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On Fourier Transforms of Functions Supported on Sets of Finite Lebesgue Measure

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Let G be a locally compact abelian group and $\hat{G}$ its dual group. Denote the Haar measure on G and $\hat{G}$ by m and $\hat{m}$, respectively. In the paper by Matolcsi and Szücs [3] the following theorem is proved:

THEOREM 1. *Let $f \in L^1(G)$ and $\hat{f}$ be its Fourier transform*

$$\hat{f}(\gamma) = \int_G (-x, \gamma) f(x)\, dm(x),$$

where $(x, \gamma) = \gamma(x)$ establishes the duality between G and $\hat{G}$. Let $A = \{x \in G;\ f(x) \neq 0\}$ and $B = \{\gamma \in \hat{G}; \hat{f}(\gamma) \neq 0\}$. Then

$$m(A) \cdot \hat{m}(B) < 1 \quad \Rightarrow \quad f = 0 \qquad a.e.\ |m|.$$

The intention of this note is to improve this result in the case $G = \mathbb{R}^n$, $\hat{G} = \hat{\mathbb{R}}^n$. For $f \in L^1(\mathbb{R}^n)$, the Fourier transform of f can be written

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-i\langle x, \xi\rangle}\, dx, \qquad \xi \in \hat{\mathbb{R}}^n,$$

where $\langle x, \xi\rangle = x_1\xi_1 + \cdots + x_n\xi_n$.

Our main result is:

THEOREM 2. *Let $f \in L^1(\mathbb{R}^n)$ and $\hat{f}$ be its Fourier transform, and let $A = \{x \in \mathbb{R}^n;\ f(x) \neq 0\}$ and $B = \{\xi \in \hat{\mathbb{R}}^n;\ \hat{f}(\xi) \neq 0\}$. Then*

$$m(A) < \infty \text{ and } \hat{m}(B) < \infty \quad \Rightarrow \quad f = 0 \qquad a.e.\ |m|.$$

Here m and $\hat{m}$ denote Lebesgue measure on $\mathbb{R}^n$ resp. $\hat{\mathbb{R}}^n$.

Proof. Without loss of generality we can assume $m(A) < (2\pi)^n$, since we can replace the function f by its dilatation $f_a(x) = f(ax)$. Its Fourier

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transform is $\hat{f}_a(\xi) = (1/a^n)\hat{f}(\xi/a)$, so $\hat{m}(B)$ will remain finite. Let $\varphi(\xi)$ be the characteristic function of B, i.e.,

$$\varphi(\xi) = \begin{cases} 1 & \text{if} \quad \xi \in B \\ 0 & \text{if} \quad \xi \notin B. \end{cases}$$

Form the function

$$\tilde{\varphi}(\xi) = \sum_{\nu \in \mathbb{Z}^n} \varphi(\xi - \nu).$$

The function $\tilde{\varphi}$ clearly is positive, measurable and periodic with period 1. Let $K = \{\xi \in \hat{\mathbb{R}}^n;\ 0 \leqslant \xi_i \leqslant 1,\ i = 1, 2,..., n\}$. Then

$$\int_K \tilde{\varphi}(\xi)\, d\xi = \int_{\hat{\mathbb{R}}^n} \varphi(\xi)\, d\xi = \hat{m}(B) < \infty.$$

This implies that $\tilde{\varphi}(\xi) < \infty$ a.e., i.e., for almost all ξ_0, $\varphi(\xi_0 + k) \neq 0$ only for a finite number of $k \in \mathbb{Z}^n$ and consequently:

(I) for almost all $\xi_0 \in \hat{\mathbb{R}}^n$, $\hat{f}(\xi_0 + k) \neq 0$ only for a finite number of $k \in \mathbb{Z}^n$.

Fix $\xi_0 \in \hat{\mathbb{R}}^n$ and form the function

$$\tilde{f}_{\xi_0}(x) = \sum_{\nu \in \mathbb{Z}^n} e^{-i\langle \xi_0, x - 2\pi\nu\rangle} f(x - 2\pi\nu).$$

$\tilde{f}_{\xi_0}$ has the following properties:

(i) $\tilde{f}_{\xi_0} \in L^1(\mathbb{T}^n)$, $\mathbb{T}^n$ being the n-torus $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$;

(ii) $\tilde{f}_{\xi_0}$ has the Fourier coefficients

$$(\tilde{f}_{\xi_0})\hat{}\,(k) = (1/(2\pi)^n)\hat{f}(\xi_0 + k), \qquad k \in \mathbb{Z}^n;$$

(iii) $m(\{x \in \mathbb{T}^n; \tilde{f}_{\xi_0}(x) \neq 0\}) < (2\pi)^n$.

(i) easily follows from $f \in L^1(\mathbb{R}^n)$, (ii) is the result of a simple computation and (iii) is an immediate consequence of $m(A) < (2\pi)^n$.

Together (I) and (ii) show that for almost all $\xi_0 \in \hat{\mathbb{R}}^n$, $\tilde{f}_{\xi_0}$ is a trigonometrical polynomial. But now (iii) implies that $\tilde{f}_{\xi_0} = 0$ a.e., since a non-null trigonometrical polynomial certainly cannot be zero on a set of positive measure. It follows that for almost all ξ_0, $\hat{f}(\xi_0 + k) = 0$ for all $k \in \mathbb{Z}^n$, i.e., $\hat{f} = 0$ a.e. By the uniqueness theorem for the Fourier transform $f = 0$ a.e. and the proof of the theorem is complete.

Note that the sets A and B are *not* the supports of f resp. $\hat{f}$ in the sense of distribution theory. In fact we have by definition

$$\operatorname{supp}(f) = \bar{A},$$
$$\operatorname{supp}(\hat{f}) = \bar{B},$$

where $\operatorname{supp}(f)$ as usual denotes the distributional support of f and $\bar{A}$ is the closure of A. Thus A and B could a priori be, e.g., dense open sets.

COROLLARY 1. *Theorem 2 also holds if we assume instead of $f \in L^1(\mathbb{R}^n)$ that $f \in L^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$.*

COROLLARY 2. *Let μ be a measure of finite total variation on $\mathbb{R}^n$ and let*

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x, \xi\rangle} \, d\mu(x)$$

be its Fourier transform. Suppose that $m(\operatorname{supp}(\mu)) < \infty$ and $\hat{m}(\operatorname{supp}(\hat{\mu})) < \infty$, where the supports are to be taken in the sense of distribution theory. Then $\mu = 0$.

The proofs of these corollaries are immediate.

Remark. Theorem 2 does not hold for the space of tempered distributions $\mathscr{S}'$ as the following example shows:

Let $T \in \mathscr{S}'(\mathbb{R})$ be defined by

$$T = \sum_{v=-\infty}^{\infty} \delta_{(v)},$$

where $\delta_{(a)}$ denotes the Dirac measure at the point $a \in \mathbb{R}$. It is well known (Poisson's summation formula) that its Fourier transform is

$$\hat{T} = \sum_{v=-\infty}^{\infty} 2\pi\delta_{(2\pi v)}$$

so T and $\hat{T}$ are both supported on sets of measure zero.

Remark. There are proofs from 1977 of the results of this paper by Amrein and Berthier [1] based on Hilbert space methods. They refer to the preprint of this paper from 1974. Nevertheless it was thought that the original proof should be available in the literature.

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