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Constructing G^2 Continuous Curve on Freeform Surface with Normal Projection

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Abstract

This article presents a new method for G^2 continuous interpolation of an arbitrary sequence of points on an implicit or parametric surface with prescribed tangent direction and curvature vector, respectively, at every point. First, a G^2 continuous curve is constructed in three-dimensional space. Then the curve is projected normally onto the given surface. The desired interpolation curve is just the projection curve, which can be obtained by numerically solving the initial-value problems for a system of first-order ordinary differential equations in the parametric domain for parametric case or in three-dimensional space for implicit case. Several shape parameters are introduced into the resulting curve, which can be used in subsequent interactive modification so that the shape of the resulting curve meets our demand. The presented method is independent of the geometry and parameterization of the base surface. Numerical experiments demonstrate that it is effective and potentially useful in numerical control (NC) machining, path planning for robotic fibre placement, patterns design on surface and other industrial and research fields.

Keywords: Hermite interpolation; normal projection; freeform surface; G^2 continuity; ordinary differential equations

1. Introduction

In numerical control (NC) machining of parametric surface, one often needs to guide a cutting tool to change its tool-path from one to another with continuous acceleration. In blended-wing-body (BWB) design for both civilian and military aircraft, one must first construct two contact curves on wing surface and fuselage surface, according to a set of points along with the necessary geometric information, such as the tangent and curvature vectors. Those issues can be reduced to G^2 blending of two specified surface curves or G^2 interpolation of a sequence of points on surface. Actually, blending curve[1] on surfaces is a special case of interpolation curve on surfaces.

As far as G^2 interpolation on a given surface is concerned, H. Pottmann, et al.[2] presented a variational approach to spline curves on surfaces. They characterized interpolating and approximating minimizers on surfaces of arbitrary finite dimension and co-dimension. The minimizers possess a characterization which is very similar to the familiar cubic C^2 splines: the fourth derivative of a cubic vanishes and so does the tangential component of the fourth derivative of spline segments in surface. Those authors aiming at variational curve design characterized the minimizers of an intrinsic geometric counterpart to the L^2 norm of the second derivative, which is integral of the squared covariant derivative of the first derivative with respect to arc length. M. Hofer, et al.[3] discussed energy - minimizing splines in manifolds. It is good that both variational and energy-minimizing approaches can, in a sense, give an optimal solution for curve interpolating data points on surfaces and thus surface curve designed by these methods is overall tight. However the optimal solution does not have any local properties and degrees of freedom for shape control which are mostly important for interactive design in computer aided design (CAD) and computer graphics (CG). For example, to modify a certain curve segment one must change data points and restart the entire process of

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curve design. Another method being able to use in interactive design was given by E. Hartmann[1]. However in practical application, we found that, in addition to the complex implementation process (firstly construct a blending surface and then conduct surface-to-surface intersection or trace an implicit plane curve), the other drawback of the method is that the resulting curve sometimes unlikely preserves G^1 continuity at inner points between interpolation points, since the new blended surface does not always intersect the base surface transversally even if the base surface is special surface such as sphere or other quadrics. An improvement was made in Ref.[4], where surface curve design problem is reduced to that of finding the zero set of a bivariate polynomial of relatively low degree in the parameter space. However, implementation process is still complicated for it involves tracing an implicit curve in plane. Moreover, J. Pegna, et al.[5] once developed a method of orthogonally projecting a space curve onto the surface, which can be comprehensively used in surface trimming, blending and patching. However the method cannot be used for truly designing a curve on a surface, for example, fitting a sequence of data points on surface to create a surface curve. We once improved this orthogonal projection method[6]. Other related theoretic researches can be found in Refs.[7]-[11].

In this article, we develop a new method to overcome the drawbacks in the existing methods[1, 4]. As it is well known, a curve on a surface is usually expressed in the form of differential equations. Here, making use of traditional differential geometric method, we will give a new solution for designing G^2 continuous curve on a surface. We focus our investigation on the problem of G^2 continuously interpolating a range of data points on a surface with prescribed tangent direction at every point.

2. Mathematical Preliminaries

Let us begin with a differentiable parametric surface that is described by a vector-valued function of two variables as follows:

$$\boldsymbol{S}(u,v)=\left[x(u,v)\quad y(u,v)\quad z(u,v)\right]$$
$$(u,v)\in D\subset \mathbf{R}^2 \tag{1}$$

where $x(u, v)$, $y(u, v)$, $z(u, v)$ are differentiable bivariate functions of u and v, which are called the surface parameters, and D denotes the surface domain. The partial derivatives of the vector valued function $\boldsymbol{S}$ with respect to u and v are $\boldsymbol{S}_u(u, v)$, $\mathrm{S}_v(u, v)$, $(u, v) \in D$. The vector $\boldsymbol{N} = \boldsymbol{S}_u \times \boldsymbol{S}_v / |\boldsymbol{S}_u \times \boldsymbol{S}_v|$ is called the unit normal vector (or just normal vector for short) of the surface $\boldsymbol{S}$ at the corresponding point. We assume the surface $\boldsymbol{S}$ is regular, i.e., $\boldsymbol{S}_u \times \boldsymbol{S}_v \neq \mathbf{0}$ for any $(u, v) \in D$. A curve on the surface can be described by parametric equations $u= u(t)$, $v= v(t)$, $t \in [a, b]$ in the surface domain. Its equation in $\mathbf{R}^3$ can be written as the vector-valued function

$$\boldsymbol{P}(t)=\boldsymbol{S}(u(t),v(t))\qquad t\in[a,b] \tag{2}$$

Taking the derivative of $\boldsymbol{P}$ (t) with respect to t to linearize Eq. (2), we get the following corresponding equation in the tangent space of the surface:

$$\boldsymbol{P}(t)=\boldsymbol{S}_u\mathrm{d}u/\mathrm{d}t+\boldsymbol{S}_v\mathrm{d}v/\mathrm{d}t\qquad t\in[a,b] \tag{3}$$

Let $\boldsymbol{S}$ be a C^2 regular parametric surface. It follows from classical differential geometry[12] that

$$\begin{cases} I(\mathrm{d}u,\mathrm{d}v)=(\mathrm{d}\boldsymbol{P})^2=E(\mathrm{d}u)^2+2F\mathrm{d}u\,dv+G(\mathrm{d}v)^2 \\ II(\mathrm{d}u,\mathrm{d}v)=\boldsymbol{N}\cdot\mathrm{d}^2\boldsymbol{P}=e(\mathrm{d}u)^2+2f\mathrm{d}u\,\mathrm{d}v+g(\mathrm{d}v)^2 \end{cases}$$

where I and II are known as the first and the second fundamental form respectively, while "·" indicates the scalar product (same as below). In addition, along the curve $\boldsymbol{P}(t)$, the normal vector $\boldsymbol{N}$ is

$$\boldsymbol{N}(t) = \boldsymbol{N}(u(t), v(t))$$

Then we have the following equations:

$$\left.\begin{aligned} \boldsymbol{N}_u &= a_{11}\boldsymbol{S}_u+a_{21}\boldsymbol{S}_v \\ \boldsymbol{N}v &= a_{12}\boldsymbol{S}_u+a_{22}\boldsymbol{S}_v \end{aligned}\right\} \tag{4}$$

where

$$\begin{cases} a_{11}=(fF-eG)/(EG-F^2) \\ a_{12}=(gF-fG)/(EG-F^2) \\ a_{21}=(eF-fE)/(EG-F^2) \\ a_{22}=(fF-gE)/(EG-F^2) \end{cases}$$

Further there is

$$\boldsymbol{N}'(t)=(a_{11}\boldsymbol{S}_u+a_{21}\boldsymbol{S}_v)\frac{\mathrm{d}u}{\mathrm{d}t}+$$
$$(a_{12}\boldsymbol{S}_u+a_{22}\boldsymbol{S}_v)\frac{\mathrm{d}v}{\mathrm{d}t} \tag{5}$$

Now let us consider implicit surfaces $f(x, y, z)=0$, where the first partial derivatives $f_x=\partial f/\partial x$, $f_y=\partial f/\partial y$, $f_z=\partial f/\partial z$ are continuous and not all zero, everywhere on the surface, i. e., the surface is regular. The vector $\nabla f = [f_x \quad f_y \quad f_z]$ is called the gradient of the implicit surface at the point (x, y, z). The vector $\boldsymbol{N}=\nabla f/|\nabla f|$ is the unit normal vector of the implicit surface at the point (x, y, z). The Jacobian of the gradient ∇f is called the Hessian of f. Write it as $H(f) = J(\nabla f)$. A curve on the implicit surface is described by parametric form $\overline{\boldsymbol{P}}(t)=\left[x(t)\quad y(t)\quad z(t)\right]$. It is characterized by the following equation $f(x(t), y(t), z(t))=0$. Linearizing it, we get

$$f_x \mathrm{d}x/\mathrm{d}t + f_y \mathrm{d}y/\mathrm{d}t + f_z \mathrm{d}z/\mathrm{d}t = 0 \tag{6}$$

In addition, similarly to Eq. (5) we have

$$\boldsymbol{N}'(t) = \boldsymbol{P}'(t) H(f)[I - \boldsymbol{N}^T(t)\boldsymbol{N}(t)]/|\nabla f| \tag{7}$$

Finally one vector identity should be mentioned here for applications in the following discussion. For any three vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ it follows that

$$(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c} = (\boldsymbol{a}\cdot\boldsymbol{c})\,\boldsymbol{b} - (\boldsymbol{b}\cdot\boldsymbol{c})\,\boldsymbol{a} \tag{8}$$

3. Computing Normal Projection Curve

Considering the complexity of the base surface, the projection curve may be comprised of several disconnected segments. Moreover, for a space curve, the normal projection may result in a set of different projection curves on the surface. We mainly describe how to trace one such connected curve. In addition, if not stated otherwise, we focus our discussion on a kind of base surface without boundary.

Let $\boldsymbol{C}(t)$ be the parametric representation of a space curve that we want to project normally onto a surface $\boldsymbol{S}$. The projection curve $\boldsymbol{P}(t)$ thus inherits the parameter of the space curve $\boldsymbol{C}(t)$ and is continuous. J. Pegna[5] once discussed the computation of first ordered projection curve. Here we deduce the equation of the projection curve in another way. Assume that the space curve $\boldsymbol{C}(t)$ is continuous and differentiable. The normal projection is characterized by

$$[\boldsymbol{C}(t) - \boldsymbol{P}(t)]\times\boldsymbol{N}(t) = 0 \tag{9}$$

Taking the derivative of Eq.(9) with respect to t and using Eq. (3) and Eq. (5), we obtain

$$\left\{\left[a_{11}(\boldsymbol{P}-\boldsymbol{C})-\boldsymbol{N}\right]\times\boldsymbol{S}_u + a_{21}(\boldsymbol{P}-\boldsymbol{C})\times\boldsymbol{S}_v\right\}\frac{\mathrm{d}u}{\mathrm{d}t} + \left\{a_{12}(\boldsymbol{P}-\boldsymbol{C})\times\boldsymbol{S}_u + \left[a_{22}(\boldsymbol{P}-\boldsymbol{C})-\boldsymbol{N}\right]\times\boldsymbol{S}_v\right\}\frac{\mathrm{d}v}{\mathrm{d}t} = \boldsymbol{C}'\times\boldsymbol{N} \tag{10}$$

Calculating the cross product of the coefficient vectors of du/dv and dv/du in Eq. (10) with both sides of Eq. (10) respectively, using Eq. (8) and then taking the dot product of N and both sides of the resulting equation, we finally have

$$\left.\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}t} &= \frac{-a_{12}\left[(\boldsymbol{C}-\boldsymbol{P})\cdot\boldsymbol{N}\right](\boldsymbol{C}'\times\boldsymbol{N})\cdot\boldsymbol{S}_u - \left[a_{22}(\boldsymbol{C}-\boldsymbol{P})\cdot\boldsymbol{N}+1\right](\boldsymbol{C}'\times\boldsymbol{N})\cdot\boldsymbol{S}_v}{\left\{(a_{11}a_{22}-a_{12}a_{21})\left[(\boldsymbol{C}-\boldsymbol{P})\cdot\boldsymbol{N}\right]^2 + (a_{11}+a_{22})\left[(\boldsymbol{C}-\boldsymbol{P})\cdot\boldsymbol{N}\right]+1\right\}|\boldsymbol{S}_u\times\boldsymbol{S}_v|}\\ \frac{\mathrm{d}v}{\mathrm{d}t} &= \frac{a_{21}\left[(\boldsymbol{C}-\boldsymbol{P})\cdot\boldsymbol{N}\right](\boldsymbol{C}'\times\boldsymbol{N})\cdot\boldsymbol{S}_v + \left[a_{11}(\boldsymbol{C}-\boldsymbol{P})\cdot\boldsymbol{N}+1\right](\boldsymbol{C}'\times\boldsymbol{N})\cdot\boldsymbol{S}_u}{\left\{(a_{11}a_{22}-a_{12}a_{21})\left[(\boldsymbol{C}-\boldsymbol{P})\cdot\boldsymbol{N}\right]^2 + (a_{11}+a_{22})\left[(\boldsymbol{C}-\boldsymbol{P})\cdot\boldsymbol{N}\right]+1\right\}|\boldsymbol{S}_u\times\boldsymbol{S}_v|}\end{aligned}\right\} \tag{11}$$

if the space curve $\boldsymbol{C}(t)$ does not pass through the centers of principal curvature of the surface at the corresponding point $\boldsymbol{P}(t)$ (i. e. , the denominator in Eq. (11) is non-zero).

For system Eq. (11) to be completely determined, the following initial-valued conditions on the surface domain $\boldsymbol{D}$ must be added.

$$\left.\begin{aligned}u(0) &= u_0\\ v(0) &= v_0\end{aligned}\right\} \tag{12}$$

Once the system of Eqs. (11)-(12) is solved by numerically integrating u and v, the space parametric equation of the projection curve can be got by substituting u and v into Eq.(2).

In the implicit case, the normal projection is also characterized by Eq.(9). Taking the derivative of Eq.(9) with respect to t, taking the cross product of $\boldsymbol{N}$ and both sides of the resulting equation and using Eqs.(6)-(8), we obtain

$$\boldsymbol{P}'(t) = [\boldsymbol{C}'(t) - (\boldsymbol{C}'(t)\cdot\boldsymbol{N})\boldsymbol{N}]\left\{I + \left[(\boldsymbol{C}(t)-\boldsymbol{P}(t)\cdot\boldsymbol{N})\right]H(\mathrm{f})\cdot(\boldsymbol{I}-\boldsymbol{N}^{\mathrm{T}}\boldsymbol{N})/|\nabla f|\right\}^{-1} \tag{13}$$

if the space curve $\boldsymbol{C}(t)$does not pass through the centers of principal curvature of the surface at the corresponding point $\boldsymbol{P}(t)$ (i. e. , the existence of the inverse matrix in Eq.(13) is guaranteed).

Similarly to the parametric case, adding an initial value conditions to Eq. (13) as follows:

$$\left.\begin{aligned}x(0) &= x_0\\ y(0) &= y_0\\ z(0) &= z_0\end{aligned}\right\} \tag{14}$$

we get an initial value problem for a first-order system of ordinary differential equations. Numerically integrating it, the projection curve can be obtained.

4. Surface Curve Design

4. 1. G^2 continuous curves

Specify a sequence of the points $\boldsymbol{P}_i$(i =1, 2,…, n) on a $C^r(r\geq 3)$ regular surface $\boldsymbol{S}$. Assume that its corresponding parameterizations are described by a sequence of real numbers t_i(i =0, 1, …, n). Also specify a sequence of tangent directions $\boldsymbol{T}_i$(i =0, 1,…, n) on the surface $\boldsymbol{S}$ and give a sequence of vectors k_i. Construct a curve lying on the surface S and passing the points $\boldsymbol{P}_i$(i =1, 2, …, n) on condition that the curve's tangent directions and curvature vectors at the points $\boldsymbol{P}_i$ are $\boldsymbol{T}_i$ and $\boldsymbol{k}_i$ respectively. We only consider one

pair of triplets, such as($\boldsymbol{P}_i$, $\boldsymbol{T}_i$, k_i) and ($\boldsymbol{P}_i$ +1, $\boldsymbol{T}_i$ +1, k_i +1). Our tactic is to construct a space curve segment $\boldsymbol{C}_i(t)$ with the corresponding triplets($\boldsymbol{P}_i$, $\boldsymbol{T}_i$, $\boldsymbol{C}''_i(0)$)and($\boldsymbol{P}_i$+1,$\boldsymbol{T}_i$ +1,$\boldsymbol{C}''_{i+1}$ (1)) and then project it onto the surface S to get the curve segment $\boldsymbol{P}_i$ (t).

First we would like to give the following proposition:

Lemma 1 For a space $\boldsymbol{C}^2$ curve $\boldsymbol{r}(t)$, $\boldsymbol{k}$ is curvature vector if and only if the following holds

$$\boldsymbol{r}''(t) = \mu \boldsymbol{r}'(t) / |\boldsymbol{r}'(t)| + |\boldsymbol{r}'(t)|^2 \boldsymbol{k} \tag{15}$$

where μ is an arbitrary constant.

Let $\boldsymbol{S}(u,v) = [x(u,v) \quad y(u,v) \quad z(u,v)]$ be a C^2 regular parametric surface. We have the following conclusions.

Lemma 2 Let $\boldsymbol{T}$ correspond to du/dv and $\boldsymbol{N}$ be a unit tangent vector and a unit normal vector of $\boldsymbol{S}$ at the point $\boldsymbol{P}$ respectively. Then $\boldsymbol{k}$ ($\boldsymbol{T} \perp \boldsymbol{k}$) can be the curvature vector of a curve lying on S, passing $\boldsymbol{P}$ and having $\boldsymbol{T}$ as its tangent vector at $\boldsymbol{P}$ if and only if the following holds

$$\boldsymbol{k} \cdot \boldsymbol{N} = \left. \frac{II(\mathrm{d}u, \mathrm{d}v)}{I(\mathrm{d}u, \mathrm{d}v)} \right|_{\boldsymbol{P}}$$

From Lemma 2 we know that $\boldsymbol{k}_i$ described in the first paragraph in Subsection 4.1 cannot be arbitrary vectors and must satisfy

$$\boldsymbol{k}_i \cdot \boldsymbol{N}_i = \left. \frac{II(\mathrm{d}u_i, \mathrm{d}v_i)}{I(\mathrm{d}u_i, \mathrm{d}v_i)} \right|_{\boldsymbol{P}_i}$$

Lemma 3 Let $\boldsymbol{T}$ and $\boldsymbol{N}$ be a unit tangent vector and a unit normal vector of S at the point $\boldsymbol{P}$ respectively. $\boldsymbol{k}$ is the curvature vector of some or other curve lying on $\boldsymbol{S}$, passing $\boldsymbol{P}$ and having $\boldsymbol{T}$ as its tangent vector at $\boldsymbol{P}$. $\boldsymbol{r}(t)$is an arbitrary curve lying on $\boldsymbol{S}$, passing $\boldsymbol{P}$ and having $\boldsymbol{T}$ as its tangent vector at P. Then it follows at $\boldsymbol{P}$ that

$$|\boldsymbol{r}'(t)|^2 \boldsymbol{k} \cdot \boldsymbol{N} = \boldsymbol{r}''(t) \cdot \boldsymbol{N}$$

Theorem Let $\boldsymbol{T}$ and $\boldsymbol{N}$ be a unit tangent vector and a unit normal vector of $\boldsymbol{S}$ at the point $\boldsymbol{P}$ respectively. $\boldsymbol{k}$ is the curvature vector of some or other curve lying on $\boldsymbol{S}$, passing $\boldsymbol{P}$ and having $\boldsymbol{T}$ as its tangent vector at $\boldsymbol{P}$.$\boldsymbol{c}(t)$is a C^2 space curve passing $\boldsymbol{P}$ and having $\boldsymbol{T}$ as its tangent vector at $\boldsymbol{P}$. $\boldsymbol{r}(t)$ is the normal projection curve of $\boldsymbol{c}(t)$ onto $\boldsymbol{S}$. Then at $\boldsymbol{P}$ the following equalities hold:

1) $\boldsymbol{c}'(t) = \boldsymbol{r}'(t)$

2) $\boldsymbol{c}''(t) - \boldsymbol{r}''(t) = (\boldsymbol{c}''(t) \cdot \boldsymbol{N} - |\boldsymbol{r}'(t)|^2 \boldsymbol{k} \cdot \boldsymbol{N})\boldsymbol{N}$

Proof The projection is characterized by

$$\boldsymbol{c}(t) - \boldsymbol{r}(t) = \alpha(t)\boldsymbol{N}(t) \tag{16}$$

By taking the derivative of Eq. (16) with respect to t, the relationship between the corresponding tangent vectors of space curve and its projection curve on the surface $\boldsymbol{S}$ can be deduced as follows:

$$\boldsymbol{c}'(t) - \boldsymbol{r}'(t) = \alpha'(t)\boldsymbol{N}(t) + \alpha(t)\boldsymbol{N}'(t) \tag{17}$$

From Eq.(16), we have $\alpha(t) = 0$ at $\boldsymbol{P}$. Thus from Eq.(17), at $\boldsymbol{P}$ it follows that

$$\boldsymbol{c}'(t) - \boldsymbol{r}'(t) = \alpha'(t) m \boldsymbol{N}$$

Therefore $\alpha'(t) = 0$ and at $\boldsymbol{P}$ there is

$$\boldsymbol{c}'(t) = \boldsymbol{r}'(t) \tag{18}$$

In addition, from Eq.(17), the relationship between the corresponding second derivative vectors of space curve and its projection curve on the surface $\boldsymbol{S}$ can be described as follows:

$$\boldsymbol{c}''(t) - \boldsymbol{r}''(t) = \alpha''(t)\boldsymbol{N}(t) + 2\alpha'(t)\boldsymbol{N}'(t) + \alpha(t)\boldsymbol{N}''(t)$$

Further, at $\boldsymbol{P}$ it follows that

$$\boldsymbol{c}''(t) - \boldsymbol{r}''(t) = \alpha''(t)\boldsymbol{N} \tag{19}$$

From Eqs.(18)-(19) and Lemma 3, the equality 2) can be easily proved.

Let the normal vectors of the surface $\boldsymbol{S}$ at $\boldsymbol{P}_i, \boldsymbol{P}_{i+1}$ be $\boldsymbol{N}_i, \boldsymbol{N}_{i+1}$ respectively. We would rather assume that the $\boldsymbol{P}_i, \boldsymbol{P}_{i+1}$ are neighboring in the following sense [1].The absolute value of the angles between the normal vectors $\boldsymbol{N}_i, \boldsymbol{N}_{i+1}$ should be less than 180°. For a fixed i, take $\boldsymbol{C}_i(t_i) = \boldsymbol{P}_i$, $\boldsymbol{C}_i(t_{i+1}) = \boldsymbol{P}_{i+1}$, $\boldsymbol{C}'_i(t_i) = \lambda_{i1}\boldsymbol{T}_i$ and $\boldsymbol{C}'_i(t_{i+1}) = \lambda_{i2}\boldsymbol{T}_{i+1}$, where λ is a positive constant. From Theorem, we further take

$$\begin{cases} \boldsymbol{C}''_i(t_i) = \mu_{i1}\boldsymbol{T}_i + \lambda_{i1}^2 \boldsymbol{k}_i - \lambda_{i1}^2 (\boldsymbol{k}_i \cdot \boldsymbol{N}_i)\boldsymbol{N}_i \\ \boldsymbol{C}''_i(t_{i+1}) = \mu_{i2}\boldsymbol{T}_{i+1} + \lambda_{i2}^2 \boldsymbol{k}_{i+1} - \\ \qquad \lambda_{i2}^2 (\boldsymbol{k}_{i+1} \cdot \boldsymbol{N}_{i+1})\boldsymbol{N}_{i+1} \end{cases}$$

Then taking $F_0(s), F_1(s), G_0(s), G_1(s), H_0(s), H_1(s)$ as one group of five degree Hermite blending functions, we design a space quintic Hermite curve $\boldsymbol{C}_i(t)$ as follows:

$$\begin{aligned} \boldsymbol{C}_i(s) = {} & F_0(s)\boldsymbol{P}_i + F_1(s)\boldsymbol{P}_{i+1} + G_0(s)\lambda_{i1}\boldsymbol{T}_i + \\ & G_1(s)\lambda_{i2}\boldsymbol{T}_{i+1} + H_0(s)\big[\mu_{i1}\boldsymbol{T}_i + \lambda_{i1}^2 \boldsymbol{k}_i - \\ & \lambda_{i1}^2 (\boldsymbol{k}_i \cdot \boldsymbol{N}_i)\boldsymbol{N}_i\big] + H_1(s)\big[\mu_{i2}\boldsymbol{T}_{i+1} + \\ & \lambda_{i2}^2 \boldsymbol{k}_{i+1} - \lambda_{i2}^2 [\boldsymbol{k}_{i+1} \cdot \boldsymbol{N}_{i+1}]\boldsymbol{N}_{i+1}\big] \end{aligned}$$

where $s = (t - t_i)/(t_{i+1} - t_i)$ and $t \in [t_i, t_{i+1}]$.

With the method described in Subsection 3.1, we then project $\boldsymbol{C}_i(t)$ normally onto surface $\boldsymbol{S}$, i.e., we can obtain the first-order ordinary differential equations of the projection curve $\boldsymbol{P}_i(t)$ with initial conditions for both parametric and implicit representations of the base surface $\boldsymbol{S}$. For a fixed i, the projection curve $\boldsymbol{P}_i(t)$ is just the desired curve that interpolates the data points $\boldsymbol{P}_i, \boldsymbol{P}_{i+1}$ with the prescribed tangent directions $\boldsymbol{T}_i, \boldsymbol{T}_{i+1}$ and curvature vectors

$\boldsymbol{k}_i, \boldsymbol{k}_{i+1}$ at corresponding points.

As a matter of fact, from Theorem we have $\boldsymbol{P}_i'(t_i) = \boldsymbol{C}_i'(t_i) = \lambda_{i1}\boldsymbol{T}_i$ and $\boldsymbol{P}_i''(t_i) = \mu_{i1}\boldsymbol{T}_i + \lambda_{i1}^2\boldsymbol{k}_i$. Then from Lemma 1 the curve $\boldsymbol{P}_i(t)$ has the curvature vector $\boldsymbol{k}_i$ at the point $\boldsymbol{P}_i$. Similarly the conclusions hold at the point $\boldsymbol{P}_{i+1}$.

Letting $i = 1, 2, \cdots, n-1$, we get a sequence of space curve segments $\boldsymbol{C}_i(t)$. Then projecting every curve segment $\boldsymbol{C}_i(t)$ normally onto the surface $\boldsymbol{S}$, we finally obtain a sequence of projection curve segments $\boldsymbol{P}_i(t)$, $i = 1, 2, \cdots, n-1$. The composite curve

$$\boldsymbol{P}(t) = \bigcup_{i=0}^{n-1} \boldsymbol{P}_i(t), \quad t \in [t_0, t_s] \tag{20}$$

is just what we want.

4.2. G^2 blending curves

The method for interpolating a sequence of points on surface can be used directly in constructing a G^2 blending curve (transition curve) between two given curves parametrically or implicitly on a parametric surface. The G^2 blending curve is completely determined by the tangents and curvatures at the two ends of the transition curve segment and has nothing to do with the global geometry and representation of two given curves. In contrast to general interpolation problem, we must first specify two points on two curves, compute the tangent directions or curvatures of two surface curves at the two points respectively, and use them as interpolation conditions. Then the remainder for us to do is similar to dealing with the interpolation issue. As for curvature computation of surface curves with all kinds of expression form, one can use the formula given out in Ref. [1].

4.3. Degenerate cases

How do we do when one of curve segments $\boldsymbol{C}_i(t)$ happens to pass one center of principle curvature of the surface at a point of its orthogonal projective curve $\boldsymbol{P}_i(t)$, which means the corresponding system of equations collapses? We give two ways as answers to the question above. One way is to insert additional interpolation data into the sequence of points $\boldsymbol{P}_i, i = 0, \cdots, s$. For example, if $\boldsymbol{C}_i(t_0)$ happens to be one center of principal curvature of the surface $\boldsymbol{S}$ at the point $\boldsymbol{P}_i(t_0)$, then insert $\boldsymbol{P}_i(t_0)$ between the points $\boldsymbol{P}_i$ and $\boldsymbol{P}_{i+1}$.

Let $\boldsymbol{C}_i(t) - \boldsymbol{P}_i(t) = \alpha(t)\boldsymbol{N}(t)$. If one of the following conclusions holds:

$$\begin{cases} \alpha(t_0) \neq \dfrac{-(\boldsymbol{C}'(t_0)\times\boldsymbol{N}(t_0))\cdot\boldsymbol{S}_u}{(\boldsymbol{C}'(t_0)\times\boldsymbol{N}(t_0))\cdot\boldsymbol{N}_u} \\ \alpha(t_0) \neq \dfrac{-(\boldsymbol{C}'(t_0)\times\boldsymbol{N}(t_0))\cdot\boldsymbol{S}_v}{(\boldsymbol{C}'(t_0)\times\boldsymbol{N}(t_0))\cdot\boldsymbol{N}_v} \end{cases}$$

then take the unit projective vector $\boldsymbol{T}_{i0}$ of $\boldsymbol{C}_i'(t_0)$ onto the tangent plane of surface at $\boldsymbol{P}_i(t_0)$ as tangent direction. Of course, here we cannot use Eq. (11) or Eq.(17) to compute $\boldsymbol{T}_{i0}$. Obviously the direction of the orthogonal projective vector of $\boldsymbol{C}_i'(t)$ onto the tangent plane of surface at the corresponding point is determined by du/dv or dv/du. For example, from Eq.(3) we have

$$\boldsymbol{P}_i'(t) \,//\, \boldsymbol{S}_u \frac{\mathrm{d}u}{\mathrm{d}v} + \boldsymbol{S}_v \tag{21}$$

Moreover, take

$$\left.\frac{\mathrm{d}u}{\mathrm{d}v}\right|_{t=t_0} = \lim_{t\to t_0} \frac{\dfrac{\mathrm{d}u}{\mathrm{d}t}}{\dfrac{\mathrm{d}v}{\mathrm{d}t}} = \left.\frac{-a_{12}\left[(\boldsymbol{C}-\boldsymbol{P})\cdot\boldsymbol{N}\right](\boldsymbol{C}'\times\boldsymbol{N})\cdot\boldsymbol{S}_u - \left[a_{22}(\boldsymbol{C}-\boldsymbol{P})\cdot\boldsymbol{N}+1\right](\boldsymbol{C}'\times\boldsymbol{N})\cdot\boldsymbol{S}_v}{a_{21}\left[(\boldsymbol{C}-\boldsymbol{P})\cdot\boldsymbol{N}\right](\boldsymbol{C}'\times\boldsymbol{N})\cdot\boldsymbol{S}_v + \left[a_{11}(\boldsymbol{C}-\boldsymbol{P})\cdot\boldsymbol{N}+1\right](\boldsymbol{C}'\times\boldsymbol{N})\cdot\boldsymbol{S}_u}\right|_{t=t_0} \tag{22}$$

Then $\boldsymbol{T}_{i0}$ can be obtained with Eqs.(21)-(22). Otherwise we estimate a tangent vector at $\boldsymbol{P}_i(t_0)$ with those at $\boldsymbol{P}_i$ and $\boldsymbol{P}_{i+1}$ through the method presented in Ref.[1].We must additionally estimate the curvature vector at $\boldsymbol{P}_i(t_0)$ with the method[1]. Finally, restart the above-mentioned interpolation process between data points $\boldsymbol{P}_i$ and $\boldsymbol{P}_i(t_0)$, and $\boldsymbol{P}_i(t_0)$ and $\boldsymbol{P}_{i+1}$.

However, another way is to make the space curve segment far enough from the surface $\boldsymbol{S}$, which has two non-zero principal curvatures everywhere. Take $\alpha(0)$ and $\alpha(1)$ as positive or negative constants, whose absolute values are large enough, and let $\boldsymbol{C}_i'(0)$ and $\boldsymbol{C}_i'(1)$ be perpendicular to $\boldsymbol{N}_i$ and $\boldsymbol{N}_{i+1}$ respectively.

Set

$$\begin{cases} \boldsymbol{C}_i(0) = \alpha(0)\boldsymbol{N}_i + \boldsymbol{P}_i \\ \boldsymbol{C}_i(1) = \alpha(1)\boldsymbol{N}_{i+1} + \boldsymbol{P}_{i+1} \\ \boldsymbol{C}_i'(0) = \alpha(0)N_i'(0) + \lambda_{i1}\boldsymbol{T}_i \\ \boldsymbol{C}_i'(1) = \alpha(1)N_{i+1}'(1) + \lambda_{i2}\boldsymbol{T}_{i+1} \\ \boldsymbol{C}_i''(0) = \mu_{i1}\boldsymbol{T}_i + \lambda_{i1}^2\boldsymbol{k}_i + [\alpha(0)\boldsymbol{N}_i' \cdot \\ \quad \boldsymbol{N}_i' - \lambda_{i1}^2(\boldsymbol{k}_i \cdot \boldsymbol{N}_i)]\boldsymbol{N}_i + \alpha(0)\boldsymbol{N}_i''(0) \\ \boldsymbol{C}_i''(1) = \mu_{i2}\boldsymbol{T}_{i+1} + \lambda_{i2}^2\boldsymbol{k}_{i+1} + [\alpha(1)\boldsymbol{N}_{i+1}' \cdot \boldsymbol{N}_{i+1}' - \\ \quad \lambda_{i2}^2(\boldsymbol{k}_{i+1} \cdot \boldsymbol{N}_{i+1})]\boldsymbol{N}_{i+1} + \alpha(1)\boldsymbol{N}_{i+1}''(1) \end{cases}$$

to construct Hermite curve $\boldsymbol{C}_i(t)$ as above and then project it onto $\boldsymbol{S}$, where $\boldsymbol{N}_i'(0)$ and $\boldsymbol{N}_{i+1}'(1)$ can be calculated by

$$\boldsymbol{N}'(t) = \boldsymbol{P}'(t)\begin{bmatrix} \boldsymbol{S}_u \\ \boldsymbol{S}_v \\ \boldsymbol{S}_u \times \boldsymbol{S}_v \end{bmatrix}^{-1}\begin{bmatrix} a_{11} & a_{21} & 0 \\ a_{12} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \boldsymbol{S}_u \\ \boldsymbol{S}_v \\ \boldsymbol{S}_u \times \boldsymbol{S}_v \end{bmatrix}$$

for the parametric case or by Eq.(7)for the implicit case.

$\boldsymbol{N}_i''(0)$ and $\boldsymbol{N}_{i+1}''(1)$ can be calculated by taking the second derivative of Eq.(5) and Eq.(9) with respect to t, and using Eq.(5) for the parametric case, or by taking the second derivative of Eq.(7) and Eq.(9) with respect to t, and using Eq.(7) for the implicit case.

Remark 1 The constants λ , μ provide some freedoms in designing curve. Consequently, they can be used as control parameters to modify the desired curve so that its shape meets our demand better. However one must pay particular attention to the fact that the overlarge value of these parameters might cause an undesirable cups or loop on the curve.

Remark 2 If the base surface is a sphere, then the orthogonal projection curve of a space curve (not passing the spherical center) onto the sphere can be described by an explicit equation instead of by a procedural curve[13].

5. Numerical Integration

The presented method makes use of numerical integration techniques feasible for first-order explicit ordinary differential equations (ODEs) systems associated with initial value problems. The desired surface curves for all cases can be obtained by solving their corresponding initial value problems. Nevertheless, there are no analytical solutions for the system of first order ODEs presented in the article except some special surfaces such as sphere[13], etc. In general, to deal with the most typical surface in CAD we can solve the systems very efficiently by using standard numerical techniques. For example, the ODEs solver of MATLAB[14] or Numerical Recipes[15], based on Runge-Kutta, Adams-Bashforth and other numerical methods can be used to solve these systems. In addition, these solvers provide user with good controls of tolerance[16].

Generally speaking, the presented method works well for any parametric or implicit surface. It should be emphasized, however, that some special cases need a careful analysis. For instance, if the surface is composed of several piecewise continuous surface patches, some kind of continuity conditions must be considered to ensure that the differential model is still valid in the neighborhood of each patch boundary.

In practical application, we may demand that the resulting curve should be described in the standard form such as B-spline or non-uniform rational B-splines (NURBS). However, the preceding numerical integrating yields an array of points in parametric domain and hence in the surface. Fortunately, using, for example, a cubic B-spline to interpolate those points on the surface we can create a closed form B-spline approximation of projection curve. Here the particular approximation method [17-18] developed by F.E. Wolter and J. Qu can be used to obtain good accuracy. Moreover, based on the method [19] we can also get the B-spline approximation of the resulting curve with the array of points in the parametric domain.

6. Implementation Examples

Actually the presented method can be applied to any implicit or parametric surface, including Bezier, B-spline and NURBS surface that are popular in CAD and CG. However, for the sake of simplicity, we only take a Bézier surface as the base surface in the parametric and implicit surface cases respectively to demonstrate their effectiveness. The control points of Bézier surface are(–4, 4, –1), (–4, 0, 3), (–4, –4, –1), (0, 4, 3), (0, 0, 6), (0, –4, 3), (4, 4, –1), (4, 0, 3), (4, –4, –1).

Fig.1 shows how to design a G^2 continuous curve on Bézier surface $\boldsymbol{S}$, which interpolates only a pair of points, such as $\boldsymbol{P}_0$, $\boldsymbol{P}_1$, with the prescribed unit vectors $\boldsymbol{T}_0$, $\boldsymbol{T}_1$ and curvature vectors $\boldsymbol{k}_1$,$\boldsymbol{k}_2$ at corresponding points on Bézier surface $\boldsymbol{S}$, where the curve $\boldsymbol{P}(t)$is the desired curve. We first design a space curve $\boldsymbol{C}(t)$with Hermite interpolation method and then project orthogonally the curve $\boldsymbol{C}(t)$onto Bézier surface to create the curve $\boldsymbol{P}(t)$.Fig.2 shows a G^2 continuous curve on Bézier surface $\boldsymbol{S}$. The curve is defined by three points $\boldsymbol{P}_0$, $\boldsymbol{P}_1$, $\boldsymbol{P}_2$ with the prescribed unit vectors $\boldsymbol{T}_0$, $\boldsymbol{T}_1$, $\boldsymbol{T}_2$ and curvature vectors $\boldsymbol{k}_0$, $\boldsymbol{k}_1$, $\boldsymbol{k}_2$ at corresponding points. Fig.3 illustrates the G^2 continuous

blending curve of two curves on surface S, i.e., the blending curve segment $\boldsymbol{\Gamma}_{01}$ and curves $\boldsymbol{\Gamma}_0$ have common tangent direction and common curvature vectors at $\boldsymbol{P}_0$, $\boldsymbol{\Gamma}_1$, and the same is true for $\boldsymbol{\Gamma}_{01}$ and curves $\boldsymbol{\Gamma}_1$ at $\boldsymbol{P}_1$. Fig.4 indicates a G^2 continuous initial path designed by the presented method for robotic fibre placement in producing fibre-reinforced composite components.

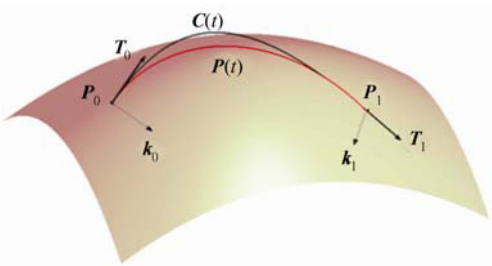

Fig.1 Design procedure for G^2 interpolation curve on Bézier surface.

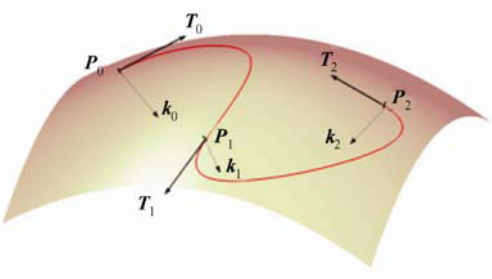

Fig.2 G2 continuous curve on Bézier surface.

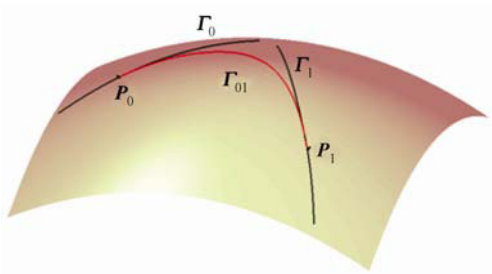

Fig.3 G^2 blending curve of two curves on Bézier surface.

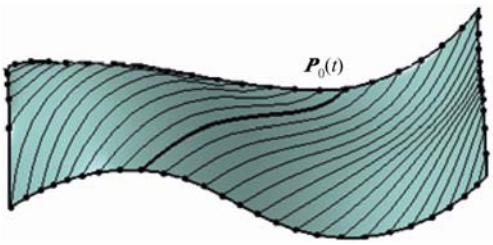

Fig.4 An initial path $\boldsymbol{P}_0$(t)designed by our method over a mould surface.

7. Conclusions

This article develops a method for constructing a G^2 continuous curve lying on a surface.

(1) The method has good local properties and reasonably good flexibility in shape control of the resulting curves, in contrast to the most related methods for curves design on surfaces. Compared with the existing method for G^2 interpolation and blending on surfaces, the presented one gives directly the first-order ordinary differential equations of the desired interpolation curve, thus avoiding using any surface/surface intersection algorithms(which is usually a troublesome process), and can guarantee the existence of the interpolation curve.

(2) The method is also applicable to the feature design or pattern design on surfaces in some related industrial fields.Other potential applications might appear in path planning for robotic fibre placement toward producing fibre-reinforced composite components.

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