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Moduli spaces of metric graphs of genus 1 with marks on vertices ☆

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A B S T R A C T

In this paper we study homotopy type of certain moduli spaces of metric graphs. More precisely, we show that the spaces $MG_{1,n}^{\mathfrak{v}}$, which parametrize the isometry classes of metric graphs of genus 1 with n marks on vertices are homotopy equivalent to the spaces $TM_{1,n}$, which are the moduli spaces of tropical curves of genus 1 with n marked points. Our proof proceeds by providing a sequence of explicit homotopies, with key role played by the so-called scanning homotopy. We conjecture that our result generalizes to the case of arbitrary genus.

1. Introduction

The moduli spaces of metric graphs with marks on vertices $MG_n^{\mathfrak{v}}$ were recently defined in [4]. The main motivation for their introduction was that they serve as universes in which the moduli spaces of tropical curves of genus g with n marked points $TM_{g,n}$ can be most conveniently defined. The latter were discovered by Mikhalkin, see [5], as an important concept in the field of tropical geometry, see [6]. The complementary study in [4] was dedicated to the topological properties of the spaces $TM_{g,n}$, paying special attention to the case of genus 1.

Here we take one step back into the broader framework where the study of the moduli spaces of metric graphs with marks on vertices is of interest in its own right. We again limit ourselves to the case of genus 1. Our main result states that the space $MG_{1,n}^{\mathfrak{v}}$ is homotopy equivalent to $TM_{1,n}$, which means that in this case the topological information is already encapsulated in the tropical case. We prove this by describing a sequence of explicit homotopies, with the so-called *scanning homotopy* playing the central role.

2. The moduli spaces of metric graphs with marks on vertices

For the brevity of the presentation, we shall limit ourselves to the descriptive definitions of the concepts which we need in order to state and to prove our results. We refer the reader to [4] for the formally complete definitions, which at times can be somewhat technical.

Intuitively the concept of the metric graph is rather simple: one takes a usual undirected graph, loops and multiple edges are specifically allowed, and adds lengths on all the edges. While there is a standard way to associate a 1-dimensional CW

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complex to every graph, once the edge lengths are added, there is then a natural way to make this CW complex into a metric space. The isometries of the metric graphs are the isomorphisms of the underlying graph which in addition preserve edge lengths. When some points are marked with labels 1 through n on the metric graph, then the isometries are required to fix the marked points as well. This allows to define the isometry classes as equivalence classes for the corresponding equivalence relation. In this paper we limit ourselves to the situation where the marks are allowed to be placed on the graph vertices only.

Let us now fix a positive integer n, and let $MG_n^{\mathfrak{v}}$ denote the set of all isometry classes of finite metric graphs with n marks on vertices, where the vertices may have several marks. In [4] we described a natural way to equip this set with topology. The idea is that given a graph G, the points in a small open neighborhood of $[G]$ are given by the isometry classes, which have a graph representative obtained by a combination of the following deformations:

- changing the lengths of the edges of G by a small number,
- expanding vertices of G into trees, with all the edges of the tree being sufficiently short,
- if the marks on vertices are involved, distributing the marks of every vertex arbitrarily on the vertices of the tree which replaces that vertex.

The precise definition can be found in [4, Section 3]. We let $MG_{g,n}^{\mathfrak{v}}$ denote the subspace of $MG_n^{\mathfrak{v}}$ consisting of the isometry classes of connected graphs of genus g.

The spaces of special interest in tropical geometry are the moduli spaces of tropical curves of genus g with n marked points $TM_{g,n}$. These are the subspaces of $MG_{g,n}^{\mathfrak{v}}$ consisting of the isometry classes whose representatives satisfy the additional condition that for every vertex the sum of its valency with the cardinality of its marking list is at least 3.

The first tool to simplify these spaces, while preserving the homotopy type, introduced in [4], was the *shrinking bridges* strong deformation retraction. The way it works is quite simple: all the bridges in the metric graph shrink to points at the speed which is inverse proportional to the edge lengths (here a *bridge* is an edge whose deletion increases the number of connected components, cf. [1, p. 11]). One can show that passing to the isometry classes this deformation is continuous and defines a strong deformation retraction of the spaces $TM_{g,n}$. The resulting spaces consist of graphs with no bridges, which are simpler. For example, when the genus is 1 we end up simply with cycles (with marked points).

It is easy to see that the shrinking bridges strong deformation retraction still works in the more general setting of the spaces $MG_{g,n}^{\mathfrak{v}}$. Hence we can choose the spaces of the bridge-free metric graphs as the starting point of our investigation.

3. The scanning homotopy and the homotopy type of $MG_{1,n}^{\mathfrak{v}}$

As mentioned above, the main focus of this paper is to understand the homotopy type of the topological space $MG_{1,n}^{\mathfrak{v}}$ consisting of all isometry classes of connected metric graphs of genus 1 with marks 1 through n distributed on the vertices the graph, where every single vertex is allowed to have multiple marks.

For an arbitrary positive integer n, we let $\widetilde{X}_n$ denote the space obtained from $MG_{1,n}^{\mathfrak{v}}$ by shrinking bridges. Let X_n denote the subspace of $\widetilde{X}_n$ consisting of the isometry classes of all cycles of total length 1. Clearly, varying the total length of the cycle yields a homeomorphism

$$\widetilde{X}_n \cong X_n \times (0, \infty).$$

In particular, the space $\widetilde{X}_n$ and X_n are homotopy equivalent. For convenience we now proceed to describe the latter space directly.

The points. The points of X_n are isometry classes of cycles of length 1 with n marked vertices. Let us identify these cycles with a unit circle in the plane, and let us always assume that the vertex whose list of marks includes 1 is located at $(-1, 0)$. The other vertices (marked or not) can be placed on the circle arbitrarily, and to pass to the isometry classes we need to mod out by the reflection with respect to the x-axis (conjugation $\mathbb{Z}_2$-action).

The topology. We say that two points $a = (a_1, a_2)$ and $b = (b_1, b_2)$ on a unit circle are *ε-close* if the shortest path connecting a and b along the circle does not leave the vertical strip $\{(x, y) \mid a_1 - \varepsilon < x < a_1 + \varepsilon\} \subseteq \mathbb{R}^2$. See the left-hand side of Fig. 1.

Definition 3.1. Let x be an arbitrary point of X_n, and let ε be an arbitrary positive number. We now define an open ε-neighborhood $N_\varepsilon(x)$ of x. Let G be a representative graph of x. Then $y \in N_\varepsilon(x)$ if and only if y has a representative graph H such that

(1) for every mark $k \in [n]$, the vertices of G and of H labelled with k are ε-close;
(2) every vertex of H is ε-close to some vertex of G;
(3) every vertex of G is ε-close to some vertex of H.

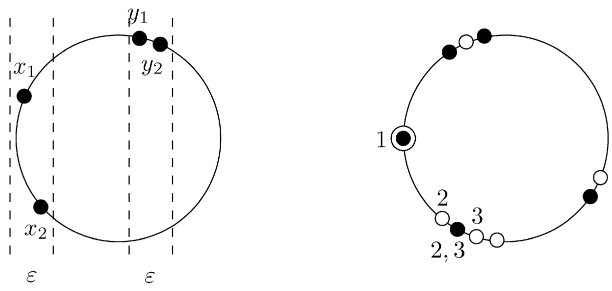

Fig. 1. On the left-hand side we show pairs of ε-close vertices. On the right-hand side we show two graphs G and H such that $H \in N_\varepsilon(G)$. The vertices of G are filled-in and the vertices of H are hollow. The vertices which seem close on the figure are not more than ε apart.

This description is obviously symmetric, so we have

$$x \in N_\varepsilon(y) \quad \Leftrightarrow \quad y \in N_\varepsilon(x),$$

for all $\varepsilon > 0$ and all $x, y \in X_n$. Furthermore, we set

$$N_\varepsilon(S) := \bigcup_{x \in S} N_\varepsilon(x),$$

for $\varepsilon > 0$ and $S \subseteq X_n$. Using these notations we have

$$N_{\varepsilon_1+\varepsilon_2}(x) = N_{\varepsilon_1}\big(N_{\varepsilon_2}(x)\big), \tag{3.1}$$

for all $\varepsilon_1, \varepsilon_2 > 0$ and $x \in X_n$.

Thinking geometrically, to obtain a point in the ε-neighborhood of a certain metric graph, we are allowed to shift the vertices, so that their x-coordinates change by at most ε, and we are allowed to merge and to split vertices in the process. The marks should follow with the corresponding vertices, we should merge the mark lists when the vertices are merged, and we can split mark lists arbitrarily when the vertices are split. See the right-hand side of Fig. 1.

The topology on X_n is now generated by the neighborhoods $N_\varepsilon(x)$ in the usual way: a subspace U of X_n is open if and only if for every $x \in U$ there exists $\varepsilon > 0$ so that the open neighborhood $N_\varepsilon(x)$ is contained in X_n.

Let Y_n be the subspace of X_n consisting of all points whose representative graph satisfies the following conditions:

- the point with coordinates $(1, 0)$ is a vertex (which might be marked);
- all other vertices of the graph are marked.

It is easy to see that the space Y_n is homotopy equivalent to the tropical moduli space $TM_{1,n}$, with the homotopy given by forgetting the vertex at $(1, 0)$, in case it is not marked.

Next, we define a map $\Phi : X_n \times [-1, 1] \to X_n$. Let $x \in X_n$, let $t \in [-1, 1]$, and let G be a metric graph with marked vertices representing the point x. Let H be the metric graph with marked vertices described by the following:

- the graph H is a cycle isometric to a unit circle;
- the marked vertices of H are the same as those of G;
- the points $(t, \sqrt{1-t^2})$ and $(t, -\sqrt{1-t^2})$ are vertices of H (marked or not);
- there are no unmarked vertices (a, b) in H satisfying $a < t$;
- the unmarked vertices (a, b) in H satisfying $a > t$ are the same as those of the graph G.

We can now set $\Phi(G, t) := H$, and accordingly $\Phi(x, t) := [H]$. It clearly does not depend on the choice of the graph representative of x. One can visualize the homotopy defined by the map Φ as vertical line "scanning" through the circle left-to-right, removing all the unmarked vertices in the process, see Fig. 2.

Theorem 3.2. *The map Φ provides a deformation retraction from X_n to Y_n. In particular, the space X_n is homotopy equivalent to Y_n, and hence the space $MG_{1,n}^{\mathfrak{v}}$ is homotopy equivalent to $TM_{1,n}$.*

Proof. The main thing is to show that the set map Φ is actually continuous. For this purpose, choose $x \in X_n$ and $t \in [-1, 1]$. Let $\varepsilon > 0$ and choose $\alpha < \varepsilon/2$. It would clearly to suffice to show that for sufficiently small ε we have

$$\Phi\big(N_\alpha(x) \times (t-\alpha, t+\alpha)\big) \subseteq N_\varepsilon\big(\Phi(x, t)\big). \tag{3.2}$$

We show (3.2) in two steps.

Step 1. Take an arbitrary point $x \in X_n$, and let G be the metric graph representing x. Furthermore, let $t_0, t_1 \in [-1, 1]$, say $t_0 < t_1$, such that $t_1 - t_0 < \varepsilon$ for some arbitrary $\varepsilon > 0$. What is the difference between the graphs $\Phi(G, t_0)$ and $\Phi(G, t_1)$?

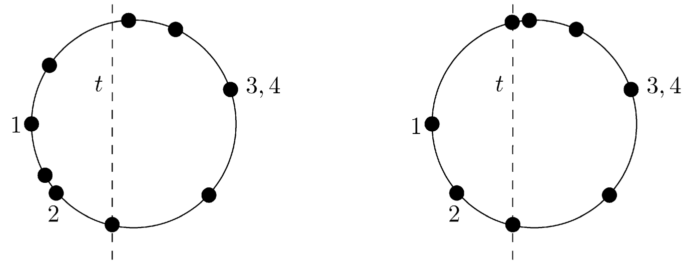

Fig. 2. On the left-hand side we show a possible graph G; on the right-hand side we show $\Phi(G,t)$.

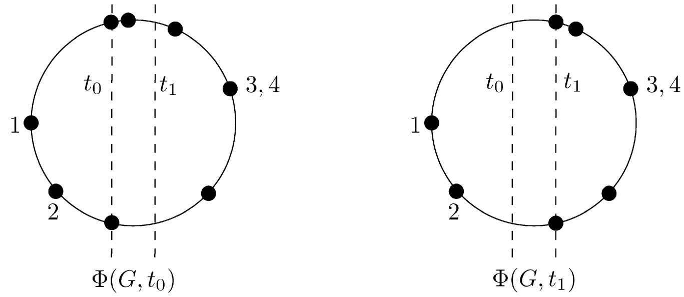

Fig. 3. We illustrate Step 1 by showing the values of $\Phi(G,-)$ for different times, using the graph G depicted on Fig. 2.

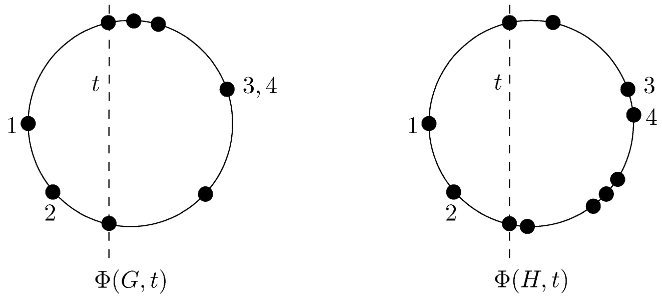

Fig. 4. We illustrate Step 2 by showing the values of $\Phi(-,t)$ for two close graphs.

The vertices, marked or not, must be the same in both graphs, if they lie to the left of t_0 or the right of t_1, i.e., in the union $\{(a,b) \mid a < t_0\} \cup \{(a,b) \mid a > t_1\}$. In the strip between t_0 and t_1 the marked vertices are the same, but the unmarked ones may be different; see Fig. 3. However, since the width of the strip is less than ε, and since the points of the circle on t_0-line are vertices of $\Phi(G,t_0)$, whereas the points of the circle on t_1-line are vertices of $\Phi(G,t_1)$, we can verify all conditions of Definition 3.1, and conclude that $\Phi([G],t_0) \in N_\varepsilon(\Phi([G],t_1))$ and vice versa.

Step 2. Fix $t \in [-1,1]$, and consider $x, y \in X_n$, and $\varepsilon > 0$, such that $y \in N_\varepsilon(x)$. Let G be a representative graph for x, and choose H to be the representative graph for y which satisfies the conditions of Definition 3.1 with respect to the chosen graph G. Again, we must ask what the difference between the graphs $\Phi(G,t)$ and $\Phi(H,t)$ is. Specifically, we want to prove that $\Phi([H],t) \in N_\varepsilon(\Phi([G],t))$. To obtain the graph H from the graph G we have to move vertices along the circle, possibly merging and splitting in the process, finally getting ε-close vertices, as described in Definition 3.1; see Fig. 4 for an illustration.

Since the marked vertices of $\Phi(H,t)$, resp. $\Phi(G,t)$, are the same as those of H, resp. G, the condition (1) of Definition 3.1 is satisfied. Furthermore, since passing from G, resp. H, to $\Phi(G,t)$, resp. $\Phi(H,t)$, removes the unmarked vertices to the left of the threshold line $\{(a,b) \mid a = t\}$, the conditions (2) and (3) could theoretically be violated, if there were points to the right of t whose ε-close partner vertex has just been removed. However, by construction the graphs $\Phi(H,t)$ and $\Phi(G,t)$ have both points of the unit circle, whose x-coordinate is t, as vertices. So one of these vertices is ε-close to every vertex in the strip $\{(a,b) \mid t < a < t+\varepsilon\}$. It shows, that all conditions of Definition 3.1 are satisfied, and hence $\Phi([H],t) \in N_\varepsilon(\Phi([G],t))$.

We can now combine the two steps as follows. Let $x \in X_n$, $\alpha > 0$, $t \in [-1,1]$, and consider $(y, t+\delta) \in N_\alpha(x) \times (t-\alpha, t+\alpha)$, in particular, we have $\delta \in (-\alpha,\alpha)$. By what we proved in Step 1, we have $\Phi(y,t+\delta) \in N_{|\delta|}(\Phi(y,t))$, while it follows from Step 2 that $\Phi(y,t) \in N_\alpha(\Phi(x,t))$. Combined with (3.1), these yield $\Phi(y,t+\delta) \in N_{\alpha+|\delta|}(\Phi(x,t))$. Since $\delta \in (-\alpha,\alpha)$ and $\alpha < \varepsilon/2$, we conclude that $\Phi(y,t+\delta) \in N_\varepsilon(\Phi(x,t))$.

By construction, the map Φ provides a homotopy between $\Phi(-,0) = \mathrm{id}_{X_n}$ and $\iota \circ \Phi(-,1) : X_n \to X_n$, where $\iota : Y_n \hookrightarrow X_n$ denotes the inclusion map. It follows that $\Phi(-,1) : X_n \to Y_n$ is a deformation retraction (see [3, Section 6.4], or [2]), and, in particular, X_n is homotopy equivalent to Y_n. As mentioned before the theorem, the space X_n is homotopy equivalent to $MG^{\mathfrak{v}}_{1,n}$, while the space Y_n is homotopy equivalent to $TM_{1,n}$, hence the proof is now finished. □

It is curious to note that the homotopy Φ gives a deformation retraction, but not a strong deformation retraction. While being ordinary in the classical algebraic topology, this is a somewhat peculiar in the context of the combinatorial algebraic topology. We conjecture that a stronger relation holds.

Conjecture 3.3. *The space Y_n is a strong deformation retract of the space X_n, for all $n \geqslant 1$.*

The second conjecture is slightly more speculative, asserting that the same holds for any genus.

Conjecture 3.4. *The space $TM_{g,n}$ is the strong deformation retract of $MG^{\mathfrak{v}}_{g,n}$, for all $g \geqslant 0$, and $n \geqslant 0$.*

A natural candidate for the strong deformation retraction is provided by the map $r : MG^{\mathfrak{v}}_{g,n} \to TM_{g,n}$ which

- contracts all the edges adjacent to the leaves[1] which are unmarked, or marked with precisely one label;
- deletes all the unmarked vertices of valency 2.

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[1] Generalizing the terminology customary for trees, we use the word *leaves* to denote any vertex of valency 1, cf. [1, p. 13].