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Implicit Differential Equations Near a Singular Point*

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This paper presents general and precise results on existence and number of solutions to implicitly defined ordinary differential equations in the vicinity of a singular point, where the equation is not equivalent to an explicit one. Unlike in most, if not in all, other studies devoted to this problem in the literature, it is not assumed that either rank drop occurs on an entire neighborhood of the singular point or that the equation is a scalar one. Attention is confined to the simplest but most frequently encountered kind of singular points. It is shown that such "standard" singular points split into two complementary classes: those from which exactly two distinct solutions emanate and those at which exactly two distinct solutions terminate. Whether a point of the latter class is eventually encountered is unaffected by slight modifications of the (nonsingular) initial condition and further evolution of the system governed by the singular ODE then requires using a suitable jump condition according to appropriate physical criteria. Points of the former class are also shown to play a more subtle but equally important role in the dynamics. The phenomena described here are relevant in various problems from the sciences, such as phase transitions or plasticity. They should also be relevant in some aspects of the classical domain of application of differential – algebraic equations: singular perturbations of ODE's.

1. INTRODUCTION

The main object of this paper is the study of implicit initial value problems of the general form

$$F(t, x, \dot{x}) = 0, \qquad x(t_0) = x_0, \qquad \dot{x}(t_0) = y_0, \tag{1.1}$$

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where $F = F(t, x, y)$ and $F(t_0, x_0, y_0) = 0$. Of course, we shall assume that $\partial_y F(t_0, x_0, y_0)$ is not invertible, so that the differential equation is not equivalent to an explicit one (multiform equation). This will be done after embedding (1.1) into a more general class of singular problems.

An examination of the literature reveals that multiform equations have been investigated either in the scalar case or in the assumption that $\partial_y F$ remains singular on an entire neighborhood of the point of interest. The study of the scalar case, apparently motivated by the example of Clairaut equation, began with the work of Darboux [7] in 1873 and related questions are considered in Tricomi [16]. More recently, the scalar case was reconsidered in the framework of catastrophe and singularity theories with contributions, e.g., by Thom [15] and Lak Dara [12]. Chaperon [6] and Arnold [2] contain a summary of the available results.

For the case when $\partial_y F$ is singular on an entire neighborhood, relevant references are Gantmacher [8] and Campbell [4] among others. An up to date account of the state of the art is to be found in Griepentrog and März [9].

In this paper, we shall not confine ourselves to the scalar case, nor assume that $\partial_y F$ is singular on an entire neighborhood. Our approach somewhat bridges the gap between the "pure" and "applied" points of view briefly summarized above. No attempt to a classification is made, but the singularity we consider is the most commonly encountered one, and is properly identified as such through transversality arguments (in the finite-dimensional case).

We believe it important to point out right now that limitation to a narrow class of singularities is justified *a priori* by elementary considerations. First and foremost, it must be observed that, in practice, when an evolution problem such as (1.1) is considered, the initial condition (t_0, x_0, y_0) will in general not be a point where $\partial_y F$ is singular. In any case, most arbitrarily small perturbations of (t_0, x_0, y_0) will reinstate the nonsingular character in the equation (except when $\partial_y F$ is singular on an entire neighborhood). However, irrespective of the initial condition, it cannot be excluded that the trajectory eventually meets a singular point at some later and unspecified time. While this gives full justification to the study of singular problems, the same reasoning as above pursued one step further yields that particular consideration should be given to singular points provided that the event of a trajectory eventually going through one of them is not affected by slight modifications of the initial condition. Elementary considerations lead to the conclusion that singular points whose projections onto the (t, x)-space occupy a variety with codimension one are especially relevant, because all other points are "missed" by trajectories emanating at "most" initial conditions.

As we shall see, there is essentially one kind of singular points satisfying the requirement that their projections occupy a hypersurface. This family of *standard singular points* will be shown to split into two complementary classes of "attracting" and "repelling" points (in a sense to be specified later). Both kinds of points seem to play an important role: the numerical work by Porsching [13] on phase transitions clearly indicates the presence of "attracting" ones. On the other hand, "repelling" points seem to be responsible for a strange "bifurcation looking" phenomenon in a problem of metal forming investigated by Cavendish *et al.* [5] and reported by Hall and Rheinboldt [11]. A simple example in which the same feature can be reproduced is given in Section 5.

Our analysis heavily relies on properties of determinants and matrices of cofactors. For this reason, it is carried out in the finite dimensional case up to and including Section 5. The infinite dimensional case is considered in Section 6, but it is not treated through a reduction argument of Lyapunov–Schmidt type, for it is easily seen that such a procedure *does not preserve* the differential structure of the problem. Instead, we have observed that a simple way exists to generalize a pair determinant-matrix of cofactors for operators close enough to a given Fredholm operator with index zero. This idea, which permits to use the same approach as in the finite-dimensional case, is only briefly discussed and will be developed elsewhere.

2. Reduction to Canonical Form and Related Notions

Given a mapping $F = F(t, x, y)$ defined on a neighborhood of (t_0, x_0, y_0) in $\mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^n$, with values in $\mathbf{R}^n$ and verifying $F(t_0, x_0, y_0) = 0$, consider the differential equation

$$\begin{gathered} F(t, x, \dot{x}) = 0, \\ x(t_0) = x_0, \qquad \dot{x}(t_0) = y_0. \end{gathered} \tag{2.1}$$

With the usual trick, one may as well confine attention to the autonomous case when F is independent of t, so that Eq. (2.1) becomes

$$\begin{gathered} F(x, \dot{x}) = 0, \\ x(0) = x_0, \qquad \dot{x}(0) = y_0. \end{gathered} \tag{2.2}$$

Now, differentiating $F(x, \dot{x}) = 0$, one finds

$$\partial_x F(x, \dot{x})\dot{x} + \partial_y F(x, \dot{x})\ddot{x} = 0. \tag{2.3}$$

Setting $\dot{x} = y$, we obtain

$$\begin{pmatrix} I & 0 \\ 0 & \partial_y F(x, y) \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -\partial_x F(x, y)\, y \end{pmatrix}. \tag{2.4}$$

Equations (2.3) and (2.4) being equivalent, any solution to (2.4) satisfying the initial condition $x(0) = x_0$, $y(0)(=\dot{x}(0)) = y_0$ yields x such that $F(x, \dot{x}) = F(x_0, y_0) = 0$, namely solves (2.2).

Clearly, introducing the notation $X = (x, y)$, Eq. (2.4) is one of the general form

$$\begin{aligned} A(X)\dot{X} &= G(X), \\ X(0) &= X_0, \end{aligned} \tag{2.5}$$

where A and G are mappings from a neighborhood of X_0 to $\mathscr{L}(\mathbf{R}^n)$ and $\mathbf{R}^n$, respectively. When A is continuous and $A(X_0)$ is invertible, Eq. (2.5) is equivalent to the explicit differential equation $\dot{X} = A^{-1}(X)\, G(X)$, $X(0) = X_0$. However, in the only case of interest here when the mapping F above is such that $\partial_y F(x_0, y_0) \notin \text{Isom}(\mathbf{R}^n)$, the corresponding matrix $A(X_0)$ is certainly not invertible. For notational convenience, we shall return to a lower case variable x and consider the problem of solving

$$\begin{aligned} A(x)\dot{x} &= G(x), \\ x(0) &= x_0, \end{aligned} \tag{2.6}$$

when $A(x_0) \in \mathscr{L}(R^n)$ is singular.

Irrespective of the invertibility of $A(x)$, one has

$$(\det A(x))I = A(x)(\text{adj } A(x)) = (\text{adj } A(x))\, A(x), \tag{2.7}$$

where adj $A(x)$ denotes the adjugate (transpose of the matrix of cofactors) of $A(x)$. From now on, we shall use the notation

$$f(x) = \det A(x), \tag{2.8}$$

$$C(x) = \text{adj } A(x). \tag{2.9}$$

Using (2.7), it follows that every solution to (2.6) is also a solution to

$$\begin{aligned} f(x)\, \dot{x} &= C(x)\, G(x), \\ x(0) &= x_0, \end{aligned} \tag{2.10}$$

and $f(x_0) = 0$ since $A(x_0)$ is singular. Conversely, every solution to (2.10) verifying $f(x(t)) \neq 0$ for $|t| > 0$ small enough is also a solution to (2.6). Setting

$$H(x) = C(x)\, G(x), \tag{2.11}$$

we see that (2.10) is a special case of

$$\begin{aligned} f(x)\, \dot{x} &= H(x), \\ x(0) &= x_0, \end{aligned} \tag{2.12}$$

where f is a real-valued mapping from a neighborhood of $x_0 \in \mathbf{R}^n$ satisfying $f(x_0)=0$, and H is a mapping from the same neighborhood of x_0 with values in $\mathbf{R}^n$. Equation (2.12) will be referred to as the *canonical form* of the singular Eq. (2.6).

For a generic function $f \in \mathscr{C}^1(\mathbf{R}^n, \mathbf{R})$ (possibly defined only on some open subset of $\mathbf{R}^n$) it never happens that both f and f' vanish simultaneously. In other words, a generic f does not have 0 as a critical value. Although we shall not need to restrict our theory to a generic function f, nevertheless this shows that the examination of Eq. (2.12) in the assumption

$$f'(x_0) \neq 0, \tag{2.13}$$

is of primary importance when $f(x_0)=0$. Whenever $f(x_0)=0$ and (2.13) holds, we shall say that x_0 is a *noncritical singular point* of Eq. (2.12). From the implicit function theorem, the set of singular points near a noncritical one is a hypersurface which, by continuity, consists only of noncritical points. Also, the null-space of the linear form $f'(x_0)$ at a noncritical singular point x_0 is a hyperplane in $\mathbf{R}^n$. Then, for a generic choice of the mapping $H \in \mathscr{C}^1(\mathbf{R}^n, \mathbf{R}^n)$ (restriction to $\mathscr{C}^1$ mappings H is unnecessary at this stage, but will be needed later), one has

$$f'(x_0)\, H(x_0) \neq 0. \tag{2.14}$$

We have found it convenient to call *standard singular point* of Eq. (2.12) a noncritical singular point for which (2.14) holds. A given Eq. (2.12) may perfectly have many singular points, none of whose is standard. But all of them become standard after replacing f and H by arbitrarily small perturbations. This statement remains essentially, but not completely, true when f and H are bound to have the form prescribed in (2.8) and (2.11), respectively. In other words, to justify the terminology introduced above, one must answer the question: How general is it to assume that the point x_0 in Eq. (2.6) is a standard singular point of the canonical form (2.12)? To do this, one first needs to characterize standard singular points of the canonical form (2.12) of Eq. (2.6) in terms of the data A and G. This is done in

PROPOSITION 2.1. *The point x_0 is a standard singular point of the canonical form* (2.12) *if and only if*

(i) $\dim \operatorname{Ker} A(x_0)=1$,

(ii) $(A'(x_0)\, e_0)\, e_0 \notin \operatorname{Range} A(x_0)$, $\quad \forall e_0 \in \operatorname{Ker} A(x_0)-(0)$,

(iii) $G(x_0) \notin \operatorname{Range} A(x_0)$.

Proof. Suppose that x_0 is a standard singular point of the canonical form (2.12) of Eq. (2.6). Identifying $A(x)$ with its columns $a_1(x), \cdots, a_n(x)$, one has

$$f(x)=A(a_1(x), ..., a_n(x)),$$

where A is a n-linear antisymmetric form. Hence, for $h \in \mathbf{R}^n$

$$f'(x)\,h = \sum_{i=1}^{n} A(a_1(x), \ldots, a_{i-1}(x), a_i'(x)\,h, a_{i+1}(x), \ldots, a_n(x)).$$

Since $A(x_0)$ is singular, at most $(n-1)$ columns are linearly independent for $x = x_0$. On the other hand, if each of the n families of $(n-1)$ columns of $A(x_0)$ are linearly dependent, it is obvious from the above formula that $f'(x_0) = 0$, a contradiction with x_0 being noncritical. Hence rank $A(x_0) = n-1$, namely, dim Ker $A(x_0) = 1$. Next, using (2.8) and (2.9), relation (2.7) reads

$$f(x)\,I = A(x)\,C(x) = C(x)\,A(x).$$

In particular, $A(x_0)\,C(x_0) = C(x_0)\,A(x_0) = 0$. As $C(x_0) \neq 0$ since one at least among the n^2 principal minors of $A(x_0)$ is nonzero, this yields

$$\text{Ker } C(x_0) = \text{Range } A(x_0) \qquad \text{and} \qquad \text{Range } C(x_0) = \text{Ker } A(x_0).$$

Now, the condition $f'(x_0)\,H(x_0) \neq 0$ requires $H(x_0) \neq 0$. But $H(x_0) = C(x_0)\,G(x_0)$ and hence $G(x_0) \notin \text{Range } A(x_0)$. Also, the former relation shows that $H(x_0)$ is a (nonzero) element of Ker $A(x_0)$. Differentiating the identity $f(x)\,I = C(x)\,A(x)$, one finds

$$(f'(x_0)\,e_0)\,I = (C'(x_0)\,e_0)\,A(x_0) + C(x_0)(A'(x_0)\,e_0).$$

Thus, $(f'(x_0)\,e_0)\,e_0 = C(x_0)(A'(x_0)\,e_0)\,e_0$ and hence $(A'(x_0)\,e_0)\,e_0 \notin$ Range $A(x_0)$ from $f'(x_0)\,e_0 \neq 0$.

Conversely, if (i) of the proposition holds, then $C(x_0)$ has rank one with Ker $C(x_0) = \text{Range } A(x_0)$ and Range $C(x_0) = \text{Ker } A(x_0)$. Thus, $H(x_0) = C(x_0)\,G(x_0)$ is a nonzero element of Ker $A(x_0)$ as soon as (iii) is satisfied. If so, one finds again $(f'(x_0)\,e_0)\,e_0 = C(x_0)(A'(x_0)\,e_0)\,e_0$ and $f'(x_0) \neq 0$ from (ii) and Ker $C(x_0) = \text{Range } A(x_0)$. ∎

From now on, we shall refer to x_0 as being a standard singular point of Eq. (2.6) whenever conditions (i), (ii), and (iii) of Proposition 2.1 are fulfilled. This is of course equivalent to saying that x_0 is a standard singular point of the canonical form (2.12) of Eq. (2.6).

When Eq. (2.6) is derived from an equation of the form (2.2), namely $A(x)$ and $G(x)$ must be replaced by (x, y)-dependent families as indicated in (2.4), it is readily checked that conditions (i) through (iii) in Proposition 2.1 are equivalent to

$$\dim \text{Ker } \partial_y F(x_0, y_0) = 1,$$

$$\partial_{yy} F(x_0, y_0)\,e_0^2 \notin \text{Range } \partial_y F(x_0, y_0), \qquad \forall e_0 \in \text{Ker } \partial_y F(x_0, y_0) - \{0\},$$

$$\partial_x F(x_0, y_0)\,y_0 \notin \text{Range } \partial_y F(x_0, y_0).$$

Using this characterization and now turning to the case when Eq. (2.2) is derived from a nonautonomous equation as in (2.1), the above conditions become

$$\dim \operatorname{Ker} \partial_y F(t_0, x_0, y_0) = 1, \tag{2.15}$$

$$\begin{gathered}\partial_{yy} F(t_0, x_0, y_0)\, e_0^2 \notin \operatorname{Range} \partial_y F(t_0, x_0, y_0), \\ \forall e_0 \in \operatorname{Ker} \partial_y F(t_0, x_0, y_0) - \{0\},\end{gathered} \tag{2.16}$$

$$\partial_t F(t_0, x_0, y_0) + \partial_x F(t_0, x_0, y_0)\, y_0 \notin \operatorname{Range} \partial_y F(t_0, x_0, y_0). \tag{2.17}$$

Note if F is independent of t that the previous conditions are recovered: no discrepancy is introduced by reducing Eq. (2.1) to the form (2.2). Also, if $n = 1$, the above conditions may be rewritten as being $\partial_y F(t_0, x_0, y_0) = 0$, $\partial_{yy} F(t_0, x_0, y_0) \neq 0$ and $\partial_t F(t_0, x_0, y_0) + \partial_x F(t_0, x_0, y_0)\, y_0 \neq 0$. They characterize (t_0, x_0, y_0) as what is called a *simple fold point* in [6, 12, 16] and a regular singular point in [2] (a rather inappropriate terminology suggesting a connection with Fuchs–Frobenius theory which does not exist).

PROPOSITION 2.2. *Suppose that $F = F(t, x, y)$ is of class $\mathscr{C}^2$ and conditions* (2.15) *to* (2.17) *hold. Then, the $\mathbf{R}^n \times \mathbf{R}$-valued mapping*

$$(F, \det \partial_y F) \tag{2.18}$$

has full rank at (t_0, x_0, y_0), so that the solutions to the system

$$\begin{aligned} F(t, x, y) &= 0, \\ \det \partial_y F(t, x, y) &= 0, \end{aligned} \tag{2.19}$$

consist of a n-dimensional submanifold of $\mathbf{R}^{2n+1}$ near (t_0, x_0, y_0). Moreover, this manifold is diffeomorphic to its projection to the (t, x)-space, which is then a hypersurface M of $\mathbf{R} \times \mathbf{R}^n$ containing (t_0, x_0).

Proof. To prove that the Jacobian matrix of the mapping (2.18) at (t_0, x_0, y_0) has full rank $n+1$, it is equivalent to show that its transpose has maximum rank $n+1$, namely that

$$\begin{aligned} [\partial_t F(t_0, x_0, y_0)]^T h + \alpha\, \partial_t \phi(t_0, x_0, y_0) &= 0, \\ [\partial_x F(t_0, x_0, y_0)]^T h + \alpha [\partial_x \phi(t_0, x_0, y_0)]^T &= 0, \\ [\partial_y F(t_0, x_0, y_0)]^T h + \alpha [\partial_y \phi(t_0, x_0, y_0)]^T &= 0, \end{aligned}$$

if and only if $h = 0 \in \mathbf{R}^n$ and $\alpha = 0 \in \mathbf{R}$, where we have set $\phi(t, x, y) = \det \partial_y F(t, x, y)$.

To see that $\alpha = 0$, note that $[\partial_y \phi(t_0, x_0, y_0)]^T \notin \text{Range}\, [\partial_y F(t_0, x_0, y_0)]^T = [\text{Ker}\, \partial_y F(t_0, x_0, y_0)]^\perp$. Otherwise, $\partial_y \phi(t_0, x_0, y_0)\, e_0 = 0$ for every $e_0 \in \text{Ker}\, \partial_y F(t_0, x_0, y_0)$. But this is impossible. Indeed, by definition of ϕ, one has

$$\begin{aligned}\phi(t, x, y)\, I &= [\text{adj}\, \partial_y F(t, x, y)]\, \partial_y F(t, x, y) \\ &= \partial_y F(t, x, y) [\text{adj}\, \partial_y F(t, x, y)],\end{aligned} \tag{2.20}$$

and $\text{adj}\, \partial_y F(t_0, x_0, y_0) \neq 0$ from (2.15). Hence

$$\begin{aligned}\text{Ker}\, (\text{adj}\, \partial_y F(t_0, x_0, y_0)) &= \text{Range}\, \partial_y F(t_0, x_0, y_0), \\ \text{Range}\, (\text{adj}\, \partial_y F(t_0, x_0, y_0)) &= \text{Ker}\, \partial_y F(t_0, x_0, y_0).\end{aligned} \tag{2.21}$$

Differentiating (2.20) wrt y in the direction $e_0 \in \text{Ker}\, \partial_y F(t_0, x_0, y_0)$ yields the relation

$$(\partial_y \phi(t_0, x_0, y_0)\, e_0)\, e_0 = [\text{adj}\, \partial_y F(t_0, x_0, y_0)]\, \partial_{yy} F(t_0, x_0, y_0)\, e_0^2 \neq 0,$$

as soon as $e_0 \neq 0$ from (2.16) and (2.21). In particular, $\partial_y \phi(t_0, x_0, y_0)\, e_0 \neq 0$.

Next, we must show that for $h \in \mathbf{R}^n$, one has $[\partial_t F(t_0, x_0, y_0)]^T h = 0 \in \mathbf{R}$ and $[\partial_x F(t_0, x_0, y_0)]^T h = [\partial_y F(t_0, x_0, y_0)]^T h = 0 \in \mathbf{R}^n$ if and only if $h = 0$. These assumptions are equivalent to h being orthogonal to the vector $\partial_t F(t_0, x_0, y_0)$ and to the spaces $\text{Range}\, \partial_x F(t_0, x_0, y_0)$ and $\text{Range}\, \partial_y F(t_0, x_0, y_0)$. In particular, h is orthogonal to the space $\text{Range}\, \partial_y F(t_0, x_0, y_0)$ and to the vector $\partial_t F(t_0, x_0, y_0) + \partial_x F(t_0, x_0, y_0)\, y_0$. But then, from (2.15) and (2.17), h is orthogonal to $\text{Range}\, \partial_y F(t_0, x_0, y_0) \oplus \text{span}\, \{\partial_t F(t_0, x_0, y_0) + {}_x F(t_0, x_0, y_0)\, y_0\} = \mathbf{R}^n$. This proves $h = 0$.

From the above, the system (2.19) is solved through the implicit function theorem near the point (t_0, x_0, y_0). To complete the proof, it suffices to show that the tangent space to the solution set of (2.19) at (t_0, x_0, y_0) contains no "vertical" vector of the form $(0, 0, h)$, $h \in \mathbf{R}^n$. Equivalently, we must show that $\partial_y F(t_0, x_0, y_0)\, h = 0$ and $\partial_y \phi(t_0, x_0, y_0)\, h = 0$ if and only if $h = 0$. The first equation means $h = e_0 \in \text{Ker}\, \partial_y F(t_0, x_0, y_0)$. The second equation ensures that $e_0 = 0$ since $\partial_y \phi(t_0, x_0, y_0)\, e_0 \neq 0$ for $e_0 \neq 0$ in $\text{Ker}\, \partial_y F(t_0, x_0, y_0)$ was seen earlier in the proof. ∎

Regarding Eq. (2.1), Proposition 2.2 means that whenever (t_0, x_0, y_0) with $F(t_0, x_0, y_0) = 0$ is a standard singular point of the corresponding Eq. (2.6)—henceforth abbreviated as standard singular point of Eq. (2.1)—there is a hypersurface M through (t_0, x_0) such that for each singular point (t, x, y) of (2.1) close enough to (t_0, x_0, y_0) one has $(t, x) \in M$. Further, for each $(t, x) \in M$, there is a unique $y \in \mathbf{R}^n$ such that

(t, x, y) is a singular point of (2.1) and (t, x, y) is close to (t_0, x_0, y_0). Of course, all the singular points close to (t_0, x_0, y_0) are standard, too.

Remark 2.1. If F is independent of t, Proposition 2.2 remains obviously valid but the above interpretation regarding Eq. (2.2) may be slightly modified: the hypersurface M lies in $\mathbf{R}^n$ (hence $\dim M = n-1$) and contains x_0. In other words, the variable t can be eliminated from the statement. This is consistent with the origin in time being irrelevant in an autonomous system. ▮

For a generic F of class $\mathscr{C}^\infty$, it follows from Thom's transversality theorem and the usual stratification of the set of noninvertible $n \times n$ matrices (see Arnold [2] or Arnold *et al.* [3]) that the set of points (t, x, y) such that $\dim \operatorname{Ker} \partial_y F(t, x, y) = 1$ is a manifold with dimension n (possibly $\emptyset$), all other singular points lying in manifolds with dimension $\leqslant n-3$. Since the additional conditions (2.16) and (2.17) are satisfied unless (t_0, x_0, y_0) assumes exceptional values, standard singular points (and their projections onto the (t, x)-space) are the only ones to occupy a variety with dimension n, a fact whose importance was already explained in the Introduction.

3. Further Reduction in the Noncritical Case

We now come back to the study of Eq. (2.6) or, equivalently, of its canonical form (2.12). Here, it is our purpose to show that another reduction is possible when x_0 is a noncritical point, not necessarily standard. However, the generic character of standard singular points stresses that solutions $x(t)$ should be sought in $\mathscr{C}^0([0, T]) \cap \mathscr{C}^1((0, T])$, for if x_0 is standard, *the derivative* $\dot{x}(0)$ *just cannot exist* in view of the standing assumptions $f'(x_0) = 0$ and $f'(x_0) H(x_0) \neq 0$. In other words, it is of primary importance to not require that the differential part of the Eq. (2.6)/(2.12) be satisfied at the singular point x_0. For simplicity of notation, $[0, T]$ (resp. $(0, T]$) must be understood as $[T, 0]$ (resp. $[T, 0)$) when $T < 0$.

A noncritical singular point x_0 of Eq. (2.12) was defined to be one at which $f'(x_0) \neq 0$. If f is $\mathscr{C}^1$, the zero set of f near x_0 coincides with a hypersurface of $\mathbf{R}^n$ through x_0 as a result of the implicit function theorem. It is well known and incidentally straightforward to check that an equivalent version of the implicit function theorem asserts the existence of a local $\mathscr{C}^1$-diffeomorphism Φ such that

$$\Phi(0) = x_0, \qquad \Phi'(0) = I, \tag{3.1}$$

and

$$f(\Phi(\tilde{x})) = l(\tilde{x}), \tag{3.2}$$

for $\tilde{x}$ on a neighborhood of the origin in $\mathbf{R}^n$, where l is the linear form

$$l = f'(x_0). \tag{3.3}$$

For $x = \Phi(\tilde{x})$ in (2.12), one finds $l(\tilde{x})\,\Phi'(\tilde{x})\,\dot{\tilde{x}} = (H \circ \Phi)(\tilde{x})$, namely, $l(\tilde{x})\,\dot{\tilde{x}} = (\Phi'(\tilde{x}))^{-1}\,(H \circ \Phi)(\tilde{x})$. The initial condition $x(0) = x_0$ amounts to $\tilde{x}(0) = 0$, so that Eq. (2.12) is equivalent to

$$\begin{aligned} l(\tilde{x})\,\dot{\tilde{x}} &= \tilde{H}(\tilde{x}), \\ \tilde{x}(0) &= 0, \end{aligned} \tag{3.4}$$

where we have set

$$\tilde{H}(\tilde{x}) = (\Phi'(\tilde{x}))^{-1}\,(H \circ \Phi)(\tilde{x}). \tag{3.5}$$

Obviously, the above transformation and its inverse preserve the regularity $\mathscr{C}^0([0, T]) \cap \mathscr{C}^1((0, T])$ of the solutions. Also, Eq. (3.4) is a special case of (2.12) where $x_0 = 0$ is noncritical and f is linear. From (3.1), (3.3) and (3.5), it is immediate that the transformation of (2.12) into (3.4) and conversely preserves standard singular points, too. Another crucial notion which is preserved but still has to be defined at this time is that of *transversal solution.* Intuitively, a transversal solution should be one that intersects the zero set of f transversely at $x = x_0$. As the solutions typically are not $\mathscr{C}^1$ at x_0, an appropriate definition must be made, which is as follows

DEFINITION 3.1. The solution $x \in \mathscr{C}^0([0, T]) \cap \mathscr{C}^1((0, T])$ to Eq. (2.12) is said to be transversal if

(i) $f'(x(t))\,\dot{x}(t) \neq 0$ for $t \in (0, T]$ and $|t| > 0$ small enough,

(ii) $\lim_{t \to 0} \dfrac{x(t) - x_0}{f(x(t))}$ exists. ∎

Note from (i) in Definition 3.1 that $f(x(t)) \neq 0$ for $t \in (0, T]$ and $|t|$ small enough. In particular, $(x(t) - x_0)/f(x(t))$ is well defined in (ii). Also, this ensures, when (2.12) is the canonical form of (2.6), that a transversal solution to (2.12) is a solution to (2.6) as well. Definition 3.1 can then be extended to equations having the form (2.6) provided that x_0 is a noncritical singular point. The interpretation is that such a solution intersects the set of singular points of $A(\cdot)$ transversely at $x = x_0$. At first sight, it may not be clear that Definition 3.1 has this meaning. But consider the case when x is a $\mathscr{C}^1$ function near x_0: transversality is equivalent to $\dot{x}(0) \notin \operatorname{Ker} f'(x_0)$, namely, $f'(x_0)\, \dot{x}(0) \neq 0$, which guarantees (i). Also,

$$\frac{x(t)-x_0}{f(x(t))} = \frac{t\dot{x}(0)+o(t)}{tf'(x_0)\,\dot{x}(0)+o(t)},$$

tends to $\dot{x}(0)/f'(x_0)\,\dot{x}(0)$ as t tends to 0, which is (ii). Conversely, if x is $\mathscr{C}^1$ and transversal in the sense of Definition 3.1, then $f'(x_0)\,\dot{x}(0) \neq 0$ from the above relation.

The fact that the transformations from (2.12) to (3.4), and conversely, preserve transversal solutions deserves a proof, given in

PROPOSITION 3.1. *Assuming that x_0 is a noncritical singular point of Eq.* (2.12), *the function $x \in \mathscr{C}^0([0, T]) \cap \mathscr{C}^1((0, T])$ is a transversal solution if and only if, with Φ as in* (3.1)–(3.2), *$\tilde{x} = \Phi^{-1}(x)$ is a transversal solution to* (3.4).[1]

Proof. Assuming that x is transversal, we shall prove that $\tilde{x}$ is transversal, too. The converse will follow by simply reversing the arguments. From (3.2), $f(x(t)) = l(\Phi^{-1}(x(t))) = l(\tilde{x}(t))$. Hence $f'(x(t))\,\dot{x}(t) = l(\dot{\tilde{x}}(t)) \neq 0$ for $t \in (0, T]$, and $|t|$ small enough. Next, using $\Phi'(0) = I$ (cf. (3.1)), write

$$\frac{\tilde{x}(t)}{l(\tilde{x}(t))} = \frac{\Phi^{-1}(x(t))}{f(x(t))} = \frac{x(t)-x_0+o(\|x(t)-x_0\|)}{f(x(t))}.$$

As $\lim_{t\to 0}(x(t)-x_0)/f(x(t))$ exists by hypothesis, the term $o(\|x(t)-x_0\|)/f(x(t))$ tends to 0 with t. This shows that

$$\lim_{t\to 0} \frac{\tilde{x}(t)}{l(\tilde{x}(t))} = \lim_{t\to 0} \frac{x(t)-x_0}{f(x(t))},$$

and the proof is complete. ∎

Finding the transversal solutions to Eq. (3.4) can be done according to a procedure that we now explain. The general principle consists in arriving at a suitable splitting of the equation in two parts that can be solved successively. The technicalities are as follows: suppose that $\tilde{x} \in \mathscr{C}^0([0, T]) \cap \mathscr{C}^1((0, T])$ is a transversal solution to Eq. (3.4). Then, from $l(\dot{\tilde{x}}(t)) \neq 0$ for $t \in (0, T]$ and $|t|$ small enough, and after shrinking $|T|$ if necessary, the mapping

$$t \in [0, T] \to \lambda(t) = l(\tilde{x}(t)), \tag{3.6}$$

[1] Of course, it is implicit that $|T|$ is small enough for $x(t)$ to lie on a neighborhood in which Φ^{-1} is a diffeomorphism.

is continuous and monotone, hence a homeomorphism of $[0, T]$ to some interval $[0, \lambda_0]$ with $|\lambda_0| > 0$ (again, $[0, \lambda_0]$ must be understood as $[\lambda_0, 0]$ if $\lambda_0 < 0$). Moreover, λ is a $\mathscr{C}^1$ diffeomorphism of $(0, T)$ to $(0, \lambda_0)$ since λ' does not vanish on $(0, T]$.

As $\lambda(t) \neq 0$ for $t \in (0, T)$, one may write

$$\tilde{x}(t) = \lambda(t)\, \tilde{y}(t), \tag{3.7}$$

where $\tilde{y} \in \mathscr{C}^0([0, T]) \cap \mathscr{C}^1((0, T])$. To see that $\tilde{y}$ is continuous at the origin, recall that $\tilde{x}(t)/\lambda(t) = \tilde{x}(t)/l(\tilde{x}(t))$ has a limit as t tends to 0 from the hypothesis that $\tilde{x}$ is transversal. Thus, in view of (3.6) and (3.7)

$$l(\tilde{y}(t)) = 1, \qquad \forall t \in [0, T]. \tag{3.8}$$

From (3.7) and for $t \in (0, T)$, it is obvious that $\dot{\tilde{x}} = \dot{\lambda}\tilde{y} + \lambda\dot{\tilde{y}}$. Therefore, the equation $l(\tilde{x})\, \dot{\tilde{x}} = \tilde{H}(\tilde{x})$ may be rewritten as

$$\lambda\dot{\lambda}\tilde{y} + \lambda^2\dot{\tilde{y}} = \tilde{H}(\lambda\tilde{y}). \tag{3.9}$$

At this stage, observe that the initial condition $\tilde{x}(0) = 0$ is guaranteed by $\lambda(0) = 0$ and $\tilde{y}(0) \in \mathbf{R}^n$. On the other hand, one infers from (3.8) that $l(\dot{\tilde{y}}(t)) = 0$, for every $t \in (0, T]$. Together with (3.8) and (3.9), this yields

$$\lambda\dot{\lambda} = l(\tilde{H}(\lambda\tilde{y})). \tag{3.10}$$

A crucial point is now that the function $\lambda(\cdot)$ defined in (3.6) being a homeomorphism of $[0, T]$ to $[0, \lambda_0]$ and a diffeomorphism of $(0, T]$ to $(0, \lambda_0]$, one has

$$\tilde{y}(t) = \tilde{u}(\lambda(t)), \tag{3.11}$$

with $\tilde{u} \in \mathscr{C}^0([0, \lambda_0]) \cap \mathscr{C}^1((0, \lambda_0])$. Upon differentiating (3.11), one gets $\dot{\tilde{y}} = (d\tilde{u}/d\lambda)\, \dot{\lambda}$ so that (3.9) may be rewritten as $\lambda\dot{\lambda}(\tilde{u} + \lambda\, d\tilde{u}/d\lambda) = \tilde{H}(\lambda\tilde{u})$. But (3.10) and (3.11) give $\lambda\dot{\lambda}$ in terms of λ and $\tilde{u}(\lambda)$, namely,

$$\lambda\dot{\lambda} = l(\tilde{H}(\lambda\tilde{u})), \tag{3.12}$$

and hence

$$\tilde{u} + \lambda \frac{d\tilde{u}}{d\lambda} = \frac{d}{d\lambda}(\lambda\tilde{u}) = \frac{\tilde{H}(\lambda\tilde{u})}{l(\tilde{H}(\lambda\tilde{u}))}. \tag{3.13}$$

While Eq. (3.12) goes along with the initial condition $\lambda(0) = 0$, no condition except continuity is required of $\tilde{u}$ as in (3.13) at the origin.

In summary, the problem of finding the transversal solutions to Eq. (2.12) when x_0 is a noncritical singular point reduces to finding the transversal solutions to Eq. (3.4). In turn, these solutions can be determined by splitting the problem into finding the solutions $\tilde{u} \in \mathscr{C}^0([0, \lambda_0]) \cap \mathscr{C}^1((0, \lambda_0])$ to Eq. (3.13) and, next, solving Eq. (3.12) for $\lambda \in \mathscr{C}^0([0, T]) \cap \mathscr{C}^1((0, T])$ and initial condition $\lambda(0) = 0$.

4. Solutions about Standard Singular Points

In this section, we shall take up the problem of solving Eq. (3.4) when 0 is a standard singular point, namely $l(\tilde{H}(0)) \neq 0$. If so, the right-hand side of Eq. (3.13) is defined without ambiguity at the origin. Of course, our approach will be as summarized at the end of Section 3.

Solving Eq. (3.13) is straightforward. Indeed, setting $\lambda\tilde{u}(\lambda) = \tilde{v}(\lambda)$, one has, equivalently

$$\frac{d\tilde{v}}{d\lambda} = \frac{\tilde{H}(\tilde{v})}{l(\tilde{H}(\tilde{v}))}.$$

If $\tilde{H}$ is of class $\mathscr{C}^1$ (or, more generally, Lipschitz-continuous) near the origin and $\tilde{v}(0)$ is given, this equation has a unique solution $\tilde{v} \in \mathscr{C}^1([-\lambda_0, \lambda_0])$ where $\lambda_0 > 0$ is small enough. But $\tilde{u}(\lambda) = \tilde{v}(\lambda)/\lambda$ is continuous at the origin if and only if $\tilde{v}(0) = 0$. Existence and uniqueness of a solution $\tilde{u} \in \mathscr{C}^0([-\lambda_0, \lambda_0])$ to (3.13) follows. Obviously, $\tilde{u}$ is $\mathscr{C}^1$ away from the origin. Note also that $\tilde{u}(0) = (d\tilde{v}/d\lambda)(0)$, namely

$$\tilde{u}(0) = \frac{\tilde{H}(0)}{l(\tilde{H}(0))}.$$

With $\tilde{u}$ as has just been obtained, the next step consists in solving the equation

$$\lambda\dot{\lambda} = l(\tilde{H}(\lambda\tilde{u}(\lambda))),$$
$$\lambda(0) = 0.$$

Since this equation also reads

$$\frac{d}{dt}(\lambda^2) = 2l(\tilde{H}(\lambda\tilde{u}(\lambda))), \tag{4.1}$$

and from $l(\tilde{H}(0)) \neq 0$, a solution $\lambda \in \mathscr{C}^0([0, T])$ with $T > 0$ can only exist if $l(\tilde{H}(0)) > 0$. Similarly, a continuous solution can exist for negative T only if $l(\tilde{H}(0)) < 0$. Therefore, the sign of $l(\tilde{H}(0))$ determines whether solutions must be sought over $[0, T]$ with $T > 0$ or $T < 0$. In what follows, we shall assume $l(\tilde{H}(0)) > 0$, so that solutions may exist on $[0, T]$ with $T > 0$ only.

Since (4.1) ensures that λ^2 is necessarily a strictly increasing $\mathscr{C}^1$ function on $[0, T]$ it follows from $\lambda(0)=0$ that $\lambda(t)$ is either positive or negative on $(0, T]$ (assuming $T>0$ small enough). In other words, $\lambda(t)=(\lambda^2(t))^{1/2}$ or $\lambda(t)=-(\lambda^2(t))^{1/2}$ on the entire interval $[0, T]$. Thus, the problem comes down to solving the equation

$$\begin{aligned} \dot{\mu} &= 2l(\tilde{H}(\pm\mu^{1/2}\,\tilde{u}(\pm\mu^{1/2}))), \\ \mu(0) &= 0. \end{aligned} \tag{4.2}$$

A solution $\lambda=\mu^{1/2}$ (resp. $\lambda=-\mu^{1/2}$) is obtained when the plus (resp. minus) sign is chosen in (4.2), and all the solutions λ have this form. Existence of at least one solution to Eq. (4.2), for either choice of sign in it, follows from the fact that the right-hand side is a continuous function of $\mu\in[0, \mu_0]$ with $\mu_0>0$ small enough. More precisely, to make standard results available (see, e.g., [10]), one may first extend the right-hand side of (4.2) for negative values of μ by setting it equal to $2l(\tilde{H}(0))$. This does not affect continuity and one gets existence of a $\mathscr{C}^1$ solution defined on some neighborhood of the origin, which satisfies (4.2) for $t>0$ from $\mu(0)=0$ and $\dot{\mu}$ being positive near the origin.

In general, continuity of the right-hand side of an explicit differential equation does not guarantee uniqueness of the solution. Uniqueness is standard only under hypotheses of local Lipschitz continuity which are not fulfilled by the right-hand side of (4.2). Showing that uniqueness is true here will then require a little extra work. Consider for instance the case when the plus sign is chosen in (4.2), namely μ is characterized by

$$\begin{aligned} \dot{\mu} &= g(\mu^{1/2}), \\ \mu(0) &= 0, \end{aligned} \tag{4.3}$$

with

$$g(\lambda)=l(\tilde{H}(\lambda\tilde{u}(\lambda))). \tag{4.4}$$

The function g is $\mathscr{C}^1$ near the origin as soon as $\tilde{H}$ is $\mathscr{C}^1$. Indeed, $\tilde{u}$ is $\mathscr{C}^1$ away from the origin and so is then g. Next, for $\lambda\neq 0$

$$(dg/d\lambda)(\lambda)=l(\tilde{H}'(\lambda\tilde{u}(\lambda))\,\tilde{H}(\lambda\tilde{u}(\lambda)))/l(\tilde{H}(\lambda\tilde{u}(\lambda))),$$

as is easily seen from $\tilde{u}$ solving Eq. (3.13). Thus, since $\tilde{u}$ is continuous at the origin and from $\tilde{u}(0)=\tilde{H}(0)/l(\tilde{H}(0))$,

$$\lim_{\lambda\to 0}\frac{dg}{d\lambda}(\lambda)=l(\tilde{H}'(0)\,\tilde{H}(0))/l(\tilde{H}(0)).$$

On the other hand,

$$\frac{1}{\lambda}(g(\lambda)-g(0))=\frac{1}{\lambda}l(\tilde{H}(\lambda\tilde{u}(\lambda))-\tilde{H}(0))$$
$$=l\left(\int_0^1 \tilde{H}'(s\lambda\tilde{u}(\lambda))\,\tilde{u}(\lambda)\,ds\right).$$

Thus, $\tilde{u}$ being continuous one finds, as desired

$$\lim_{\lambda\to 0}\frac{1}{\lambda}(g(\lambda)-g(0))=l(\tilde{H}'(0)\,\tilde{u}(0))$$
$$=l(\tilde{H}'(0)\,\tilde{H}(0))/l(\tilde{H}(0)).$$

If μ_1 and μ_2 are two solutions to (4.3) defined on $[0, T]$, and $T>0$ is small enough for both $\mu_1(\cdot)$ and $\mu_2(\cdot)$ to be positive and increasing on $[0, T]$, it follows that

$$|\mu_1(t)-\mu_2(t)|\leqslant Ct\max_{s\in[0,T]}|\mu_1^{1/2}(s)-\mu_2^{1/2}(s)|, \tag{4.5}$$

where $C>0$ is a constant depending only on g' that can be taken independent of $T>0$ small enough. If μ_1 and μ_2 do not coincide on $[0, T]$, one has

$$\max_{s\in[0,T]}|\mu_1^{1/2}(s)-\mu_2^{1/2}(s)|=|\mu_1^{1/2}(t_0)-\mu_2^{1/2}(t_0)|>0, \tag{4.6}$$

for some $t_0\in(0, T]$. Taking $t=t_0$ in (4.5) and using (4.6), one gets

$$\mu_1^{1/2}(t_0)+\mu_2^{1/2}(t_0)\leqslant Ct_0. \tag{4.7}$$

Now, from (4.3), it is clear that

$$\mu_j(t)=tg(0)+o(t), \qquad j=1, 2,$$

for $t\in[0, T]$ small enough. As $g(0)>0$ by hypothesis, it can be assumed that $T>0$ is so small as

$$\mu_j(t)\geqslant\frac{t}{2}g(0), \qquad \forall t\in[0, T].$$

In particular, for $t=t_0$ and due to (4.7), one finds $[2g(0)]^{1/2}\,t_0^{1/2}\leqslant Ct_0$. As $t_0>0$, this means $t_0\geqslant 2g(0)/C^2$. But $t_0\leqslant T$ and a contradiction is reached provided that T is chosen according to $T<2g(0)/C^2$ in the first place (recall that C is independent of $T>0$ small enough). Of course, $g(0)\neq 0$ is essential to uniqueness: $g(\lambda)=\lambda$ yields an obvious counterexample to uniqueness in (4.3).

The above shows that any two solutions to (4.3) coincide on some interval $[0, T]$ with $T>0$. Then, they coincide as long as they are defined since the function $g(\mu^{1/2})$ is locally Lipschitz continuous away from the origin. The same arguments obviously apply when the minus sign is chosen in (4.2).

Regarding Eq. (3.4) with $\tilde{H}$ being $\mathscr{C}^1$ and $l(\tilde{H}(0)) \neq 0$, we thus have that it possesses exactly two transversal[2] solutions in $\mathscr{C}^0([0, T]) \cap \mathscr{C}^1((0, T])$, both defined either for $T>0$ or for $T<0$ with $|T|$ small enough. Moreover, whether $T>0$ or $T<0$ depends only on the sign of $l(\tilde{H}(0))$. The two solutions $\tilde{x}(t)$ are distinct, which follows from $\tilde{x}(t) = \lambda(t)\, \tilde{u}(\lambda(t))$, and $\lambda(t) = \pm \mu^{1/2}(t)$ according to the choice for the sign in (4.2). Together with $\tilde{u}(0) = \tilde{H}(0)/l(\tilde{H}(0)) \neq 0$, this shows that the two solutions $\tilde{x}(t)$ start tangent to $\tilde{u}(0)$, but in opposite directions. Also, the derivative $\dot{\tilde{x}}(t)$ blows up as t tends to 0. To see this, it suffices to show that $l(\dot{\tilde{x}}(t))$ tends to infinity as t tends to 0. But this is obvious from $l(\dot{\tilde{x}}(t)) = \dot{\lambda}(t)$ (cf. (3.6)) and $2\lambda(t)\, \dot{\lambda}(t) = \dot{\mu}(t)$ with $\lambda(0) = 0$ and μ solving Eq. (4.2) so that $\dot{\mu}(0) \neq 0$. These results are summarized in

THEOREM 4.1. *Assume that $\tilde{H}$ is class $\mathscr{C}^1$ and $l(\tilde{H}(0)) \neq 0$ (i.e. 0 is a standard singular point). Then, Eq. (3.4) has exactly two transversal solutions in $\mathscr{C}^0([0, T]) \cap \mathscr{C}^1((0, T])$, both defined either for $T>0$ or for $T<0$ depending only on the sign of $l(\tilde{H}(0))$, and with $|T|>0$ small enough. These solutions are distinct and their derivatives blow up at $t=0$.*

Figure 4.1 provides a schematic representation of the transversal solutions to Eq. (3.4) in the assumptions of Theorem 4.1. As usual, the arrows represent evolution in increasing time.

To complete the study of Eq. (3.4) in the hypotheses of Theorem 4.1, it remains to examine whether nontransversal solutions may exist. A negative answer is given in

THEOREM 4.2. *In the assumptions of Theorem 4.1, Eq. (3.4) has no nontransversal solution $\tilde{x} \in \mathscr{C}^0([0, T]) \cap \mathscr{C}^1((0, T])$.*[3]

Proof. Let $\tilde{x} \in \mathscr{C}^0([0, T]) \cap \mathscr{C}^1((0, T])$ denote any solution of Eq. (3.4) with $|T|>0$ arbitrarily small. According to Definition 3.1, one must first check that $l(\dot{\tilde{x}}(t)) \neq 0$ for $t \neq 0$ and $|t|$ small enough. This is immediate from $l(\tilde{x})\dot{\tilde{x}} = \tilde{H}(\tilde{x})$, yielding

$$l(\tilde{x}(t))\, l(\dot{\tilde{x}}(t)) = l(\tilde{H}(\tilde{x}(t))), \tag{4.8}$$

[2] Recall that the procedure we have followed is justified for transversal solutions only.

[3] For any $|T|>0$, but this is obvious from transversality being independent of the interval of definition.

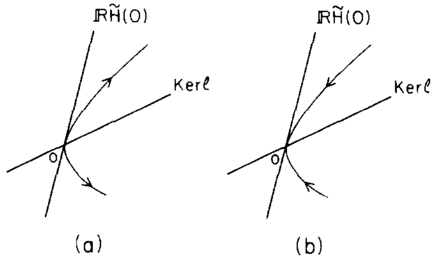

FIG. 4.1. Transversal solutions of Eq. (3.4).

and from $l(\tilde{H}(0)) \neq 0$. Proving that the limit

$$\lim_{t \to 0} \frac{\tilde{x}(t)}{l(\tilde{x}(t))}, \tag{4.9}$$

necessarily exists requires slightly more sophisticated arguments. To begin with, (4.8) may be rewritten as

$$\frac{d}{dt}(l(\tilde{x}))^2 = 2l(\tilde{H}(\tilde{x})) \qquad \text{on} \quad (0, T],$$

so that $(l(\tilde{x}))^2$ is $\mathscr{C}^1$ on $(0, T]$. From sign considerations, it easily follows that $(l(\tilde{x}(t)))^2$, hence $\tilde{x}(t)$, is defined only for $t \geqslant 0$ (resp. $t \leqslant 0$) if $l(\tilde{H}(0)) > 0$ (resp. < 0). In the sequel, we assume $l(\tilde{H}(0)) > 0$. Therefore,

$$l(\tilde{x}(t)) = \pm (\tilde{\alpha}(t))^{1/2}, \tag{4.10}$$

where $\tilde{\alpha}$ is the $\mathscr{C}^1$ function of $t \in [0, T] (T > 0)$

$$\tilde{\alpha}(t) = \int_0^t 2l(\tilde{H}(\tilde{x}(s)))\, ds.$$

If T is small enough, the function $\tilde{\alpha}$ is positive on $(0, T]$ and strictly increasing. In particular, continuity requires the sign in (4.10) to be the same on the entire interval $[0, T]$. Suppose for instance that

$$l(\tilde{x}(t)) = (\tilde{\alpha}(t))^{1/2}, \tag{4.11}$$

so that the equation $l(\tilde{x})\, \dot{\tilde{x}} = \tilde{H}(\tilde{x})$ may be rewritten as $\dot{\tilde{x}} = \tilde{H}(\tilde{x})/\tilde{\alpha}^{1/2}$. Introducing the function

$$\tilde{\beta}(t) = \frac{2\tilde{H}(\tilde{x}(t))}{\dot{\tilde{\alpha}}(t)},$$

defined and continuous on $[0, T]$ with $\tilde{\beta}(0) = H(0)/l(\tilde{H}(0))$, one has

$$\dot{\tilde{x}} = \tilde{\beta} \frac{d}{dt}(\tilde{\alpha}^{1/2}). \tag{4.12}$$

Hence, using (4.11), one finds

$$\frac{\tilde{x}(t)}{l(\tilde{x}(t))}-\frac{\tilde{H}(0)}{l(\tilde{H}(0))}=\frac{\tilde{x}(t)}{(\tilde{\alpha}(t))^{1/2}}-\tilde{\beta}(0)$$
$$=\frac{1}{(\tilde{\alpha}(t))^{1/2}}\int_0^t [\tilde{\beta}(s)-\tilde{\beta}(0)]\frac{d}{dt}(\tilde{\alpha}^{1/2})(s)\,ds. \quad (4.13)$$

In this formula, convergence of the integral $\int_0^t \tilde{\beta}(s)\,d/dt\,(\tilde{\alpha}^{1/2})(s)\,ds$ is guaranteed by continuity of $\tilde{x}$ at the origin since, from (4.12)

$$\tilde{x}(t)-\tilde{x}(\varepsilon)=\int_\varepsilon^t \tilde{\beta}(s)\frac{d}{dt}(\tilde{\alpha}^{1/2})(s)\,ds,$$

for every $\varepsilon>0$ small enough. Since $\tilde{\alpha}$, hence $\tilde{\alpha}^{1/2}$, is increasing on $[0, T]$, relation (4.13) yields a fortiori

$$\left|\frac{\tilde{x}(t)}{l(\tilde{x}(t))}-\frac{\tilde{H}(0)}{l(\tilde{H}(0))}\right|\leqslant \max_{s\in[0,t]}|\tilde{\beta}(s)-\tilde{\beta}(0)|,$$

so that existence of the limit in (4.9) (i.e. $\tilde{H}(0)/l(\tilde{H}(0))$) follows from continuity of $\tilde{\beta}$ at the origin. The same conclusion is reached if $l(\tilde{x}(t))=-(\tilde{x}(t))^{1/2}$ (compare with (4.11)) and also if $l(\tilde{H}(0))<0$. ∎

5. Interpretation of the Results

Combining Proposition 3.1 and Theorems 4.1 and 4.2 immediately yields a statement regarding Eq. (2.6) when x_0 is a standard singular point. However, this statement makes references to the sign of $l(\tilde{H}(0))$ where $l=f'(x_0)$ and $\tilde{H}(0)=H(x_0)=C(x_0)\,G(x_0)$ with $f(x)=\det A(x)$ and $C(x_0)=\operatorname{adj} A(x_0)$. The sign of $l(\tilde{H}(0))$ will now be determined in terms of the data in Eq. (2.6). Recall, as a result of x_0 being a standard singular point, that $H(x_0)$ is a nonzero element e_0 of $\operatorname{Ker} A(x_0)$ (cf. Section 2). Also, in the proof of Proposition 2.1, we obtained that $l(e_0)\,e_0=C(x_0)(A'(x_0)\,e_0)\,e_0$. Hence

$$\operatorname{sgn} l(\tilde{H}(0))=\operatorname{sgn}(C(x_0)(A'(x_0)\,e_0)\,e_0, e_0)$$
$$=\operatorname{sgn}((A'(x_0)\,e_0)\,e_0, C(x_0)^T e_0).$$

Note that $C(x_0)^T e_0$ is a (nonzero) element e_0^* of $\operatorname{Ker} A(x_0)^T$. This follows from $C(x_0)^T$ being the adjugate of $A(x_0)^T$ and $\dim \operatorname{Ker} A(x_0)^T=1$. In this notation

$$\operatorname{sgn} l(\tilde{H}(0))=\operatorname{sgn}((A'(x_0)\,e_0)\,e_0, e_0^*). \quad (5.1)$$

On the other hand, $e_0(=H(x_0))=C(x_0)\,G(x_0)$, so that

$$(G(x_0), e_0^*)=(G(x_0), C(x_0)^T e_0)$$
$$=(C(x_0)\,G(x_0), e_0)=(e_0, e_0)>0.$$

Using this, relation (5.1) is equivalent to

$$\operatorname{sgn} l(\tilde{H}(0))=\operatorname{sgn}(G(x_0), e_0^*)(A'(x_0)\,e_0)\,e_0, e_0^*). \tag{5.2}$$

In this form, the right-hand side is independent of the nonzero elements e_0 and e_0^* of $\operatorname{Ker} A(x_0)$ and $\operatorname{Ker} A(x_0)^T$, respectively, which can then be arbitrarily chosen. With this, we can now state

THEOREM 5.1. *Suppose that A and G are of class $\mathscr{C}^1$ and that x_0 is a standard singular point. Then, Eq. (2.6) has exactly two solutions in $\mathscr{C}^0([0, T])\cap\mathscr{C}^1((0, T])$, both defined for $T>0$ or $T<0$ depending only on the sign of $(G(x_0), e_0^*)((A'(x_0)\,e_0)\,e_0, e_0^*)$ where e_0 and e_0^* are arbitrary nonzero elements of* $\operatorname{Ker} A(x_0)$ *and* $\operatorname{Ker} A(x_0)^T$, *respectively, and provided that $|T|>0$ is small enough. Moreover, these solutions are distinct and their derivatives blow up at $t=0$.*

Recall that the conditions characterizing x_0 as a standard singular point, expressed in terms of A and G, are listed in Proposition 2.1. Also, the solutions are defined for $T>0$ (resp. $T<0$) if $(G(x_0), e_0^*)((A'(x_0)\,e_0)\,e_0, e_0^*)>0$ (resp. <0). The solutions can be pictured as in Fig. 4.1, except that 0 is replaced by x_0 and Ker l by the hypersurface of singular points of $A(x)$ near x_0. The "tangent" line $\mathbf{R}\tilde{H}(0)=\mathbf{R}H(x_0)=\mathbf{R}C(x_0)\,G(x_0)$ has only to be shifted through x_0.

The constrast between the two diagrams in Fig. 5.1 is evident. On Fig. 5.1 (a), the point x_0 cannot be reached in increasing time: no solution to (2.6) starting at any point will ever go through the "repelling" point x_0 at any later time. The same comment is true regarding every singular point near x_0 since such a point is necessarily standard and the sign condition $(G(x_0), e_0^*)((A'(x_0)\,e_0)\,e_0, e_0^*)>0$ is unaffected by small variations of x_0.

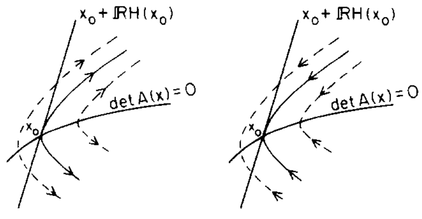

FIG. 5.1. Solutions to Eq. (2.16) near a standard singular point.

This is because the one dimensional null-space varies smoothly with the standard singular point. While such points are harmless to evolution in positive time, just the opposite is true for those at which $(G(x_0), e_0^*)$ $((A'(x_0)e_0)e_0, e_0^*) < 0$. Indeed, any point on the solid trajectory pictured on Fig. 5.1 (b) will reach the "attracting" point x_0 in finite time and *cannot continue beyond it.* In practice, when Eq. (2.6) represents a mathematical model, this means that a "catastrophe" of some kind occurs as x_0 is met and that, in any case, further evolution of the system is not governed by Eq. (2.6) alone. Again, the same conclusion is true if x_0 is replaced by any neighboring singular point. The main consequence of this remark is that the set of those initial data for which the aforementioned "catastrophe" will happen at some later time (i.e. an "attracting" standard singular point like x_0 will be met) is certainly not negligible. Moreover, since existence of standard singular points fulfilling the sign condition of Fig. 5.1 (b) is unaffected by small perturbations of A and G, their presence can only mean—assuming the model is accurate—that sudden and discontinuous phenomena must take place in finite time for a non negligible set of initial data.

The above comments are valid in particular when Eq. (2.6) is derived from an equation such as (2.1). The condition ensuring that (t_0, x_0, y_0) is a standard singular point are listed in (2.15), (2.16), and (2.17). An elementary calculation shows that the corresponding discussion is based on the sign of the quantity

$$-(\partial_{yy}F(t_0, x_0, y_0)e_0^2, e_0^*)(\partial_t F(t_0, x_0, y_0) + \partial_x F(t_0, x_0, y_0)y_0, e_0^*),$$

where e_0 and e_0^* denote arbitrary nonzero elements of Ker $\partial_y F(t_0, x_0, y_0)$ and $[\text{Range } \partial_y F(t_0, x_0, y_0)]^\perp$, respectively.

Two things must be mentioned regarding the solutions to Eq. (2.1) near a standard singular point, which are not a priori evident from the reading of Theorem 5.1. First, $F \in \mathscr{C}^2$ is required and the solutions obtained are of class $\mathscr{C}^1$ on their interval of definition $[t_0, t_0 + T]$. This is not in contradiction with Theorem 5.1 since the variable x in (2.12) represents the

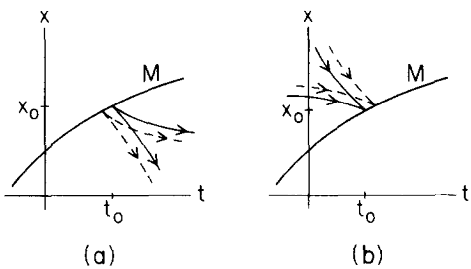

FIG. 5.2. Solutions to Eq. (2.1) in the (t, x)-space.

triple (t, x, y), where y occupies the slot of the derivative $\dot{x}$ in Eq. (2.1). A solution $x \in \mathscr{C}^0([0, T])$ thus means a solution $(t, x, \dot{x}) \in \mathscr{C}^0([t_0, t_0 + T])$ of Eq. (2.1). Accordingly, the component x is $\mathscr{C}^1$ on $[t_0, t_0 + T]$ and $\mathscr{C}^2$ away from t_0, and its second derivative blows up at $t = t_0$. Next, Fig. 5.1 represents the curves $(t, x(t), \dot{x}(t))$, so that $(t, x(t))$ is recovered after projection onto the (t, x)-space. This introduces a singularity, for the "tangent" line $\mathbf{R}H(x_0)$ corresponds with the "vertical" line $\{(0, 0)\} \times \operatorname{Ker} \partial_y F(t_0, x_0, y_0)$. This results in the diagram for $(t, x(t))$ having the shape of a semi-cubic parabola as indicated in Fig. 5.2 above, in agreement with what is known for $n = 1$. The manifold M is the same as in Proposition 2.2. The multiform character of the equation justifies that the trajectories intersect those emanating/terminating at nearby singular points.

Above, we have emphasized the importance of the "attracting" standard singular points regarding evolution in increasing time. But it would be a mistake to underestimate the role of the "repelling" singular points in the dynamics. The following example is meant to duplicate the strange bifurcation-like phenomenon observed in a problem of metal forming by Hall and Rheinboldt already mentioned in the Introduction. The peculiar aspect is that trajectories originating at arbitrarily small perturbations of some symmetric initial condition all largely deviate from the symmetric trajectory after approximately the same time. A model equation exhibiting such a behavior is as follows.

Consider the sytem in the two real variables x_1, x_2

$$\begin{aligned} \dot{x}_1 &= x_1, \\ x_2 \dot{x}_2 &= x_2 + x_1^2, \end{aligned} \tag{5.3}$$

which has the form required in (2.6) upon setting $x = (x_1, x_2)$ and

$$A(x) = \begin{pmatrix} 1 & 0 \\ 0 & x_2 \end{pmatrix}, \qquad G(x) = \begin{pmatrix} x_1 \\ x_2 + x_1^2 \end{pmatrix}.$$

The singular points of this system are those with $x_2 = 0$, i.e. the points of the x_1-axis. Also, it is straightforward to check that all of them except the origin are standard and "repelling".

Now, associate with (5.3) the initial condition at $t = 0$ and the point $(0, b)$ of the x_2-axis. The first equation yields $x_1(t) \equiv 0$ and the second one has the solution $x_2(t) = t + b$. The corresponding trajectory is the half-line $[b, \infty)$ on the x_2-axis. In particular, if $b < 0$, the origin is reached at time $t = -b > 0$, when the trajectory enters the upper half-plane and remains in it.

Next, with $b < 0$ as above, associate with (5.3) the initial condition at

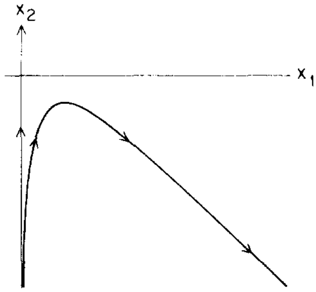

FIGURE 5.3

$t=0$ and the point (ε, b) with $|\varepsilon|>0$ arbitrarily small. The value of ε is the only difference with the previous initial conditions. The first equation yields $x_1(t)=\varepsilon e^t$. If $\varepsilon>0$ (resp. <0), the point $(x_1(t), x_2(t))$ thus moves to the right (resp. left) of (ε, b). Therefore, if the trajectory ever enters the upper half-plane, it must do so at a point of the x_1-axis with $|x_1|>\varepsilon$. But this is impossible from $(x_1,0)$ being a repelling point for $x_1 \neq 0$. As a result, the trajectory emanating at (ε, b) remains in the lower half-plane irrespective of $\varepsilon \neq 0$.

The examination of Fig. 5.3 above clearly shows why the phenomenon may be taken for a bifurcation—although it is not one—on the basis of numerical results. It also demonstrates that large discrepancies may be observed in finite time between an ideal and hence irrealistic situation and the slightest of its perturbation: the trajectory pictured on Fig. 5.3 that deviates from the vertical axis has been obtained with $b=-5$ and $\varepsilon=10^{-2}$.

6. REMARKS ON THE INFINITE-DIMENSIONAL CASE

Because our approach heavily relies on properties of determinants and adjugates, it seems that the method we have used previously cannot be extended to the infinite dimensional setting. Still, it is the purpose of this section to indicate how simple tools can be introduced that allow for an identical analysis. The key observation is that existence of determinants and adjugates was not used for arbitrary matrices, but only for those that are "close" to the matrix $A(0)$. This may be exploited as follows: let X be a Banach space and $A \in \mathscr{L}(X)$ Fredholm with index zero. Then, one can find a neighborhood of A in $\mathscr{L}(X)$ such that smooth (even analytic) mappings

$$\delta: \quad \mathscr{U} \to \mathbf{R},$$

$$\Delta: \quad \mathscr{U} \to \mathscr{L}(X),$$

exist that generalize the notions of determinant and adjugate, respectively. This generalization possesses the property that, for $B \in \mathscr{U}$, one has $\delta(B) \neq 0$ if and only if $B \in \text{Isom}(X)$ and

$$\delta(B)\, I = \Delta(B)\, B = B\Delta(B), \qquad \forall\, B \in \mathscr{U}.$$

The pair (δ, Δ) is by no means unique and cannot be extended to the whole space $\mathscr{L}(X)$. To obtain a pair (δ, Δ) as above, one can use decompositions $X = X_1 \oplus X_2$, $X = Y_1 \oplus Y_2$ where $X_1 = \text{Ker}\, A$, $Y_2 = \text{Range}\, A$, and X_2 and Y_1 are topological complements of X_1 and Y_2, respectively. These decompositions allow for a decomposition of $B \in \mathscr{L}(X)$ in the form

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

As $B_{22} \in \text{Isom}(X_2, Y_2)$ for B close enough to A in $\mathscr{L}(X)$, appropriate choices for δ and Δ are dictated by performing a "block LU decomposition" of B under which B becomes block triangular. Details are omitted for brevity and will be presented elsewhere. As far as we are concerned here, the main properties of the pair (δ, Δ) are

$$\dim \text{Ker}\, A = 1 \Leftrightarrow \text{Ker}\, A = \text{Range}\, \Delta(A) \text{ and } \text{Range}\, A = \text{Ker}\, \Delta(A) \tag{6.1}$$

and

$$\dim \text{Ker}\, A = 1 \Leftrightarrow \delta'(A) \neq 0. \tag{6.2}$$

Going back to a parametrized family $A(x) \in \mathscr{L}(X)$ with $x \in X$ and $A(\cdot)$ being $\mathscr{C}^1$ near some point x_0 such that $A(x_0)$ is Fredholm with index zero, consider a pair (δ, Δ) as above associated with $A = A(x_0)$. For x near x_0, one may then set

$$f(x) = \delta(A(x)).$$

On the other hand, let $G(x) \in X$ with G being $\mathscr{C}^1$ near x_0. Setting

$$H(x) = \Delta(A(x))\, G(x),$$

one can derive from (6.1) and (6.2) that the analog of Proposition 2.1 holds, namely that $f'(x_0)\, H(x_0) \neq 0$ if and only if

(i) $\dim \text{Ker}\, A(x_0) = 1$,

(ii) $(A'(x_0)\, e_0)\, e_0 \notin \text{Range}\, A(x_0), \qquad \forall e_0 \in \text{Ker}\, A(x_0) - \{0\}$,

(iii) $G(x_0) \notin \text{Range}\, A(x_0)$.

The above equivalence is all that is needed to extend the definition of standard singular points and to follow exactly the same procedure as in Sections 3 and 4 to generalize the results to the Banach space setting. Caution must only be exercised regarding genericity statements. Indeed, for one thing, genericity involves a notion of measure that does not exist on an infinite dimensional Banach space. This difficulty can be overcome (Smale's density theorem; see Abraham and Robbin [1]) but, still, other problems arise when one wants to speak of generic mappings between Banach spaces. However, the basic fact remains that standard singular points form a smooth hypersurface in X since the implicit function theorem is still available.

The hypothesis $A(x) \in \mathscr{L}(X)$ can be generalized to $A(x) \in \mathscr{L}(X, Y)$ with X and Y different Banach spaces, while $G(x) \in Y$ for consistency. This is immediate, for existence of a Fredholm operator with index zero between X and Y (i.e. $A(x_0)$) guarantees that X and Y are isomorphic, and it is straightforward to check that none of the assumptions depends on the isomorphism used to identify X and Y.

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