Note

On Montel’s theorem and Yang’s problem ✤

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Abstract

Let \( \mathcal{F} \) be a family of meromorphic functions defined in a domain \( D \), and let \( \psi (\neq 0) \) be a function meromorphic in \( D \). For every function \( f \in \mathcal{F} \), if (1) \( f \) has only multiple zeros; (2) the poles of \( f \) have multiplicity at least 3; (3) at the common poles of \( f \) and \( \psi \), the multiplicity of \( f \) does not equal the multiplicity of \( \psi \); (4) \( f(z) \neq \psi(z) \), then \( \mathcal{F} \) is normal in \( D \). This gives a partial answer to a problem of L. Yang, and generalizes Montel’s theorem. Some examples are given to show the sharpness of our result.

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1. Introduction

Let \( D \) be a domain in \( \mathbb{C} \). Let \( \mathcal{F} \) be a family of meromorphic functions defined in \( D \). \( \mathcal{F} \) is said to be normal in \( D \), in the sense of Montel, if for any sequence \( f_n \in \mathcal{F} \) there exists a subsequence \( f_{n_j} \), such that \( f_{n_j} \) converges spherically locally uniformly in \( D \), to a meromorphic function or \( \infty \) (see [5,7,9]).

Let \( f \) be a nonconstant meromorphic function and \( z_0 \in \mathbb{C} \). If \( f(z_0) = z_0 \), then \( z_0 \) is called a fixed point of \( f(z) \). In the theory of complex dynamical systems, it is well known...
that the Julia set is the closure of repulsive periodic points. Thus, if there is no periodic point in a domain, the domain must be contained in the Fatou set. Related to this kind of situation, Yang [10] proposed

**Problem Y.** Let $D$ be a domain in the complex plane, $k$ a positive integer. Let $F$ be a family of entire functions. Assume that for every $f \in F$ both $f$ and its iteration $f^k$ of order $k$ have no fix point in $D$. Is $F$ normal in $D$?

By using an interesting method and the tool of the Ahlfors theory of covering surfaces, Essén and Wu [4] proved the following result.

**Theorem A.** Let $D$ be a domain in the complex plane, and let $F$ be a family of holomorphic functions defined in $D$. If, for every function $f \in F$, there exists $k = k(f) > 1$ such that $f^k$ has no fix point in $D$, then $F$ is normal in $D$.

Essén and Wu also indicated that Theorem A is false for $k = 1$ by an example (for details, see [4]).

In this note, we shall show that Theorem A is valid for $k = 1$ if $f \in F$ has no simple zeros. In fact, we prove the following more general results.

**Theorem 1.** Let $F$ be a family of holomorphic functions defined in a domain $D$. Let $\psi (\not\equiv 0)$ be a function holomorphic in $D$. If, for every function $f \in F$, $f$ has only multiple zeros and $f(z) \neq \psi(z)$, then $F$ is normal in $D$.

**Remark 1.** The condition that $f$ has only multiple zeros is necessary in Theorem 1.

**Example 1.** Let $D = \{ z: |z| < 1 \}$, $\psi(z) = z$, and

$$F = \{ f_n(z) = e^{nz} + z: z \in D, \ n = 1, 2, \ldots \}.$$ 

Obviously, for each $f_n \in F$, $f_n(z) = e^{nz} + z \neq z$, and $f_n(z)$ has only simple zeros. On the other hand, we have

$$f_n'(0) = \frac{|f_n'(0)|}{1 + |f_n(0)|^2} \to \infty, \quad \text{as } n \to \infty.$$

Hence by Marty’s criterion, $F$ is not normal in $D$.

For the case of families of meromorphic functions, we prove the following theorem.

**Theorem 2.** Let $F$ be a family of meromorphic functions defined in a domain $D$. Let $\psi (\not\equiv 0)$ be a function meromorphic in $D$. For every function $f \in F$, if

1. $f$ has only multiple zeros;
2. the poles of $f$ have multiplicity at least 3;
(3) at the common poles of \( f \) and \( \psi \), the multiplicity of \( f \) does not equal the multiplicity of \( \psi \);
(4) \( f(z) \neq \psi(z) \),

then \( \mathcal{F} \) is normal in \( D \).

**Remark 2.** The classical Montel’s theorem [5,7,9] states that a family of meromorphic functions \( \mathcal{F} \) is normal in a domain \( D \) if there are three distinct points \( a, b, c \in \hat{\mathbb{C}} \) such that each \( f \in \mathcal{F} \), \( f \neq a, b, c \) in \( D \). If \( a, b, c \) are replaced by three meromorphic functions \( a(z), b(z), c(z) \) in \( D \) avoiding each other, Montel’s theorem still holds. This observation seems first to have been made by Fatou. (In fact, D. Bargmann, M. Bonk, A. Hinkkanen and G.J. Martin [1] have proved that Montel’s theorem is still valid if the meromorphic functions \( a(z), b(z), c(z) \) are replaced by three arbitrary continuous functions avoiding each other in \( D \).) Clearly, Theorem 2 is a generalization of Montel’s theorem.

**Remark 3.** The condition (1) cannot be omitted in Theorem 2.

**Example 2.** Let \( D = \{ z : |z| < 1 \} \), \( \psi(z) = z \), and
\[
\mathcal{F} = \left\{ f_n(z) = \frac{nz^4 - 1}{nz^3} : z \in D, \ n = 1, 2, \ldots \right\}.
\]
Clearly, for \( f_n \in \mathcal{F} \), the zeros of \( f_n \) are simple, and the pole of \( f_n \) have multiplicity 3, and
\[
f_n(z) = z - \frac{1}{nz} \neq z.
\]
However, we have \( f_n^*(\frac{1}{n}) \to \infty \) as \( n \to \infty \). Then, by Marty’s criterion, \( \mathcal{F} \) is not normal in \( D \).

**Remark 4.** Condition (3) is necessary in Theorem 2, as is shown by the following example.

**Example 3.** Let \( k \geq 3 \) be a positive integer, \( D = \{ z : |z| < 1 \} \), \( \psi(z) = \frac{1}{z^k} \), and
\[
\mathcal{F} = \left\{ f_n(z) = \frac{1}{n z^k} : z \in D, \ n = 2, 3, \ldots \right\}.
\]
For each \( f_n \in \mathcal{F} \), we have
(1) \( f_n(z) - \psi(z) = \frac{1-n}{nz} \neq 0 \);
(2) \( f_n \) has no zero;
(3) \( z = 0 \) is the common pole of \( f_n \) and \( \psi \) and with the same multiplicity \( k \geq 3 \).

But \( \mathcal{F} \) is not normal in \( D \).

**Remark 5.** It is not clear to me whether the condition (2) in Theorem 2 is best possible. Here we can only give the following example to show that the poles of \( f \in \mathcal{F} \) must have multiplicity at least 2.
Example 4. Let \( D = \{z: |z| < 1\} \), \( \psi(z) = \frac{1}{z} \), and
\[
\mathcal{F} = \left\{ f_n(z) = -\frac{nz^2}{nz^3 - 1}; z \in D, \ n = 1, 2, \ldots \right\}.
\]
Clearly, for \( f_n \in \mathcal{F} \), we have

1. the zeros of \( f_n \) are multiple, and the poles of \( f_n \) are simple;
2. \( f_n \) and \( \psi \) have no common pole;
3. and
\[
f_n(z) = -\frac{nz^2}{nz^3 - 1} = \frac{1}{z - \frac{1}{nz^2}} \neq \frac{1}{z},
\]
in \( D \). However, since \( f_n^\#((\frac{1}{n})^{\frac{1}{2}}) \rightarrow \infty \), by Marty’s criterion, \( \mathcal{F} \) is not normal in \( D \).

2. Some lemmas

To prove our results, we need the following lemmas.

Lemma 1. Let \( f \) be a nonconstant rational function. If \( f(z) \neq 1 \), then
\[
f(z) = 1 + \frac{1}{p(z)},
\]
where \( p(z) \) is nonconstant polynomial.

Proof. Obviously, \( f \) cannot be a polynomial. Then we may assume
\[
f(z) = h(z) + \frac{q(z)}{p(z)},
\]
where \( h(z) \) is a polynomial, \( p(z) \), \( q(z) \) are two coprime polynomials with \( \deg q(z) < \deg p(z) \). If \( \deg h(z) \geq 1 \), then there exists \( z_0 \in \mathbb{C} \) such that
\[
f(z_0) - 1 = \frac{p(z_0)(h(z_0) - 1) + q(z_0)}{p(z_0)} = 0,
\]
since \( \deg(p(z)(h(z) - 1) + q(z)) > \deg p(z) \). Hence \( \deg h(z) = 0 \), that is, \( h(z) = a \) (a constant). We claim that \( a = 1 \). Otherwise, \( a = 1 + b (b \neq 0) \). Since \( f - 1 = (bp + q)/p \) and \( p(z) \), \( q(z) \) are coprime, then \( f - 1 = 0 \) has a solution in \( \mathbb{C} \), a contradiction. Thus \( f = 1 + q/p \). Furthermore, we have \( q/p \neq 0 \). It implies that \( q(z) \) must be a nonzero constant. Without loss of generality, we assume \( q = 1 \). Lemma 1 is proved.

Lemma 2. Let \( f \) be a rational function, and \( f(z) \neq 1 \). If all zeros and poles of \( f \) are multiple with the possible exception of one pole (or one zero), then \( f(z) \) is a constant.
Proof. We first prove the case that all zeros and poles of \( f \) are multiple with the possible exception of one pole. Suppose that \( f \) is not a constant. Since \( f(z) \neq 1 \), by Lemma 1, we have

\[
f(z) = 1 + \frac{1}{p(z)},
\]

where \( p(z) \) is nonconstant polynomial with \( \deg p(z) = n \geq 1 \). We claim that \( f \) has at least two poles. Otherwise, \( f(z) \) has only one pole \( z_0 \) with order of \( n \). Then

\[
f(z) = a(z - z_0)^n + \frac{1}{a(z - z_0)^n},
\]

where \( a \) is a nonzero constant. It follows that \( f(z) \) has \( n \) simple zeros, a contradiction. Let the poles of \( f(z) \) be \( a_1, a_2, \ldots, a_h \) and the related orders be \( \alpha_1, \alpha_2, \ldots, \alpha_h \). Obviously, the poles of \( f(z) \) are the zeros of \( p(z) \) and with the same order. Then \( \alpha_1 + \alpha_2 + \cdots + \alpha_h = n \).

By the assumption, we have that \( \alpha_i \geq 2 \) (for \( i = 1, 2, \ldots, h - 1 \)) with the possible exception of \( \alpha_h \). Thus

\[
h \leq \frac{n}{2} + 1 - \frac{\alpha_h}{2} \leq \frac{n}{2} + \frac{1}{2}.
\]

(1)

Let the zeros of \( p(z) + 1 \) be \( b_1, b_2, \ldots, b_k \) and the related orders be \( \beta_1, \beta_2, \ldots, \beta_k \). Clearly, the zeros of \( f(z) \) are the same with the zeros of \( p(z) + 1 \) (with same order), and \( a_i \neq b_j \) (for \( i = 1, 2, \ldots, h \); \( j = 1, 2, \ldots, k \)). Then

\[
\beta_1 + \beta_2 + \cdots + \beta_k = \deg(p(z) + 1) = n.
\]

It follows from the assumption that

\[
k \leq \frac{n}{2}.
\]

(2)

Since all zeros of \( f(z) \) are multiple, then \( b_1, b_2, \ldots, b_k \) must be the zeros of \( p'(z) \) with orders \( \beta_1 - 1, \beta_2 - 1, \ldots, \beta_k - 1 \), respectively. In addition, \( a_1, a_2, \ldots, a_h \) are also the zeros of \( p'(z) \) with orders \( \alpha_1 - 1, \alpha_2 - 1, \ldots, \alpha_h - 1 \), respectively (if \( \alpha_h - 1 = 0 \), it means that \( \alpha_h \) is not the zero of \( p'(z) \)). Thus we have

\[
(\alpha_1 - 1) + (\alpha_2 - 1) + \cdots + (\alpha_h - 1) + (\beta_1 - 1) + (\beta_2 - 1) + \cdots + (\beta_k - 1) \leq \deg p'(z) = n - 1,
\]

so that \( h + k \geq n + 1 \). By (1) and (2), we have

\[
\frac{n}{2} + \frac{1}{2} + \frac{n}{2} \geq h + k \geq n + 1.
\]

(3)

This is impossible.

If all zeros and poles of \( f \) are multiple with the possible exception of one pole, then we have

\[
h \leq \frac{n}{2},
\]

(1')

and

\[
k \leq \frac{n}{2} + 1 - \frac{\beta_h}{2} \leq \frac{n}{2} + \frac{1}{2}.
\]

(2')

Hence, we also have (3), a contradiction. Lemma 2 is proved.  \( \Box \)
The well-known Zalcman’s lemma plays a very important role in the study of normal families. It has also undergone various extensions and improvements. The following is one up-to-date version, which is due to Pang and Zalcman [6] (cf. [2,3,8,11]).

**Lemma 3.** Let \( k \) be a positive integer and let \( F \) be a family of functions meromorphic in a domain \( D \), such that each function \( f \in F \) has only zeros of multiplicity at least \( k \), and suppose that there exists \( A \geq 1 \) such that \( \| f^{(k)}(z) \| \leq A \) whenever \( f(z) = 0 \). If \( F \) is not normal at \( z_0 \in D \), then, for each \( 0 \leq \alpha \leq k \), there exist a sequence of points \( z_n \in D \), \( z_n \to z_0 \), a sequence of positive numbers \( \rho_n \to 0 \), and a sequence of functions \( f_n \in F \) such that

\[
g_n(\zeta) = \frac{f_n(z_n + \rho_n \zeta)}{\rho_n^\alpha} \to g(\zeta)
\]

locally uniformly with respect to the spherical metric, where \( g \) is a nonconstant meromorphic function on \( \mathbb{C} \), all of whose zeros have multiplicity at least \( k \), such that \( g^k(\zeta) \leq g^k(0) = kA + 1 \). Moreover, \( g \) has order at most \( 2 \) (for the holomorphic case, \( g \) has order at most \( 1 \)). Here, \( g^k(\zeta) = |g'(\zeta)|/(1 + |g(\zeta)|^2) \) is the spherical derivative.

### 3. Proof of the theorems

Obviously, Theorem 1 is the special case of Theorem 2. Here we still give the proof of Theorem 1, since we shall use a different but simple method to prove it.

**Proof of Theorem 1.** Since normality is a local property, it is enough to show that \( F \) is normal at each \( z_0 \in D \). We distinguish two cases.

**Case 1.** \( \psi(z_0) \neq 0 \). Suppose that \( F \) is not normal at \( z_0 \). By Lemma 3, there exist a sequence of functions \( f_n \in F \), a sequence of complex numbers \( z_n \to z_0 \) and a sequence of positive numbers \( \rho_n \to 0 \), such that

\[
g_n(\zeta) = f_n(z_n + \rho_n \zeta) \to g(\zeta)
\]

converges uniformly on compact subsets of \( \mathbb{C} \), where \( g(\zeta) \) is a nonconstant entire function on \( \mathbb{C} \), and all zeros of \( g(\zeta) \) are multiple, and the order of \( g(\zeta) \) is at most one. Since

\[
f_n(z_n + \rho_n \zeta) - \psi(z_n + \rho_n \zeta) \to g(\zeta) - \psi(z_0)
\]

converges uniformly on compact subsets of \( \mathbb{C} \), and noting the fact that \( f_n(z_n + \rho_n \zeta) - \psi(z_n + \rho_n \zeta) \neq 0 \), Hurwitz’s theorem implies that \( g(\zeta) \neq \psi(z_0) \) for all \( \zeta \in \mathbb{C} \). Note that the order of \( g(\zeta) \) is at most one, we have

\[
g(\zeta) = \psi(z_0) + e^{az + b},
\]

where \( a \) (\( \neq 0 \)) and \( b \) are constants. Thus, we see that \( g(\zeta) \) has only simple zeros, a contradiction.

**Case 2.** \( \psi(z_0) = 0 \). There exists \( r > 0 \) such that \( \psi(z) \neq 0 \) in \( \Delta'(z_0, r) = \{ z : 0 < |z - z_0| < r \} \subset D \). By Case 1, we know that \( F \) is normal in \( \Delta'(z_0, r) \). Suppose that \( F \) is
not normal at $z_0$. Then there exists a sequence $\{f_n\} \subset F$ such that $f_n$ converges locally uniformly in $\Delta'(z_0, r)$, but not in $\Delta(z_0, r)$. The maximum modulus theorem implies that $f_n \to \infty$ on compact subsets of $\Delta'(z_0, r)$. Let $\max_{|z-z_0|=r/2} |\psi(z)| = M$, then there exists $n_0$ such that $|f_n(z)| > M$ on $|z-z_0|=r/2$ for $n \geq n_0$. Thus for $n \geq n_0$, we have that

$$|f_n(z) - \psi(z) - f_n(z)| = |\psi(z)| \leq |f_n(z)|$$

on $|z-z_0| = r/2$. On the other hand, we know that $f_n(z)$ has a zero in $\Delta(z_0, r/2)$ for all large $n$ (otherwise, we can deduce that $F$ is normal at $z_0$). Rouché’s theorem implies that $f_n(z) - \psi(z)$ must have a zero in $\Delta(z_0, r/2)$, a contradiction. This completes the proof of Theorem 1.

**Proof of Theorem 2.** Since normality is a local property, it is enough to show that $F$ is normal at each $z_0 \in D$. We distinguish three cases.

**Case 1.** $\psi(z_0) \neq 0, \infty$. Suppose that $F$ is not normal at $z_0$. By Lemma 3, there exist a sequence of functions $f_n \in F$, a sequence of complex numbers $z_n \to z_0$ and a sequence of positive numbers $\rho_n \to 0$, such that

$$g_n(\xi) = f_n(z_n + \rho_n \xi) \to g(\xi)$$

converges spherically uniformly on compact subsets of $\mathbb{C}$, where $g(\xi)$ is a nonconstant meromorphic function on $\mathbb{C}$, and all zeros of $g(\xi)$ are multiple. In addition, by Hurwitz’s theorem, all poles of $g(\xi)$ have multiplicity at least 3. Since

$$f_n(z_n + \rho_n \xi) - \psi(z_n + \rho_n \xi) \to g(\xi) - \psi(z_0)$$

converges spherically uniformly on compact subsets of $\mathbb{C}$, and noting the fact that $f_n(z_n + \rho_n \xi) - \psi(z_n + \rho_n \xi) \neq 0$, Hurwitz’s theorem implies that $g(\xi) \neq \psi(z_0)$ for all $\xi \in \mathbb{C}$. By Nevanlinna’s first and second fundamental theorems (see [5,9]), we have

$$T(r, g) \leq \tilde{N}(r, g) + \tilde{N} \left( r, \frac{1}{g} \right) + \tilde{N} \left( r, \frac{1}{g - \psi(z_0)} \right) + S(r, g)$$

$$\leq \frac{1}{3} N(r, g) + \frac{1}{2} N \left( r, \frac{1}{g} \right) + S(r, g)$$

$$\leq \frac{1}{3} T(r, g) + \frac{1}{2} T \left( r, \frac{1}{g} \right) + S(r, g)$$

$$\leq \frac{5}{6} T(r, g) + S(r, g),$$

here $T(r, g)$, $N(r, g)$ and $\tilde{N}(r, g)$ are standard notations of [5,9], with $S(r, g)$ denoting any term which is $o(T(r, g))$ as $r \to \infty$, possibly outside a set of finite measure. Thus $g(\xi)$ is not transcendental. However, Lemma 2 implies that $g(\xi)$ is a constant, a contradiction.

**Case 2.** $\psi(z_0) = 0$. There exists $r > 0$ such that $\psi(z) \neq 0, \infty$ in $\Delta'(z_0, r) = \{ z : 0 < |z - z_0| < r \} \subset D$. By Case 1, we know that $F$ is normal in $\Delta'(z_0, r)$.

Consider the family $\mathcal{G} = \{ g(z) = f(z)/\psi(z) : f \in F, z \in \Delta(z_0, r) \}$. Note that $f(z_0) \neq \psi(z_0) = 0$, then $f(z_0) \neq 0$. Thus, for any $g \in \mathcal{G}$, $g(z_0) = f(z_0)/\psi(z_0) = \infty$. In addition,
all zeros of $g(\zeta)$ are multiple, and all poles of $g(\zeta)$ have multiplicity at least 3 with the possible exception at $z = \zeta_0$.

We first prove that the family $G$ is normal in $\Delta(z_0, r)$. Suppose, on the contrary, that $G$ is not normal at $z_1 \in \Delta(z_0, r)$. Then by Lemma 3, there exist a sequence of functions $g_n \in G$, a sequence of complex numbers $z_n \rightarrow z_1$ and a sequence of positive numbers $\rho_n \rightarrow 0$, such that

$$G_n(\zeta) = g_n(z_n + \rho_n \zeta) \rightarrow G(\zeta)$$

converges spherically uniformly on compact subsets of $\mathbb{C}$, where $G(\zeta)$ is a nonconstant meromorphic function on $\mathbb{C}$, and all zeros of $G(\zeta)$ have multiplicity at least 2, and all poles of $G(\zeta)$ have multiplicity at least 3 with the possible exception at $z = \zeta_0$. Note that $g(z) = f(z)/\psi(z) \neq 1$, by Hurwitz’s theorem, we have $G(\zeta) \neq 1$ for all $\zeta \in \mathbb{C}$. Using Nevanlinna’s first and second fundamental theorems and similarly as Case 1, we know that $G(\zeta)$ is not transcendental. It follows from Lemma 2 that $G(\zeta)$ is a constant, a contradiction.

Next we prove that this implies that $F$ is normal at $z_0$. Suppose that $F$ is not normal at $z_0$. Since $G$ is normal in $\Delta(z_0, r)$, then the family $G$ is equicontinuous in $\Delta(z_0, r)$ with respect to the spherical distance. On the other hand, $g(z_0) = \infty$ for each $g \in G$. Thus, there exists $\delta > 0$ such that $|g(z)| \geq 1$ for all $g \in G$ and $z \in \Delta(z_0, r)$. It follows that $f(z) \neq 0$ for all $f \in F$ and $z \in \Delta(z_0, r)$. Since $F$ is normal in $\Delta'(z_0, r)$, then the family $1/F = \{1/f : f \in F\}$ is holomorphic in $\Delta(z_0, r)$ and normal in $\Delta'(z_0, r)$, but it is not normal at $z = z_0$. Thus, there exists a sequence $\{1/f_n\} \subset 1/F$ which converges locally uniformly in $\Delta'(z_0, r)$, but none of whose subsequences converges uniformly in a neighborhood of $z_0$. The maximum modulus principle implies that $1/f_n \rightarrow \infty$ on compact subsets of $\Delta'(z_0, r)$. Hence, $f_n \rightarrow 0$ uniformly on compact subsets of $\Delta'(z_0, r)$. Note that $g_n = f_n/\psi$, we see that $g_n \rightarrow 0$ uniformly on compact subsets of $\Delta'(z_0, r)$. But we already know that $|g_n(z)| \geq 1$ for $z \in \Delta(z_0, r)$ in the above, a contradiction.

Case 3. $\psi(z_0) = \infty$. There exists $r > 0$ such that $\psi(z) \neq 0, \infty$ in $\Delta'(z_0, r) = \{z : 0 < |z - z_0| < r\} \subset D$. Consider the family $G$ as in Case 2. Here, we need note that:

(a) If $z_0$ is not a pole of $f$ for $f \in F$, then, for $g \in G$, $g(z_0) = f(z_0)/\psi(z_0) = 0$, so that $z_0$ is a zero of $g$.

(b) If $z_0$ is also a pole of $f$ for $f \in F$, by the assumption, we know that $z_0$ is either a pole or a zero of $g$ for $g \in G$.

Thus, we conclude that $\zeta = z_0$ is a zero or a pole of $G(\zeta)$. With the possible exception at $z_0$, all zeros of $G(\zeta)$ are multiple and all poles of $G(\zeta)$ have multiplicity at least 3. After this, we can use almost the same argument as in Case 2 to prove Case 3. Theorem 2 is proved.

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