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A Matricial Boundary Value Problem Which Appears in the Transport Theory

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We study the boundary value problem $T\phi' = -A\phi$, $(I-P)\,\phi(0)$, $P\phi(\tau)=0$, where $T=T^*$, $A=A^*$, $P^2=P$, $P^*T=TP$, T invertible. The motivation to consider such problem comes from the transport theory. The behavior of values of τ for which there is a nontrivial solution (exceptional values) is investigated using the indicator function. This is an analytic hermitian valued function of the real parameter t which reflects, in particular, the characteristic properties of exceptional values. © 1988 Academic Press, Inc.

1. INTRODUCTION

Let T and A be linear bounded selfadjoint operators in a Hilbert space H such that $\operatorname{Ker} T=\{0\}$ and $I-A$ is compact. Denote by P_+ (P_-) the orthogonal projection corresponding to the positive (negative) part of the spectrum of T (so that $P_\pm T = TP_\pm$, $P_+ + P_- = I$, $\sigma(T|_{\operatorname{Im} P_+})\subset[0,\infty)$, $\sigma(T|_{\operatorname{Im} P_-})\subset(-\infty,0]$). Consider the following boundary value problem:

$$(T\phi)'(t) = -A\phi(t)+f(t);\qquad 0<t<\tau;\tag{1.1}$$

$$\lim_{t\downarrow 0} P_+\phi(t)=\phi_+;\qquad \lim_{t\uparrow\tau} P_-\phi(t)=\phi_-;\qquad \tau>0 \text{ is given.}\tag{1.2}$$

Here $f(t)$ is a given H-valued function of the real variable t; $\phi_\pm\in\operatorname{Im} P_\pm$

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are given vectors. Concrete boundary value problems of type (1.1), (1.2) appear in the theory of radiation transfer and neutron transport (see, e.g., [2, 3]). Abstract boundary value problems of type (1.1), (1.2) have been extensively studied recently (see [8, 11]). For positive semidefinite A, it turns out that the problem (1.1), (1.2) has always a unique solution [8]. However, in the case A is not positive semidefinite, this is not always so (this case is physically meaningful and important; in the problem of radiation transfer, it reflects the presence of sources of radiation in the medium). So it is of interest to obtain information concerning those values of t for which the homogeneous boundary value problem

$$(T\phi)'(t) = -A\phi(t); \qquad 0 < t < \tau; \tag{1.3}$$

$$\lim_{t \downarrow 0} P_+ \phi(t) = 0, \qquad \lim_{t \uparrow \tau} P_- \phi(t) = 0 \tag{1.4}$$

has a nontrivial solution.

In this paper, we shall consider the boundary value problem (1.3), (1.4) in the finite dimensional case ($\dim H < \infty$). This case is important from the physical point of view (it appears in description of transport phenomena when only a finite number of scattering angles is feasible; see, e.g., [12]), and is, of course, much more tractable mathematically. We also believe that investigation of the finite dimensional case will help to understand better the infinite dimensional problem.

In fact, we shall study here the more general problem

$$T\phi'(t) = -A\phi(t), \qquad 0 < t < \tau \tag{1.5}$$

$$(I - P_-)\,\phi(0) = 0; \qquad P_-\,\phi(\tau) = 0, \tag{1.6}$$

where T and A are $n \times n$ hermitian matrices with invertible T, and P_- is a projection (not necessarily orthogonal) such that $P_-^* T = TP_-$. Clearly, (1.3), (1.4) (with finite dimensional H) is a particular case of (1.5), (1.6). The reason to consider the more general problem is that (1.5), (1.6) behaves well (i.e., transforms to a boundary value problem of the same type) under the transformation $T \to S^*TS$, $A \to S^*AS$, $P_- \to S^{-1}P_-S$ with invertible S, while (1.3), (1.4) does not.

The values of $\tau(>0)$ for which there is a nontrivial solution to (1.5), (1.6) will be called *exceptional values*. Observe that for the nonexceptional values, the corresponding inhomogeneous problem is well posed. We shall show in Section 2 that there is an analytic (on the real variable t) hermitian value matrix function $V(t)$ whose properties are intimately related to the properties of the exceptional values. In particular, the exceptional values are precisely the (real) zeros of $\det V(t)$. The function $V(t)$ will be called the *indicator* of the problem (1.5), (1.6). Recently obtained results on perturbations of analytic hermitian matrix functions (see [6]), when applied to the indicator, allow us to obtain certain facts concerning the behavior of exceptional values under small perturbations of the initial data T, A, and P_-.

Of particular interest, motivated by the transport theory background, are cases when A is semidefinite, or A has only one negative eigenvalue. These cases are investigated in detail in Sections 3 and 4. Here we assume, in addition, that the subspace $\operatorname{Im} P_-$ is maximal T-negative (which obviously holds in the boundary problem (1.3), (1.4)), and we take the full advantage of the possibility to apply the transformation $T \to S^*TS$, $A \to S^*AS$, $P_- \to S^{-1}P_-S$. One of the obtained results, perhaps not unexpected, is that for positive semidefinite A, there are no exceptional values. The underlying reason is that in this case the indicator is negative definite for $t=0$ and decreases monotonically for positive t.

We shall use the notation $\operatorname{diag}(Z_1, ..., Z_k)$ or $Z_1 \oplus Z_2 \oplus \cdots \oplus Z_k$ for the block diagonal matrix

$$\begin{bmatrix} Z_1 & 0 & \cdots & 0 \\ 0 & Z_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & Z_k \end{bmatrix}.$$

I_m stands for the $m \times m$ unit matrix. When convenient, $n \times m$ complex matrices will be considered as linear transformations $\mathbb{C}^m \to \mathbb{C}^n$ with the norm induced by the standard euclidean norms in $\mathbb{C}^n$ and $\mathbb{C}^m$.

2. The Indicator

Consider the problem (1.5), (1.6) where T and A are selfadjoint $n \times n$ matrices, $\operatorname{Ker} T = (0)$, and P_- is a projection with $P_-^* T = TP_-$. A number of $\tau > 0$ will be called *exceptional* if (1.5), (1.6) has a nonzero solution. Introduce the function

$$V(t) = P_-^* [Te^{-tT^{-1}A}] P_- : \operatorname{Im} P_- \to \operatorname{Im} P_-^*.$$

The function $V(t)$ will be called the *indicator* of the boundary value problem (1.5), (1.6); it will play a crucial role in our analysis of exceptional points. Observe that $V(0)$ is always invertible.

The significance of the indicator is immediately seen from the following theorem.

THEOREM 2.1. *The number $\tau > 0$ is exceptional if and only if the indicator*

$V(t)$ is not invertible for $t=\tau$. Moreover, the dimension of the space of solutions of (1.5), (1.6) *coincides with* dim Ker $V(\tau)$.

Proof. The general solution of $T\phi' = -A\phi$ is $\phi(t) = e^{-tT^{-1}A}\phi(0)$. Assume that $\phi(t)$ satisfies also the boundary conditions (1.6). Then

$$\phi(\tau) = e^{-\tau T^{-1}A}P_{-}\phi(0),$$

so

$$0 = P_{-}\phi(\tau) = P_{-}[e^{-\tau T^{-1}A}P_{-}]\,\phi(0),$$

which is equivalent to

$$0 = TP_{-}\phi(\tau) = TP_{-}[e^{-\tau T^{-1}A}P_{-}]\,\phi(0) = V(\tau)\,\phi(0). \tag{2.1}$$

If $\phi(\tau) \neq 0$, then $P_{-}\phi(0) = 0$, and (2.1) implies that $V(\tau)$ is not invertible. Conversely, if $V(\tau)$ is not invertible, then there is ϕ_{+} in Im P_{-} such that $\phi_{+} \neq 0$ but $V(\tau)\phi_{+} = 0$. Then

$$\phi(t) = e^{-tT^{-1}A}\phi_{+}$$

satisfies (1.5), (1.6) and is not identically zero.

The above argument proves also the second statement of the theorem. ∎

The following basic property of the indicator lies at the heart of our approach.

THEOREM 2.2. *Let $f_1, ..., f_p$ be an orthonormal basis in* Im P_{-}. *Then $P^*_{-}f_1, ..., P^*_{-}f_p$ is a basis in* Im P^*_{-}, *and with respect to these bases, the indicator $V(t)$ is a hermitian matrix for all $t > 0$.*

Theorem 2.2 follows from a general statement on hermitian matrices and projections:

PROPOSITION 2.3. *Let B be a hermitian $n \times n$ matrix, and let P be a projection. If $f_1, ..., f_p$ is an orthonormal basis in* Im P, *then $P^*f_1, ..., P^*f_p$ is a basis in* Im P^*, *and with respect to these bases the linear transformation P^*BP:* Im $P \to$ Im P^* *is given by a hermitian matrix.*

Proof. If $P^*f_1, ..., P^*f_p$ were not a basis in Im P^*, then $\langle P^*f_j, g\rangle = 0$ for some $g \in$ Im P^*, $g \neq 0$. So

$$\langle f_j, Pg\rangle = 0 \qquad \text{for} \quad j = 1, ..., p$$

and since $f_1, ..., f_p$ is a basis in Im P, this implies $Pg = 0$. We have obtained a contradiction with the fact that Ker P and Im P^* are orthogonal complements to each other.

For the proof of the second part of the proposition, observe first that without loss of generality, we can assume that

$$P = \begin{bmatrix} I & 0 \\ X & 0 \end{bmatrix}. \tag{2.2}$$

(Let us justify this step. Let $g_1, ..., g_n$ be a basis in $\mathbb{C}^n$ such that $g_1, ..., g_p$ is a basis in Ker P and $g_{p+1}, ..., g_n$ is a basis in Im P. Let $h_1, ..., h_n$ be the orthonormal basis obtained from $g_1, ..., g_n$ by Gram–Schmidt orthogonalization. In this basis, P has the form $P = \left[\begin{smallmatrix} 0 & X \\ 0 & B \end{smallmatrix}\right]$. Since $P^2 = P$ we have $\left[\begin{smallmatrix} X \\ B \end{smallmatrix}\right] B = \left[\begin{smallmatrix} X \\ B \end{smallmatrix}\right]$, i.e., $\left[\begin{smallmatrix} X \\ B \end{smallmatrix}\right](B - I) = 0$. Since $\operatorname{Ker}\left[\begin{smallmatrix} X \\ B \end{smallmatrix}\right] = 0$ it follows that $B = I$. Interchanging the order of the basis elements yields $P = \left[\begin{smallmatrix} I & 0 \\ X & 0 \end{smallmatrix}\right]$). Then

$$[f_1, f_2, ..., f_p] = \begin{bmatrix} I \\ X \end{bmatrix} U,$$

where U is an invertible matrix such that

$$U^*[I\, X^*] \begin{bmatrix} I \\ X \end{bmatrix} U = I.$$

Now, write

$$B = \begin{bmatrix} B_1 & B_2 \\ B_2^* & B_3 \end{bmatrix}, \qquad B_1 = B_1^*, \qquad B_3 = B_3^*$$

with respect to the same orthogonal partition of $\mathbb{C}^n$ as in (2.2). Then

$$P^*BP[f_1, ..., f_k] = \begin{bmatrix} (B_1 + B_2X + X^*B_2 + XB_3^*X)\, U \\ 0 \end{bmatrix}.$$

On the other hand,

$$P^*[f_1, ..., f_k] = \begin{bmatrix} (I + X^*X)\, U \\ 0 \end{bmatrix} = \begin{bmatrix} U^{*-1} \\ 0 \end{bmatrix}.$$

So the matrix $A = [a_{ij}]_{i,j=1}^k$ which represents P^*BP with respect to the bases $f_1, ..., f_p$ and $P^*f_1, ..., P^*f_p$ is defined by the equalities

$$\begin{bmatrix} (B_1 + B_2X + X^*B_2^* + X^*B_3X) \\ 0 \end{bmatrix} u_j = \begin{bmatrix} U^{*-1} \\ 0 \end{bmatrix} \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{pj} \end{bmatrix},$$

where u_j is the jth column in U. It follows that

$$A = U^*(B_1 + B_2 X + X^* B_2^* + X^* B_3 X) U,$$

which is hermitian. ▮

The indicator is an analytic matrix function which is invertible at $t=0$. Hence, the number of real zeros of its determinant (which are exactly the exceptional values) is either finite of countable, and in the latter case, the only accumulation point of the set of exceptional values is at infinity. A simple sufficient condition for finiteness of the number of exceptional values is that all eigenvalues of $T^{-1}A$ be real. Indeed, in this case case it is easily seen that the determinant of the indicator $V(t)$ is a linear combination of functions of the type $t^j e^{-\mu t}$. Let μ_0 be the smallest among the numbers μ in the exponent of functions $t^j e^{-\mu t}$ which appear in det $V(t)$ with a nonzero coefficient. Then

$$e^{\mu_0 t} \det V(t) = p(t) + q(t),$$

where $p(t)$ is a polynomial, not identically zero, and $q(t)$ is analytic and tends to zero as t goes to infinity. Now it is clear that the number of (real) zeros of det $V(t)$ is finite.

3. Perturbations of Exceptional Values

Let $V(t)$ be the indicator of the boundary value problem (1.5), (1.6). Since $V(t)$ is hermitian and analytic (as a function of the real variable t), by Rellich's theorem (see, e.g., [9] or Chap. A.6 in [4]) there is a representation

$$V(t) = U(t)^{-1} \begin{bmatrix} \mu_1(t) & 0 & \cdots & 0 \\ 0 & \mu_2(t) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \mu_p(t) \end{bmatrix} U(t), \tag{3.1}$$

where $U(t)$ is an analytic matrix function of the real variable t which takes unitary values and $\mu_1(t), ..., \mu_p(t)$ are real analytic functions of t in $\mathbb{R}$. Clearly, $t>0$ is an exceptional point if and only if τ is a zero of at least one of the functions $\mu_j(t)$. For a given exceptional value τ let $\mu_{i_1}(t), ..., \mu_{i_r}(t)$ be all the functions among $\mu_1(t), ..., \mu_p(t)$ which vanish at τ (here $r = r(\tau)$ depends on τ). Denote by $m_1 = m_1(\tau), ..., m_r = m_r(\tau)$ the positive integers defined by the equalities

$$\mu_{i_q}^{(\alpha)}(\tau) = \begin{cases} 0 & \text{for} \quad \alpha = 0, ..., m_q - 1; \\ \neq 0 & \text{for} \quad \alpha = m_q. \end{cases}$$

The integers $m_1, \ldots, m_r$ are called the *partial multiplicities* of $V(t)$ at τ; their number r is exactly dim Ker $V(\tau)$ which coincides with the dimension of the set of solutions of (1.5), (1.6). The signs $\varepsilon_1 = \varepsilon_1(\tau), \ldots, \varepsilon_r = \varepsilon_r(\tau)$ ($+1$ or -1) of the nonzero real numbers $\mu_{i_1}^{(m_1)}(\tau), \ldots, \mu_{i_r}^{(m_r)}(\tau)$ form the *sign characteristic* of $V(t)$ at τ. For more information and other descriptions of the sign characteristic of meromorphic hermitian matrix functions, see [5]. We shall need also the quantity

$$\begin{aligned} s(\tau) = \{i \mid 1 \leqslant i \leqslant r, m_i \text{ is odd}, \varepsilon_i = +1\}^{\#} \\ - \{i \mid 1 \leqslant i \leqslant r, \quad m_i \text{ is odd}, \varepsilon_i = -1\}^{\#}, \end{aligned} \tag{3.2}$$

where by $\Delta^{\#}$ we denote the number of different elements in a finite set Δ.

We use now the notions of partial multiplicities and sign characteristic in order to describe the behaviour of exceptional values under perturbations of the initial data (i.e., T, A, and P_-) of the boundary value problem.

THEOREM 3.1. *Let τ be an exceptional value for the boundary value problem* (1.5), (1.6), *and let $\alpha < \tau < \beta$ be positive real numbers such that τ is the only exceptional value in (α, β). Then there is $\varepsilon > 0$ with the following property. For every system*

$$\begin{aligned} \tilde{T}\phi'(t) = -\tilde{A}\phi(t), \\ (I - \tilde{P}_-)\phi(0) = 0, \qquad \tilde{P}_-\phi(\tau) = 0 \end{aligned} \tag{3.3}$$

with

$$\tilde{T} = \tilde{T}^*, \qquad \tilde{A} = \tilde{A}^*, \qquad \tilde{P}_- = \tilde{P}_-^2, \qquad \tilde{P}_-^* \tilde{T} = \tilde{T}\tilde{P}_-$$

and

$$\|\tilde{T} - T\| + \|\tilde{A} - A\| + \|\tilde{P}_- - P_-\| < \varepsilon$$

the distinct exceptional values (if any) $\tilde{\tau}_1, \ldots, \tilde{\tau}_q$ of (3.3) *in* (α, β) *satisfy the inequality*

$$2 \sum_{j=1}^{q} \tilde{s}(\tilde{\tau}_j) = \operatorname{sig} V(\beta) - \operatorname{sig} V(\alpha), \tag{3.4}$$

where $\tilde{s}(\tau)$ is the quantity defined for (3.3) *analogously to* (3.2), *and* sig H *denotes the signature (i.e., the difference between the number of positive eigenvalues and the number of negative eigenvalues, counting multiplicities) of an invertible Hermitian matrix H.*

This theorem follows from a general result on perturbations of analytic hermitian matrix functions proved in [6, Theorem 2.1].

In particular, when sig $V(\beta) \neq$ sig $V(\alpha)$, there is always an exceptional value for the perturbed system (3.3) in (α, β). However, if sig $V(\alpha) =$ sig $V(\beta)$, then the perturbed system may have no exceptional values in (α, β) at all, as the following example shows.

EXAMPLE 3.1. Let

$$T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then, with respect to the orthonormal basis

$$\begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ -\frac{\sqrt{2}}{2} \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

(with which we shall work throughout this example) we have

$$Te^{-tT^{-1}A} = \begin{bmatrix} -1 + \frac{t^2}{4} & \frac{t\sqrt{2}}{2} & -\frac{t^2}{4} \\ \frac{t\sqrt{2}}{2} & 1 & -\frac{t\sqrt{2}}{2} \\ -\frac{t^2}{4} & -\frac{t\sqrt{2}}{4} & 1 + \frac{t^2}{4} \end{bmatrix}.$$

Now for a complex number x with $|x| \neq 1$ let

$$P_- = \alpha \begin{bmatrix} 1 & -\bar{x} & 0 \\ x & -|x|^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \alpha = (1 - |x|^2)^{-1}.$$

One easily checks that $P_-^2 = P_-$ and $P_-^* T = TP_-$. A straightforward computation shows that the indicator is

$$V(t) = V(t, x) = \frac{\alpha}{1 + |x|^2} \left[\frac{t^2}{4} + \sqrt{2}\, t \operatorname{Re} x + |x|^2 - 1 \right].$$

In particular,

$$V(t, -1 + i\sqrt{2}) = -\frac{1}{8} \left(\frac{t}{2} - \sqrt{2} \right)^2,$$

and clearly, for any x with

$$(\operatorname{Im} x)^2 - (\operatorname{Re} x)^2 - 1 > 0$$

the indicator $V(t, x)$ has no real zeros.

In the particular case when all partial multiplicities are equal to 1 (in such case we shall say that the exceptional value is *simple*) and all signs are the same, one can say more about the perturbed exceptional values.

THEOREM 3.2. *Let τ be a simple exceptional value for* (1.5), (1.6) *and assume that the signs $\varepsilon_i(\tau)$ are all equal. Let (α, β) be a real positive interval such that τ is the only exceptional value of* (1.5), (1.6) *in (α, β). Then for $\varepsilon > 0$ sufficiently small there are exactly* dim Ker $V(\tau)$ *exceptional values for* (3.3) *in the interval (α, β). Moreover, all these exceptional values are simple and the signs in their sign characteristics are all equal to the signs $\varepsilon_i(\tau)$.*

Theorem 3.2 is a particular case of Theorem 3.1.

If one restricts the class of admissible perturbations of the data of the boundary value problem, then sometimes it is possible to obtain more precise information about the perturbed exceptional values than in Theorem 3.1. For example:

THEOREM 3.3. *Let τ, α, β be as in Theorem* 3.1. *Let B be a hermitian matrix such that*

$$AT^{-1}B = BT^{-1}A \tag{3.5}$$

and $Be^{-\tau T^{-1}A}$ is negative definite (note that (3.5) *implies that $Be^{-\tau T^{-1}A}$ is selfadjoint). Then for $\varepsilon > 0$ small enough the exceptional values $\tau_1, ..., \tau_q$ (if any) of the boundary value problem*

$$T\phi'(t) = -(A + \varepsilon B)\,\phi(t); \qquad (I - P_-)\,\phi(0) = 0; \qquad P_-\phi(\tau) = 0 \tag{3.6}$$

which are in the interval (α, β), enjoy the following properties: Each τ_j is a simple exceptional value whose signs in the sign characteristic are all $+1$ if $\tau_j < \tau$ and all -1 if $\tau_j > \tau$. The number of τ_j's (counting multiplicities) which are smaller then τ is exactly

$$\{j \mid m_j(\tau) \textit{ is even}; \varepsilon_j(\tau) = -1\}^{\#} + \{j \mid m_j(\tau) \textit{ is odd}; \varepsilon_j(\tau) = +1\}^{\#}.$$

The number of τ_j's (counting multiplicities) which are bigger than τ is exactly

$$\{j \mid m_j(\tau) \textit{ is even}; \varepsilon_j(\tau) = -1\}^{\#} + \{j \mid m_j(\tau) \textit{ is odd}; \varepsilon_j(\tau) = -1\}^{\#}.$$

An analogous result holds if $Be^{-\tau T^{-1}A}$ is assumed to be positive definite.

In this case "+1" and "−1" in the statement of Theorem 3.3 should be interchanged.

Proof. Let $V_\varepsilon(t)$ be the indicator for (3.6). We claim that $V_\varepsilon(\tau) - V(\tau)$ is positive definite for sufficiently small $\varepsilon > 0$. Indeed, denote

$$\begin{aligned} f(\varepsilon) &= T\exp(-\tau T^{-1}(A+\varepsilon B)\} \\ &= Te^{-\tau T^{-1}A}e^{-\varepsilon\tau T^{-1}B}. \end{aligned}$$

Then

$$f'(\varepsilon)|_{\varepsilon=0} = Te^{-\tau T^{-1}A}(-\tau T^{-1}B) = -\tau Be^{-\tau T^{-1}A},$$

which is positive definite, and our claim follows. Now Theorem 3.3 follows from Theorem 5.1 in [6]. ▮

4. Semidefinite Matrix A

Throughout this and the next section, we consider the problem (1.5), (1.6) under the additional assumption that Im P_- is a *maximal T-negative* subspace. This means that, denoting by $\langle \cdot, \cdot \rangle$ the standard inner product in $\mathbb{C}^n$, we have $\langle Tx, x\rangle < 0$ for every x in Im $P_- \setminus \{0\}$ and that Im P_- is a maximal subspace in $\mathbb{C}^n$ with this property. Clearly, this is the case in the boundary value problem in transport theory, where Im P_- is spanned by all the eigenvectors of T corresponding to negative eigenvalues.

Note that the subspace Ker P_- is *maximal T-positive*. Indeed, suppose x in Ker $P_- \setminus \{0\}$ is such that $\langle Tx, x\rangle \leqslant 0$. Then, for all y in Im P_- we have

$$\langle Tx, y\rangle = \langle Tx, P_- y\rangle = \langle TP_- x, y\rangle = 0,$$

so

$$\langle Tz, z\rangle \leqslant 0$$

for every z in Span$\{x, \text{Im } P_-\}$. This contradicts the assumption that Im P_- is maximal T-negative (any subspace $M \subset \mathbb{C}^n$ which is strictly bigger than a maximal T-negative subspace must contain a vector y with $\langle Ty, y\rangle > 0$, see, e.g., Theorem I.1.3 in [7]).

Observe also that the indicator $V(t)$ is negative definite for $t = 0$.

In this section, we shall study the behavior of the indicator when the matrix A is semidefinite. The notation $X \geqslant Y$ $(X > Y)$ for hermitian matrices X and Y means that $X - Y$ is positive semidefinite (positive definite).

THEOREM 4.1. *Assume that A is positive semidefinite, and let $V(t)$ be the indicator of the boundary value problem* (1.5), (1.6). *Then*

$$V(t_2) \geqslant V(t_1) \qquad \textit{for} \quad t_1 > t_2 > 0. \tag{4.1}$$

In particular (*taking into account that $V(0)$ is negative definite*), *$V(t) < 0$ for all $t > 0$, and there are no exceptional values.*

For the case when P_- is the spectral projection corresponding to the negative eigenvalues of T, the nonexistence of exceptional values is known, even in the infinite dimensional case (see [8]).

Proof. Consider first the case when A is positive definite. Using the transformation $T \to S^*TS$, $A \to S^*AS$, $P_- \to S^{-1}P_-S$, we can assume that

$$T = \begin{bmatrix} -I_k & 0 \\ 0 & I_l \end{bmatrix}, \qquad A = \operatorname{diag}(\lambda_1, ..., \lambda_n),$$

where λ_j are positive numbers and $k + l = n$ (indeed, since A is positive definite, T and A are simultaneously diagonalizable). The conditions that $\operatorname{Im} P_-$ is maximal T-negative and $\operatorname{Ker} P_-$ is maximal T-positive translates into

$$\operatorname{Im} P_- = \operatorname{Im} \begin{bmatrix} I \\ X \end{bmatrix}, \qquad \operatorname{Ker} P_- = \operatorname{Im} \begin{bmatrix} Y \\ I \end{bmatrix}, \tag{4.2}$$

where X and Y are contractions: $\|X\| < 1$, $\|Y\| < 1$. Then

$$\operatorname{Im} P_-^* = \operatorname{Im} \begin{bmatrix} I \\ -Y^* \end{bmatrix}, \qquad \operatorname{Ker} P_-^* = \operatorname{Im} \begin{bmatrix} -X \\ I \end{bmatrix}.$$

The condition $P_-^* T = TP_-$ means $Y = X^*$. Now

$$Te^{-tT^{-1}A} = \begin{bmatrix} Q_1(t) & 0 \\ 0 & Q_2(t) \end{bmatrix},$$

where

$$Q_1(t) = \operatorname{diag}(-e^{t\lambda_1}, ..., -e^{t\lambda_k}),$$
$$Q_2(t) = \operatorname{diag}(e^{-t\lambda_{k+1}}, ..., e^{-t\lambda_n}),$$

and (up to a congruence by a constant matrix) we have

$$V(t) = Q_1(t) + X^* Q_2(t) X$$

(cf. the proof of Proposition 2.3). The derivative

$$V'(t) = \mathrm{diag}(-\lambda_1 e^{t\lambda_1}, ..., -\lambda_k e^{t\lambda_k}) + X^*\mathrm{diag}(-\lambda_{k+1} e^{-t\lambda_{k+1}}, ..., -\lambda_n e^{-t\lambda_n})X$$

is clearly negative definite, and (4.1) is proved.

Assume now A is positive semidefinite but singular. Let $\{A_m\}_{m=1}^{\infty}$ be a sequence of positive definite matrices such that $A_m \to A$ as $m \to \infty$, and let $V_m(t)$ be the indicator of the problem (1.5), (1.6) with A replaced by A_m. By the already proved part of the theorem, (4.1) holds with V replaced by V_m. Taking into account that $\lim_{m\to\infty} V_m(t) = V(t)$ for every $t \geqslant 0$, the inequality (4.1) follows. ∎

The case when A is negative semidefinite is more complicated. We start the analysis assuming first that A is negative definite.

THEOREM 4.2. *Assume that $A < 0$, and let $v = \min(k, l)$, where k is the number of negative eigenvalues of T and l is the number of positive eigenvalues of T (counting multiplicities). Then the derivative $V'(t)$ of the indicator $V(t)$ is positive definite. Moreover, there are not more than v positive zeros of* det $V(t)$ *(counted with multiplicities), and all partial multiplicities of $V(t)$ at any exceptional value are equal to* 1 *with the signs in the sign characteristic all being* $+1$.

Proof. Arguing as in the proof of Theorem 4.1, we have (up to congruence by a constant matrix)

$$V(t) = Q_1(t) + X^* Q_2(t) X,$$

where $\|X\| < 1$,

$$Q_1(t) = \mathrm{diag}(-e^{t\lambda_1}, ..., -e^{t\lambda_k}),$$

$$Q_2(t) = \mathrm{diag}(e^{-t\lambda_{k+1}}, ..., e^{t\lambda_n}),$$

and all numbers λ_j are negative. Because of the latter condition, $V'(t) > 0$ for all real t. Now use formula (3.1), and let τ be an exceptional value so that $\mu_i(\tau) = 0$ for some i. Let $y(t)$ be the ith column of $U(t)^{-1}$; then $V(\tau)\, y(\tau) = 0$. On the other hand,

$$\mu_i(t) = \langle V(t)\, y(t), y(t) \rangle$$

and consequently

$$\mu_i'(\tau) = \langle V(\tau)\, y(\tau), y'(\tau) \rangle + \langle V'(\tau)\, y(\tau), y(\tau) \rangle + \langle V(\tau)\, y'(\tau), y(\tau) \rangle = \langle V'(\tau)\, y(\tau), y(\tau) \rangle > 0.$$

It remains to prove that det $V(t)$ has not more than ν positive zeros. When $t \to \infty$ we have

$$\|Q_2(t)\| \geqslant \exp\{t \min\{|\lambda_{k+1}|, ..., |\lambda_n|\}\} I$$

and

$$Q_1(t) \to 0$$

(recall that all λ_j are negative). Hence, $V(t)$ has exactly rank X positive eigenvalues for t large. Taking into account that $\mu_i'(\tau) > 0$ for every exceptional value τ such that $\mu_i'(\tau) = 0$, it follows that det $V(t)$ has exactly rank X positive zeros. But rank $X \leqslant \nu$, and our assertion follows. ∎

COROLLARY 4.3. *Assume A is negative definite. Then there is $\varepsilon > 0$ such that for every system*

$$\tilde{T}\phi'(t) = -\tilde{A}\phi(t); (I - \tilde{P}_-)\phi(0) = 0; \tilde{P}_-\phi(t) = 0 \tag{4.3}$$

with

$$\tilde{T} = \tilde{T}^*, \tilde{A} = \tilde{A}^*, \tilde{P}_- = \tilde{P}_-^2, \tilde{P}^*\tilde{T} = \tilde{T}\tilde{P}_-$$

and

$$\|\tilde{T} - T\| + \|\tilde{A} - A\| + \|\tilde{P}_- - P_-\| < \varepsilon$$

the number of exceptional values of (4.3) (*counted with multiplicities*) *is not smaller than the number of exceptional values of* (1.5), (1.6) (*also counted with multiplicities*). *Moreover, all exceptional values of* (4.3) *are simple with all the signs* $+1$ *in the sign characteristic.*

Proof. Combine Theorems 4.2 and 3.2. ∎

It can happen that the number of exceptional values for (4.3) is bigger than that of (1.5), (1.6) (for instance, let

$$\tilde{T} = T = \begin{bmatrix} -I_k & 0 \\ 0 & I_l \end{bmatrix}, \qquad \tilde{A} = A = \operatorname{diag}(\lambda_1, ..., \lambda_n)$$

with $\lambda_j < 0$,

$$\operatorname{Im} P_- = \operatorname{Im} \begin{bmatrix} I \\ 0 \end{bmatrix}, \qquad \operatorname{Im} \tilde{P}_- = \operatorname{Im} \begin{bmatrix} I \\ X \end{bmatrix}$$

with $X \neq 0$ and $\|X\|$ small). In such case the new exceptional values are "coming from infinity."

The case when A is negative semidefinite is more complicated. Some results will be summarized in the next theorem.

THEOREM 4.4. *Assume that A is negative semidefinite and singular. Let v and $V(t)$ be as in Theorem 4.2. Then $V'(t) \geqslant 0$ for $t > 0$. If, in addition, at least one of the following conditions is satisfied*:

(i) $V'(t) > 0$ *for* $t \geqslant 0$;

(ii) Ker A *is T-negative*;

(iii) Ker A *is T-positive*;

then all the exceptional values are simple with all signs $+1$ in the sign characteristic, and the total number of exceptional values (counted with multiplicities) does not exceed v.

Proof. The inequality $V'(t) \geqslant 0$ follows by considering a sequence of negative definite matrices $\{A_m\}_{m=1}^{\infty}$ which converges to A and using the result of Theorem 4.2 (cf. the proof of Theorem 4.1). If (i) holds, then the conclusion follows by repeating the proof of Theorem 4.2.

Consider the case when either (ii) or (iii) holds. Using the transformation $T \to S^*TS$, $A \to S^*AS$, $P_- \to S^{-1}P_-S$ with invertible S, and using the canonical form for a pair of hermitian matrices (see, e.g., [10], also [4, 7]), we can assume that

$$T = \begin{bmatrix} -I_k & 0 \\ 0 & I_l \end{bmatrix}, \qquad A = \operatorname{diag}(\lambda_1, ..., \lambda_k, \mu_1, ..., \mu_l),$$

where in the case (ii) holds we have $\lambda_1 = \cdots = \lambda_p = 0$ for some $p \leqslant k$ and $\lambda_i < 0$ for $p < i \leqslant k$ and $\mu_i < 0$ for $1 \leqslant i \leqslant l$, while in the case (iii) holds we have $\mu_1 = \cdots = \mu_p = 0$ for some $p < l$ and $\lambda_i < 0$ for $1 \leqslant i \leqslant k$ and $\mu_i < 0$ for $p < i \leqslant l$. Now,

$$V(t) = \operatorname{diag}(-e^{t\lambda_1}, ..., -e^{t\lambda_k}) + X^* \operatorname{diag}(e^{-t\mu_1}, ..., e^{-t\mu_l})X,$$

so in the case (iii) holds we obtain $V'(t) > 0$ for all real t, and the conclusion follows. Assume now (ii) holds. Let

$$M = \operatorname{Im} X^* + \operatorname{Im} \begin{bmatrix} 0 \\ I_{k-p} \end{bmatrix} \subset \mathbb{C}^k;$$

then with respect to the orthogonal decomposition $\mathbb{C}^k = M^{\perp} \oplus M$ we have

$$V(t) = \begin{bmatrix} I & 0 \\ 0 & V_1(t) \end{bmatrix}, \qquad V'(t) = \begin{bmatrix} 0 & 0 \\ 0 & V_1'(t) \end{bmatrix},$$

where $V_1'(t)>0$ for $t \geqslant 0$. Applying the arguments of the proof of Theorem 4.2 to $V_1(t)$ rather than to $V(t)$ we obtain Theorem 4.4. ∎

It should be noted that for a negative semidefinite A it can happen that $\operatorname{Ker} A$ is neither T-positive nor T-negative (this case is not covered by Theorem 4.4). The canonical form for a pair of hermitian matrices shows that $\operatorname{Ker} A$ is T-definite (either T-positive or T-negative) for a negative semidefinite A if and only if $T^{-1}A$ is diagonalizable.

In any case, if A is negative semidefinite, then $T^{-1}A$ has only real eigenvalues and hence, the number of exceptional values is finite (cf. the remark at the end of Section 2).

5. The Matrix A Has One Negative Eigenvalue

Here, we study behavior of the indicator in the case when A is nonsingular and has only one negative eigenvalue (counting with multiplicities). This case is important in applications (transport in supercritical media; see, e.g., [1]). We start with general information (which follows from the canonical form for a pair of hermitian matrices [10, 4, 7]) about the eigenvalues of $T^{-1}A$ when A is nonsingular and has only one negative eigenvalue. $T^{-1}A$ may be diagonalizable or not. If $T^{-1}A$ is diagonalizable, then all its eigenvalues, with the possible exception of two (counting with multiplicities) nonreal mutually conjugate eigenvalues, are real. If $T^{-1}A$ is not diagonalizable, then all its eigenvalues are real and there is precisely one Jordan block in the Jordan form of $T^{-1}A$ of size bigger than 1. The size of this exceptional Jordan block is either 2 or 3, and in the latter case it must correspond to a positive eigenvalue. Let $x_1, ..., x_q$ be a Jordan chain of $T^{-1}A$ corresponding to a real eigenvalue λ_0:

$$(T^{-1}A-\lambda_0 I)x_1=0; \qquad x_1 \neq 0; \qquad (T^{-1}A-\lambda_0 I)x_u=x_{u-1}, \qquad u=2, ..., q.$$

(In view of the previous discussion $q \leqslant 3$.) If this chain is maximal, i.e., there is no vector x_{q+1} such that $x_1, ..., x_{q+1}$ is also a Jordan chain of $T^{-1}A$ corresponding to λ_0, then $\langle Tx_1, x_r\rangle=0$ for $r=1, ..., q-1$ and $\langle Tx_1, x_q\rangle$ is a nonzero real number.

This information is taken into account in the following statement. The number of positive eigenvalues of $V(t)$ (as a hermitian matrix for every fixed t) is always considered with multiplicities counted.

Theorem 5.1. *Assume A is nonsingular with precisely one negative eigenvalue, and let $V(t)$ be the indicator of* (1.5), (1.6).

(i) *Assume that* x_1, x_2 *is a maximal Jordan chain of* $T^{-1}A$ (*necessarily corresponding to a real eigenvalue*) *and*

$$\langle Tx_1, x_2 \rangle > 0.$$

Then $V(t)$ *has at most one positive eigenvalue for* $t \geqslant 2$ *and at most two positive eigenvalues for* $0 \leqslant t < 2$.

(ii) *Assume that* x_1, x_2 *is a maximal Jordan chain of* $T^{-1}A$ *and* $\langle Tx_1, x_2 \rangle < 0$. *Then* $V(t)$ *has at most one positive eigenvalue for* $0 \leqslant t \leqslant 2$ *and at most two positive eigenvalues for* $t > 2$.

(iii) *Assume that* $T^{-1}A$ *has a nonreal eigenvalue* μ. *If* $\operatorname{Re} \mu > 0$ *then* $V(t)$ *has at most one positive eigenvalue. If* $\operatorname{Re} \mu < 0$ *then* $V(t)$ *has at most two positive eigenvalues* (*for* $t > 0$).

(iv) *Assume that* $T^{-1}A$ *is diagonable with all eigenvalues real. Then* $V(t)$ *has at most one positive eigenvalue.*

(v) *Assume that* x_1, x_2, x_3 *is a maximal Jordan chain of* $T^{-1}A$ (*necessarily corresponding to a real eigenvalue*) *and* $\langle Tx_1, x_3 \rangle > 0$. *Then* $V(t)$ *has at most one positive eigenvalue* ($t > 0$).

(vi) *Assume that* x_1, x_2, x_3 *is a maximal Jordan chain of* $T^{-1}A$ *and* $\langle Tx_1, x_3 \rangle < 0$. *Then* $V(t)$ *has at most two positive eigenvalues for* $t < 2$ *and at most one positive eigenvalue for* $t \geqslant 2$.

Before we proceed to the proof of Theorem 5.1 (which amounts basically to checking each case by reducing first T, A to the canonical form for a pair of hermitian matrices) let us indicate how Theorem 5.1 can be used to derive information on the exceptional values. For this, we need the following proposition (where $A = A^*$ is not necessarily nonsingular with one negative eigenvalue).

PROPOSITION 5.2. *Let* $V(t)$ *be the indicator for* (1.5), (1.6) *and suppose that for each* t *in an interval* (a, b) *the hermitian matrix* $V(t)$ *has at most* k *positive eigenvalues. Assume that there are at least* $k+1$ *odd partial multiplicities of* $V(t)$ *with same sign* ε *in the sign characteristic corresponding to some exceptional values in the interval* (a, b). *Then there must be an odd partial multiplicity of* $V(t)$ *corresponding to an exceptional value in* (a, b) *whose sign in the sign characteristic is* $-\varepsilon$.

Proof. Arguing by contradiction, assume that the required partial multiplicity is absent. Consider the representation (3.1) of $V(t)$. Then the eigenvalues of $V(t)$ are the numbers $\mu_i(t)$, $i = 1, ..., p$. Due to the fact that the required partial multiplicity is absent, there is a point t in (α, β) for which $V(t)$ has at least $k+1$ positive eigenvalues. This contradicts our hypothesis. ▮

Combining Theorem 5.1 and Proposition 5.2, we obtain certain information about exceptional points in case A is nonsingular with one negative eigenvalue. Note also that in cases (i), (ii), (iv), (v), (vi) (but not necessarily in the case (iii)) of Theorem 5.1, the number of exceptional points is finite.

The rest of this section is devoted to the proof of Theorem 5.1.

Proof of Theorem 5.1. Cases (i) and (ii). Using the canonical form for a pair of hermitian matrices [10, 4, 7], we can assume that

$$T = -I_k \oplus \begin{bmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{bmatrix} \oplus I_l, \qquad A = \lambda_1 \oplus \cdots \oplus \lambda_k \oplus \begin{bmatrix} 0 & \mu \\ \mu & 1 \end{bmatrix} \oplus \nu_1 \oplus \cdots \oplus \nu_l,$$

where $\varepsilon = 1$ (in case (i)) and $\varepsilon = -1$ (in case (ii)), the numbers λ_j and ν_j are positive, and μ is a nonzero real number. Then

$$Te^{-tT^{-1}A} = -e^{\lambda_1 t} \oplus \cdots \oplus -e^{\lambda_k t} \oplus \varepsilon \begin{bmatrix} 0 & e^{-t\varepsilon\mu} \\ e^{-t\varepsilon\mu} & -t\varepsilon e^{-t\varepsilon\mu} \end{bmatrix} \oplus e^{-t\nu_1} \oplus \cdots \oplus e^{-t\nu_l}.$$

Write $Te^{-tT^{-1}A}$ in the following orthonormal basis in $\mathbb{C}^n$: The first k vectors in the basis are the standard ones $e_1, ..., e_k$ (here and in the sequel $e_q = [0, ..., 1, 0, ..., 0]^{\mathrm{T}}$ with 1 on the qth place); the next two vectors are $(1/\sqrt{2})e_{k+1} - (\varepsilon/\sqrt{2})e_{k+2}$, $(1/\sqrt{2})e_{k+1} + (\varepsilon/\sqrt{2})e_{k+2}$; the last l vectors are the standard ones $e_{n-l+1}, ..., e_n$. Then in this basis T has the form $\left[\begin{smallmatrix} -I & 0 \\ 0 & I \end{smallmatrix}\right]$ and $Te^{-tT^{-1}A}$ has the form

$$\Phi = -e^{\lambda_1 t} \oplus \cdots \oplus -e^{\lambda_k t} \oplus \varepsilon e^{-t\varepsilon\mu} \begin{bmatrix} -1-\dfrac{t}{2} & \dfrac{t}{2} \\ \dfrac{t}{2} & 1-\dfrac{t}{2} \end{bmatrix}$$
$$\oplus e^{-t\nu_1} \oplus \cdots \oplus e^{-t\nu_l}.$$

As in the proof of Theorem 4.1, write

$$\operatorname{Im} P_- = \operatorname{Im} \begin{bmatrix} I \\ X \end{bmatrix}, \qquad \|X\| < 1,$$

and then (up to congruence by a constant matrix)

$$V(t) = [I\ X^*]\, \Phi \begin{vmatrix} I \\ X \end{vmatrix}$$
$$= \left(-e^{\lambda_1 t} \oplus \cdots \oplus -e^{\lambda_k t} \oplus \varepsilon \left(-1 - \frac{t}{2} \right) e^{-t\varepsilon\mu} \right) + X^*K + K^*X$$
$$+ X^* \left[\varepsilon e^{-t\varepsilon\mu} \left(1 - \frac{t}{2} \right) \oplus e^{-t\nu_1} \oplus \cdots \oplus e^{-t\nu_l} \right] X,$$

where K is the $(l+1)\times(k+1)$ matrix with the upper right corner $\varepsilon(t/2)e^{-t\varepsilon\mu}$ and zeros elsewhere. Observe that $X^*K+K^*X=\varepsilon e^{-t\varepsilon\mu}(t/2)B$, where

$$B=\begin{bmatrix} & 0 & & x_1 \\ & & & \vdots \\ & & & x_k \\ \bar{x}_1 & \cdots & \bar{x}_k & x_{k+1}+\bar{x}_{k+1} \end{bmatrix},$$

and $[x_1, ..., x_{k+1}]^{\mathrm{T}}$ is the first column of X. So

$$V(t)=Q_1+X^*Q_2X+\varepsilon e^{-t\varepsilon\mu}\frac{t}{2}B,$$

where

$$Q_1=-e^{\lambda_1 t}\oplus\cdots\oplus -e^{\lambda_k t}\oplus\varepsilon\left(-1-\frac{t}{2}\right)e^{-t\varepsilon\mu};$$

$$Q_2=\varepsilon e^{-t\varepsilon\mu}\left(1-\frac{t}{2}\right)\oplus e^{-t\nu_1}\oplus\cdots\oplus e^{-t\nu_l}.$$

It will be convenient to rewrite $V(t)$ in the form

$$V(t)=\tilde{Q}_1+X^*Q_2X+C,$$

where

$$\tilde{Q}_1=-e^{\lambda_1 t}\oplus\cdots\oplus -e^{\lambda_n t}\oplus -1$$

and

$$C=\varepsilon e^{-t\varepsilon\mu}\frac{t}{2}B+\left(0\oplus\cdots\oplus 0\oplus\varepsilon\left(-1-\frac{t}{2}\right)e^{-t\varepsilon\mu}+1\right).$$

Consider two cases:

(1) $\varepsilon=1$. Then $\tilde{Q}_1\leqslant -I$ for all $t\geqslant 0$ and $Q_2\leqslant I$ for $t\geqslant 2$. So $\tilde{Q}_1+X^*Q_2X\leqslant 0$ for all $t\geqslant 2$. Since C has at most one positive eigenvalue, it follows that $V(t)$ has at most one positive eigenvalue for $t\geqslant 2$.

(2) $\varepsilon=-1$. As before, $\tilde{Q}_1\leqslant -I$ for all $t\geqslant 0$. Now $Q_2\leqslant I$ for $t\leqslant 2$. So $V(t)$ has at most one positive eigenvalue for $t\leqslant 2$.

To cover the remaining values of t write

$$V(t)=\tilde{Q}_1+X^*\tilde{Q}_2X+C+X^*DX,$$

where

$$\tilde{Q}_2 = 0 \oplus e^{-t\nu_1} \oplus \cdots \oplus e^{-t\nu_l},$$

and

$$D = \left(\varepsilon e^{-t\varepsilon\mu}\left(1 - \frac{t}{2}\right)\right) \oplus 0 \oplus \cdots \oplus 0.$$

Clearly, $C + X^*DX$ has at most two positive eigenvalues, and $\tilde{Q}_1 + X^*\tilde{Q}_2X$ is negative definite. So $V(t)$ has at most two positive eigenvalues.

Consider the case (iii). Using the canonical form again, we can assume that

$$T = -I_k \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus I_l, \qquad A = \lambda_1 \oplus \cdots \oplus \lambda_k \oplus \begin{bmatrix} 0 & \bar{\mu} \\ \mu & 0 \end{bmatrix} \oplus \nu_1 \oplus \cdots \oplus \nu_l,$$

where λ_j, ν_j are positive and μ is nonreal. Then

$$Te^{-tT^{-1}A} = -e^{\lambda_1 t} \oplus \cdots \oplus -e^{\lambda_k t} \oplus \begin{bmatrix} 0 & e^{-t\bar{\mu}} \\ e^{-t\mu} & 0 \end{bmatrix} \oplus e^{-t\nu_1} \oplus \cdots \oplus e^{-t\nu_l}.$$

Passing to a new basis, as in the cases (i) and (ii), we obtain

$$Te^{-tT^{-1}A} = -e^{\lambda_1 t} \oplus \cdots \oplus -e^{\lambda_k t} \oplus \begin{bmatrix} -\frac{1}{2}(u + \bar{u}) & \frac{1}{2}(\bar{u} - u) \\ \frac{1}{2}(u - \bar{u}) & \frac{1}{2}(u + \bar{u}) \end{bmatrix}$$
$$\oplus e^{-t\nu_1} \oplus \cdots \oplus e^{-t\nu_l},$$

where $u = e^{-t\mu}$. Write $V(t)$ in the form

$$V(t) = Q_1 + X^*Q_2X + X^*K + K^*X + (0 \oplus \cdots \oplus 0 \oplus (-\tfrac{1}{2}(u + \bar{u}) + 1)),$$

where

$$Q_1 = -e^{\lambda_1 t} \oplus \cdots \oplus -e^{\lambda_k t} \oplus -1;$$
$$Q_2 = e^{-t \operatorname{Re} \mu} \cos(t \operatorname{Im} \mu) \oplus e^{-t\nu_1} \oplus \cdots \oplus e^{-t\nu_l};$$

and K is the $(l+1) \times (k+1)$ matrix with the right upper corner $\frac{1}{2}(u - \bar{u})$ and zeros elsewhere. If $\operatorname{Re} \mu \geqslant 0$, then the conclusion is that $V(t)$ has at most one positive eigenvalue. If $\operatorname{Re} \mu < 0$, then $V(t)$ has at most two positive eigenvalues.

Case (iv). Here we can assume

$$T = -I_k \oplus I_l, \qquad A = \lambda_1 \oplus \cdots \oplus \lambda_k \oplus \lambda_{k+1} \oplus \cdots \oplus \lambda_{k+l},$$

where λ_j are nonzero real numbers and precisely one of them is negative. We leave the verification of this case to the reader.

Cases (v) and (vi). Here we can assume

$$T = -I_k \oplus \varepsilon \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \oplus I_l,$$

$$A = \lambda_1 \oplus \cdots \oplus \lambda_k \oplus \begin{bmatrix} 0 & 0 & \mu \\ 0 & \mu & 1 \\ \mu & 1 & 0 \end{bmatrix} \oplus \nu_1 \oplus \cdots \oplus \nu_l$$

where λ_j, ν_j, and μ are positive and $\varepsilon = \pm 1$. Applying similar ideas, as in the cases (i)–(iii), we pass to a new basis given as follows: The first k, last l, and the $(k+2)$th vector are the standard ones, the $(k+1)$th and $(k+3)$th given by

$$\left[0, ..., 0, \frac{1}{\sqrt{2}}, 0, \frac{-\varepsilon}{\sqrt{2}}, 0, ..., 0\right]^{\mathrm{T}} \quad \text{and} \quad \left[0, ..., 0, \frac{1}{\sqrt{2}}, 0, \frac{\varepsilon}{\sqrt{2}}, 0, ..., 0\right]^{\mathrm{T}},$$

respectively, where the first k and the last l coordinates are zeros.
In this basis, we have

$$T = -I_{k+1} \oplus \varepsilon \oplus I_{l+1}$$

and

$$Te^{-tT^{-1}A} = -e^{\lambda_1 t} \oplus \cdots \oplus -e^{\lambda_k t} \oplus e^{-t\varepsilon\mu} \begin{bmatrix} -1 + \frac{\varepsilon}{4} t^2 & \frac{1}{\sqrt{2}} t & -\frac{\varepsilon}{4} t^2 \\ \frac{1}{\sqrt{2}} t & \varepsilon & -\frac{1}{\sqrt{2}} t \\ -\frac{\varepsilon}{4} t^2 & -\frac{1}{\sqrt{2}} t & 1 + \frac{\varepsilon}{4} t^2 \end{bmatrix}$$
$$\oplus e^{-\nu_1 t} \oplus \cdots \oplus e^{-\nu_l t}.$$

If $\varepsilon = +1$ (i.e., the case (v) holds) we have

$$V(t) = Q_1(t) + X^* Q_2(t) X + X^* K + K^* X,$$

where

$$Q_1(t) = -e^{\lambda_1 t} \oplus \cdots \oplus -e^{\lambda_k t} \oplus e^{-t\mu}\left(-1 + \frac{1}{4} t^2\right),$$

$$Q_2(t)=\begin{bmatrix} e^{-t\mu} & \frac{-1}{\sqrt{2}}te^{-t\mu} \\ -\frac{1}{\sqrt{2}}te^{-t\mu} & \left(1+\frac{1}{4}t^2\right)e^{-t\mu} \end{bmatrix} \oplus e^{-\nu_1 t} \oplus \cdots \oplus e^{-\nu_l t},$$

and

$$K=\begin{bmatrix} 0 & \cdots & 0 & \frac{1}{\sqrt{2}}te^{-t\mu} \\ 0 & \cdots & 0 & -\frac{1}{4}t^2e^{-t\mu} \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Writing $V(t)$ in the form $\tilde{Q}_1 + X^*\tilde{Q}_2X + X^*K + X^*DX + \tilde{D}$, where

$$\tilde{Q}_1 = -e^{\lambda_1 t} \oplus \cdots \oplus -e^{\lambda_k t} \oplus -e^{-t\mu},$$

$$\tilde{D} = 0 \oplus \frac{1}{4}t^2e^{-t\mu},$$

$$\tilde{Q}_2 = e^{-t\mu} \oplus e^{-t\mu} \oplus e^{-\nu_1 t} \oplus \cdots \oplus e^{-\nu_l t},$$

$$D=\begin{bmatrix} 0 & -\frac{1}{\sqrt{2}}te^{-t\mu} \\ -\frac{1}{\sqrt{2}}te^{-t\mu} & \frac{1}{4}t^2e^{-t\mu} \end{bmatrix} \oplus 0,$$

we see that $\tilde{Q}_1 + X^*\tilde{Q}_2X < 0$. Further,

$$X^*K + K^*X + X^*DX + \tilde{D} = [I\ X^*]\begin{bmatrix} \tilde{D} & K^* \\ K & D \end{bmatrix}\begin{bmatrix} I \\ X \end{bmatrix}.$$

Now, we have (ignoring zero rows and columns)

$$\begin{bmatrix} \tilde{D} & K^* \\ K & D \end{bmatrix} = e^{t\mu}\begin{bmatrix} \frac{1}{4}t^2 & \frac{1}{\sqrt{2}}t & -\frac{1}{4}t^2 \\ \frac{1}{\sqrt{2}}t & 0 & -\frac{1}{\sqrt{2}}t \\ -\frac{1}{4}t^2 & -\frac{1}{\sqrt{2}}t & \frac{1}{4}t^2 \end{bmatrix},$$

and a computation shows that $\left[\begin{smallmatrix}\tilde{D} & K^* \\ K & D\end{smallmatrix}\right]$ has exactly one positive eigenvalue. Then $V(t)$ has at most one positive eigenvalue. If $\varepsilon = -1$, we obtain

$$V(t) = Q_1(t) + X^*Q_2(t)X + X^*K + K^*X + \tilde{D},$$

where

$$Q_1(t) = -e^{\lambda_1 t} \oplus \cdots \oplus -e^{\lambda_k t} \oplus \left(-1 - \frac{1}{4}t^2\right) e^{t\mu} \oplus -e^{t\mu},$$

$$\tilde{D} = 0 \oplus e^{t\mu} \begin{bmatrix} 0 & \dfrac{t}{\sqrt{2}} \\ \dfrac{t}{\sqrt{2}} & 0 \end{bmatrix},$$

$$Q_2 = (1 - \tfrac{1}{4}t^2)e^{t\mu} \oplus e^{-\nu_1 t} \oplus \cdots \oplus e^{-\nu_l t},$$

and K is the $(l+1) \times (k+2)$ matrix whose right upper corner is $-\frac{1}{4}t^2e^{t\mu}$ and zeros elsewhere.

Now $Q_1(t) < -I$, $Q_2(t) \leqslant I$ for $t \geqslant 2$, and $X^*K + K^*X + \tilde{D}$ has at most one positive eigenvalue. So for $t \geqslant 2$, $V(t)$ has at most one positive eigenvalue.

Next, rewrite

$$V(t) = Q_1(t) + X^*\tilde{Q}_2(t)X + X^*K + KX + \tilde{D} + X^*DX, \tag{5.1}$$

where

$$\tilde{Q}_2(t) = -\tfrac{1}{4}t^2e^{t\mu} \oplus e^{-\nu_1 t} \oplus \cdots \oplus e^{-\nu_l t}$$

and

$$D = e^{t\mu} \oplus 0.$$

It is easily seen from (5.1) that $V(t)$ has at most two positive eigenvalues for $0 < t < 2$. ∎

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