, +\infty)$,

$$\lim_{z\to 0+} z^{-3/2}\Phi(z) \in (0, +\infty) \quad \text{and} \quad \lim_{z\to +\infty} z^{-3}\Phi(z) \in (0, +\infty).$$

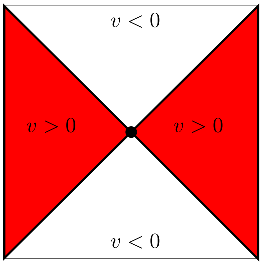

Fig. 1. Double wedge (with the set $\tilde{\Omega}$ painted as filled shape).

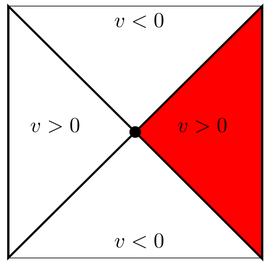

Fig. 2. Pseudo-wedge (with the set $\tilde{\Omega}$ painted as filled shape).

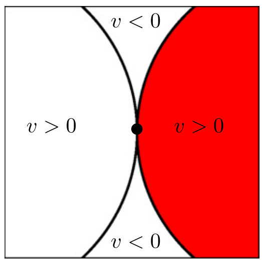

Fig. 3. Pseudo-cusp (with the set $\tilde{\Omega}$ painted as filled shape).

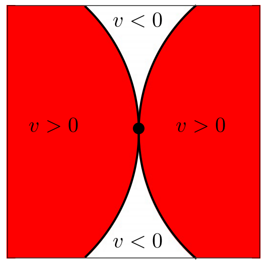

Fig. 4. Double cusp (with the set $\tilde{\Omega}$ painted as filled shape).

Theorem A. *For each axisymmetric three-dimensional solution* (u, Ω) *of problem* (1.6), *where* $\min(u, 0) \in L^\infty(\mathbb{R}^3)$, $r = \sqrt{x_1^2 + x_2^2}$, $u(x_1, x_2, x_3) = v(r, x_3)$ *and* $(x_1, x_2, x_3) \in \Omega \Leftrightarrow (r, x_3) \in \tilde{\Omega}$, *the following holds: Apart from the singular set* $\{v = 0\} \cap \{\nabla v = 0\}$ *the level set* $\{v = 0\}$ *and the boundary* $\partial\tilde{\Omega}$ *are locally* $C^{2,\alpha}$*-curves. The set* $\partial\tilde{\Omega} \cap \{\nabla v = 0\}$ *contains in each bounded subset of* $\mathbb{R}^2$ *at most finitely many singular points* (r^0, x_3^0) *with the following possible asymptotics:*

(a) *Suppose that* $r^0 = 0$. *Then one of the following three alternatives holds:*
  1. *Either* $v((r^0, x_3^0) + t(r, x_3))/t^2 \to \frac{1}{4}r^2$ *as* $t \to 0$ *or* $v((r^0, x_3^0) + t(r, x_3))/t^2 \to \frac{1}{2}x_3^2$ *as* $t \to 0$, *and* $\{v \leqslant 0\}$ *is a cusp. See Figs.* 3–5 *for the asymptotics of* $\{v > 0\}$ *and* $\tilde{\Omega}$ *in the case that* $v((r^0, x_3^0) + t(r, x_3))/t^2 \to \frac{1}{4}r^2$ *as* $t \to 0$.
  2. *There is* $\lambda \in (-\infty, 0) \cup (1, +\infty)$ *such that*

$$v\big((r^0, x_3^0) + t(r, x_3)\big)/t^2 \to \frac{1}{2}\left(\frac{\lambda}{2}r^2 + (1-\lambda)x_3^2\right) \quad \text{as } t \to 0.$$

  Then $\{v > 0\}$ *is a double wedge. See Figs.* 1–2 *for the asymptotics of* $\{v > 0\}$ *and* $\tilde{\Omega}$ *in the case that* $\lambda > 1$.
  3. *There is* $\lambda \in (0, 1)$ *such that* $v((r^0, x_3^0) + t(r, x_3))/t^2 \to \frac{1}{2}(\frac{\lambda}{2}r^2 + (1-\lambda)x_3^2)$ *as* $t \to 0$. *The complement of* $\{v > 0\}$ *and that of* $\tilde{\Omega}$ *is the single point* x^0.

(b) *Suppose that* $r^0 > 0$. *Then after rotation one of the following three alternatives holds:*
  1. *Either* $v((r^0, x_3^0) + t(x, y))/t^2 \to \frac{1}{2}x^2$ *as* $t \to 0$, *and* $\{v \leqslant 0\}$ *is a cusp. See Figs.* 3–5 *for the asymptotics of* $\{v > 0\}$ *and* $\tilde{\Omega}$.
  2. *There is* $\lambda \in (-\infty, 0) \cup (1, +\infty)$ *such that*

$$v\big((r^0, x_3^0) + t(x, y)\big)/t^2 \to \frac{1}{2}\big(\lambda x^2 + (1-\lambda)y^2\big) \quad \text{as } t \to 0.$$

  Then $\{v > 0\}$ *is a double wedge. See Figs.* 1–2 *for the asymptotics of* $\{v > 0\}$ *and* $\tilde{\Omega}$ *in the case that* $\lambda > 1$.

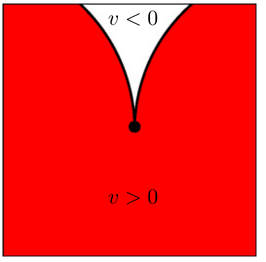

Fig. 5. Cusp (with the set $\tilde{\Omega}$ painted as filled shape).

3. *There is* $\lambda \in (0,1)$ *such that* $v((r^0, x_3^0) + t(x,y))/t^2 \to \frac{1}{2}(\lambda x^2 + (1-\lambda)y^2)$ *as* $t \to 0$. *The complement of* $\{v > 0\}$ *and that of* $\tilde{\Omega}$ *is the single point* x^0.

Remark 1.1. The asymptotics for v in Theorem A is actually valid at any point $\tilde{x}^0 \in \{v = 0\} \cap \{\nabla v = 0\}$, however at points $\tilde{x}^0 \in \{v = 0\} \cap \{\nabla v = 0\} \setminus \partial\Omega$ there is the possibility of straight line segments for $\{v \leqslant 0\}$ so that we have confined the statement to points $\tilde{x}^0 \in \partial\Omega \cap \{\nabla v = 0\}$.

2. Proof of the main result

Let (u, Ω) be a solution of (1.6) satisfying the assumptions in the statement of the theorem. By L^p- and C^α-estimates $u \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^3) \cap C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^3)$ for each $p \in (1, +\infty)$ and $\alpha \in (0,1)$. Differentiating u we obtain

$$\Delta \partial_{x_k} u = -\Phi'(u)\chi_\Omega \partial_{x_k} u \in L^\infty_{\mathrm{loc}}\big(\mathbb{R}^3\big) \tag{2.1}$$

in the sense of distributions: one way to realize this would be to observe that

$$w_\delta(x) := \begin{cases} \max(u(x) - \delta, 0), & x \in \Omega, \\ 0, & x \notin \Omega \end{cases}$$

defines a family of functions $(w_\delta)_{\delta \in (0,1)}$ which is bounded in $W^{1,2}(B_R)$ for each $R \in (0, +\infty)$. Thus

$$w(x) := \begin{cases} u(x), & x \in \Omega, \\ 0, & x \notin \Omega \end{cases}$$

is a function in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^3)$, and

$$\int_{\mathbb{R}^3} u \Delta\zeta = \int_{\mathbb{R}^3} \big(1 - \Phi(w)\big)\zeta, \qquad \int_{\mathbb{R}^3} u \Delta \partial_{x_k}\zeta = -\int_{\mathbb{R}^3} \Phi(w)\partial_{x_k}\zeta$$

for every $\zeta \in C^\infty_0(\mathbb{R}^3)$. Thus, by the assumptions $\Phi \in C^1([0,+\infty))$ and $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^3)$ as well as the fact that $w \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^3)$, $\Phi(w) \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^3)$ so that $u \in W^{3,p}_{\mathrm{loc}}(\mathbb{R}^3)$ with $p = 2$ by L^2-theory. Therefore $\partial_{x_k} u$ is a strong solution of

$$\Delta \partial_{x_k} u = \begin{cases} -\Phi'(w)\partial_{x_k} w, & x \in \Omega, \\ 0, & x \notin \Omega, \end{cases}$$

in the sense of [12, Chapter 9] which proves (2.1).

Consequently $u \in W^{3,p}_{\mathrm{loc}}(\mathbb{R}^3) \cap C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^3)$ for each $p \in (1,+\infty)$ and $\alpha \in (0,1)$. We maintain that in addition

$$\|u\|_{C^{2,\alpha}(B_1(\{u=0\}))} \leqslant C. \tag{2.2}$$

In order to see that we observe that by the assumption of our theorem,

$$u \geqslant -C \quad \text{and} \quad \Delta u \leqslant C \quad \text{in } \mathbb{R}^3.$$

Therefore $w := u + C$ satisfies $w \geqslant 0$ and

$$\int\limits_{B_2(x)} (u+C) = \int\limits_{B_2(x)} w \leqslant C_1 w(x) = C_1 C \quad \text{for every } x \in \{u=0\};$$

here C_1 is a universal constant. But then (2.2) follows by repeating the above proof using the uniform $L^1(B_2(x))$-bound.

On the other hand we infer from (1.6) that the Hessian of u satisfies

$$\left|D^2 u\right| \geqslant c > 0 \quad \text{on } \{u=0\}. \tag{2.3}$$

As $\{u=0\} \cap \{\nabla u \neq 0\}$ is by the implicit function theorem locally a $C^{2,\alpha}$-surface—the regularity of the surface can be improved to real analyticity by the methods in [7]—, we will focus on the singular set $\partial\Omega \cap \{\nabla u = 0\}$.

Let us consider a point $x^0 \in \partial\Omega \cap \{\nabla u = 0\}$. If $(x_1^0)^2 + (x_2^0)^2 > 0$ then we translate x^0 to the origin and rotate so that

$$D^2 u(0) = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix} \quad \text{for some } \lambda \in \mathbb{R}. \tag{2.4}$$

In cylindrical coordinates (where we do rotate but not translate) we obtain in this case that

$$D^2 v\big(\big(r^0, x_3^0\big)\big) = \begin{pmatrix} \lambda & 0 \\ 0 & 1-\lambda \end{pmatrix} \quad \text{for the same } \lambda \in \mathbb{R}.$$

If $r_0 = 0$, that is x^0 is on the axis of symmetry, we expand

$$v(r, x_3) = c_1 r^2 + c_2 x_3^2 + c_3 r x_3 + o\big(r^2 + x_3^2\big) \quad \text{and} \quad \partial_{r x_3} v = c_3 + o(1).$$

Suppose now towards a contradiction that $c_3 \neq 0$: then

$$\partial_{x_1 x_3} u(x_1, x_2, x_3) = \frac{x_1}{r} \partial_{r x_3} v,$$

where the right-hand side is discontinuous at $(x_1, x_2, x_3) = 0$, a contradiction. Thus

$$v(r, x_3) = c_1 r^2 + c_2 x_3^2 + o\big(r^2 + x_3^2\big) \quad \text{and}$$
$$u(x_1, x_2, x_3) = c_1\big(x_1^2 + x_2^2\big) + c_2 x_3^2 + o\big(x_1^2 + x_2^2 + x_3^2\big),$$

and using the PDE for u we obtain in the case $r_0 = 0$ that

$$u(x_1, x_2, x_3) = \frac{\lambda}{4}\left(x_1^2 + x_2^2\right) + \frac{1}{2}(1-\lambda)x_3^2 + o\left(x_1^2 + x_2^2 + x_3^2\right)$$

and that, setting $\tilde{x}^0 := (r^0, x_3^0)$,

$$D^2 v\left(\tilde{x}^0\right) = \begin{pmatrix} \frac{\lambda}{2} & 0 \\ 0 & 1-\lambda \end{pmatrix} \quad \text{for some } \lambda \in \mathbb{R}.$$

Case 1: If $0 < \lambda < 1$, then Ω^c consists in a sufficiently small ball $B_\delta(x^0)$ of only the point x^0 which is in this case a local minimum point of u.

Case 2: If $\lambda > 1$ or $\lambda < 0$, $\{v = 0\}$ consists of two C^1-curves intersecting at a nonzero angle at $\tilde{x}^0$ (cf. Fig. 1 and Fig. 2): we may assume that $\lambda > 1$, and as explained above we may assume that either

$$D^2 v\left(\tilde{x}^0\right) = \begin{pmatrix} \frac{\lambda}{2} & 0 \\ 0 & 1-\lambda \end{pmatrix}$$

or that after a rotation of the coordinates,

$$D^2 v\left(\tilde{x}^0\right) = \begin{pmatrix} \lambda & 0 \\ 0 & 1-\lambda \end{pmatrix}.$$

As in this case for sufficiently small δ, $\partial_{x_3} v < 0$ in $B_r(\tilde{x}^0) \cap \{x_3 > \delta\}$ and $\partial_{x_3} v > 0$ in $B_r(\tilde{x}^0) \cap \{x_3 < -\delta\}$, we may rescale and obtain that $\{v = 0\} \setminus \{0\}$ consists of four $C^{1,\alpha}$-graphs. The fact that

$$v\left(\tilde{x}^0 + t(r, x_3)\right)/t^2 \to \frac{1}{2}\left(\mu r^2 + (1-\lambda)x_3^2\right) \quad \text{as } t \to 0$$

with $\mu \in \{\lambda/2, \lambda\}$ and that convergence takes place in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$, implies now that the graphs have tangents as $x \to 0$ and that we may combine them to two C^1-curves intersecting at a nonzero angle at 0.

Case 3: If $\lambda = 1$ or $\lambda = 0$, then $\{v = 0\}$ consists either of two C^1-curves ending in a cusp at $\tilde{x}^0$ (cf. Fig. 5) or intersecting in a double cusp at $\tilde{x}^0$ (cf. Figs. 1–2).

In order to prove the statement of Case 3, we may assume that $x^0 = 0$ and that in the case that the original free boundary point is not on the axis of symmetry, (2.4) holds. We will first consider the case $\lambda = 1$, implying that $v(t(r, x_3))/t^2 \to \frac{1}{2}r^2$ as $t \to 0$ after rotation in the case that the original free boundary point is not on the axis of symmetry, and that $v(t(r, x_3))/t^2 \to \frac{1}{4}r^2$ as $t \to 0$ else. From (2.1) we obtain that

$$\Delta \partial_{x_3} u = c(x)\partial_{x_3} u$$

with Hölder continuous coefficients $c(x)$. Using once more the asymptotic assumptions for Φ and applying [6, Lemma 3.1] repetitively for $\beta = 3/2, 7/2, 11/2, 15/2, \ldots$ we infer that either

$$\partial_{x_3} u = p + \Gamma$$

where p is a nontrivial harmonic polynomial of degree $[\beta] + 2$ with leading term of order $\geqslant 2$ (by the fact that $u(tx)/t^2 \to \mu(x_1^2 + x_2^2)$ as $t \to 0$ with $\mu \in \{1/2, 1/4\}$) and

$$\left|\Gamma(x)\right| \leqslant C_1|x|^{\beta+2},$$

or $\partial_{x_3} u$ vanishes of infinite order at 0, that is

$$\left|\partial_{x_3} u(x)\right| \leqslant C_k |x|^k \quad \text{in } B_{r_k}(0)$$

for every $k \in \mathbb{N}$.

In the latter case we obtain by repetitive application of a well-known *strong unique continuation property* (see [13, Remark 6.7] for a very general result), that $\partial_{x_3} u \equiv 0$ in each connected component of $\{u \neq 0\}$ touching the origin, implying by our information on the blow-up limit that $u \equiv \frac{1}{4}(x_1^2 + x_2^2) + f(x_1^2 + x_2^2)$ in each connected component of $\{u \neq 0\}$ touching the origin, where $f(z) = o(z)$ as $z \to 0$. But this contradicts the assumption $\int_\Omega \Phi(u) \in (0, +\infty)$—by which Ω incidentally is non-empty—, thus proving that infinite order vanishing is not possible.

Let us return to the former case

$$\partial_{x_3} u = q + O\left(|x|^{k+\frac{1}{2}}\right)$$

where q is a nontrivial homogeneous harmonic polynomial of degree $k \geqslant 2$. It follows that in the case that the original free boundary point is on the axis of symmetry,

$$u = \frac{1}{4}\left(x_1^2 + x_2^2\right) + g\left(x_1^2 + x_2^2\right) + \int_0^{x_3} q(x_1, x_2, s)\, ds + O\left(|x|^{k+\frac{3}{2}}\right), \tag{2.5}$$

where $g(z) = o(z)$ as $z \to 0$, and that in the case that the original free boundary point is not on the axis of symmetry,

$$u = \frac{1}{2}x_1^2 + g\left(x_1^2\right) + \int_0^{x_3} q(x_1, x_2, s)\, ds + O\left(|x|^{k+\frac{3}{2}}\right), \tag{2.6}$$

where $g(z) = o(z)$ as $z \to 0$. Now, if $q((0,0,1)) \neq 0$, then $\{v \leqslant 0\}$ is in a neighborhood of $\tilde{x}^0$ a one-sided or double axially symmetric cusp (depending on the signs of $q((0,0,1))$ and $q((0,0,-1))$); see Figs. 3–5. If $q((0,0,1)) = 0$, then we will distinguish two cases, depending on whether the original free boundary point is on the axis of symmetry or not. In the case that the original free boundary point is on the axis of symmetry, we will prove that $q((0,0,1)) = 0$ is not possible. Suppose towards a contradiction that $q((0,0,1)) = 0$. We observe that q is axisymmetric and define $\tilde{q}(r, x_3) = q(x_1, x_2, x_3)$. As q is a homogeneous polynomial, $\tilde{q}(r, x_3)$ is a homogeneous polynomial, and—using the assumption $q((0,0,1)) = 0$—all powers of r in the expansion of $\tilde{q}$ are even numbers $\geqslant 2$. Let $ar^{2j}x_3^\ell$ be the leading term in the expansion of $\tilde{q}$, where a is a real constant, $j \geqslant 1$ and $\ell \geqslant 0$ are integers. Then (2.5) becomes

$$v = \frac{1}{4}r^2\left(1 + \frac{4a}{\ell+1}r^{2j-2}x_3^{\ell+1}\right) + o\left(r^2\right) + O\left(r^{k+\frac{3}{2}}\right),$$

so that v is positive for sufficiently small x_3 and sufficiently small $r > 0$. Suppose now that for

$$W^a(x_3) := \inf\left\{r > 0\colon\ v(\cdot, x_3) > 0 \text{ in } (-r, r) \setminus \{0\}\right\},$$

$\liminf_{x_3 \to X} W^a = 0$, where $X \in [-\infty, +\infty]$. Then we obtain a contradiction to (2.2) and (2.3). Thus $W^a \geqslant c > 0$ on $\mathbb{R}$. It follows that $u = 0$ on the whole line $x_1 = x_2 = 0$ and $u \geqslant d > 0$ in $\{(x_1, x_2, x_3)\colon\ \delta < x_1^2 + x_2^2 < 2\delta\}$, a contradiction to the assumption $\int_\Omega \Phi(u) \in (0, +\infty)$.

In the case that the original free boundary point is not on the axis of symmetry, we obtain by our choice of rotation that $\partial_{x_3 x_2} u = 0$ and that consequently $\partial_{x_2} q \equiv 0$. Using also the information that by homogeneity, $q(0,0,t)=0$ for every $t\in\mathbb{R}$, we may expand q, and we obtain a leading term of the form $ax_1^jx_3^\ell$ where a is a real constant, $j\geqslant 1$ and $\ell\geqslant 0$ are integers satisfying $j+\ell\geqslant 2$. Thus

$$u=\frac{1}{2}x_1^2+g\big(x_1^2\big)+\frac{a}{\ell+1}x_1^jx_3^{\ell+1}+O\big(|x|^{k+\frac{3}{2}}\big).$$

If $j\geqslant 2$, then we conclude that and we obtain that $u=0$ on $\{x_1=0\}$ and $u>0$ for sufficiently small $x_1>0$, sufficiently small $|x_2|$ and sufficiently small $x_3^2>0$. Suppose now that for

$$W^b(x_2,x_3):=\inf\big\{x>0\colon\ u(\cdot,x_2,x_3)>0 \text{ in } (-x,x)\setminus\{0\}\big\},$$

$\liminf_{(x_2,x_3)\to X}W^b=0$, where $X\in\mathbb{R}^2\cup\{\infty\}$. Then we obtain a contradiction to (2.2) and (2.3). Thus $W^b\geqslant c>0$ on $\mathbb{R}^2$. It follows that $u=0$ on $\{x_1=0\}$ and $u\geqslant d>0$ in $\{(x_1,x_2,x_3)\colon\ \delta<x_1^2<2\delta\}$, a contradiction to the assumption $\int_\Omega\Phi(u)\in(0,+\infty)$. If $j=1$, then we conclude that $\ell\geqslant 1$. If both connected components of $\{u>0\}$ are subsets of Ω, then we may again obtain a contradiction to the assumption $\int_\Omega\Phi(u)\in(0,+\infty)$. Else we translate the obtained result into cylindrical coordinates and obtain a double cusp $\{v\leqslant 0\}$ that is asymmetric with respect to the axis $x_1=0$ (note that as we have rotated, $x_1=0$ would in this case not be the axis of the axisymmetry).

The proof in the case $\lambda=0$ is similar. In this case we replace $\partial_{x_3}u$ by $\partial_{x_1}u$. In the case that the original free boundary point is not on the axis of symmetry, the above proof works with obvious changes. In the case that the original free boundary point is on the axis of symmetry, we obtain the formula

$$u=\frac{1}{2}x_3^2+h\big(x_3^2\big)+\int_0^{x_1}q(s,x_2,x_3)\,ds+O\big(|x|^{k+\frac{3}{2}}\big),$$

with $h(z)=o(z)$ as $z\to 0$. If $q((1,0,0))\neq 0$, then we obtain as before that $\{v\leqslant 0\}$ is in a neighborhood of 0 a double cusp. Let us therefore focus on the case $q((1,0,0))=0$. This time, $Q(x_1,x_2,x_3):=\int_0^{x_1}q(s,x_2,x_3)\,ds$ is a homogeneous axisymmetric polynomial satisfying $Q((t,0,0))=0$ for every $t\in\mathbb{R}$. Thus we may write $Q(x_1,x_2,x_3)=\tilde{Q}(r,x_3)$. Let $ar^{2j}x_3^\ell$ be the leading term in the expansion of $\tilde{Q}$ where a is a real constant, $j\geqslant 1$ and $\ell\geqslant 1$ are integers. If $\ell\geqslant 2$, then we conclude that

$$v=\frac{1}{2}x_3^2\big(1+2ar^{2j}x_3^{\ell-2}\big)+o\big(r^2\big)+O\big(r^{k+\frac{3}{2}}\big),$$

and we obtain that $u=0$ on $\{x_3=0\}$ and $u>0$ for sufficiently small r and sufficiently small $x_3^2>0$. Suppose now that for

$$W^c(r):=\inf\big\{x>0\colon\ v(r,x)>0 \text{ in } (-x,x)\setminus\{0\}\big\},$$

$\liminf_{r\to R}W^c=0$, where $R\in[0,+\infty]$. Then we obtain a contradiction to (2.2) and (2.3). Thus $W^c\geqslant c>0$ on $(0,+\infty)$. It follows that $u=0$ on $\{x_3=0\}$ and $u\geqslant d>0$ in $\{(x_1,x_2,x_3)\colon\ \delta<x_3^2<2\delta\}$, a contradiction to the assumption $\int_\Omega\Phi(u)\in(0,+\infty)$. If $\ell=1$, then we conclude that

$$v=\frac{1}{2}x_3^2+ar^{2j}x_3+\text{lower order terms}.$$

If both connected components of $\{u>0\}$ are subsets of Ω, then we may again obtain a contradiction to the assumption $\int_\Omega\Phi(u)\in(0,+\infty)$. Else we obtain a double cusp $\{v\leqslant 0\}$ that is asymmetric with respect to the axis $x_3=0$.

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