Journal of Number Theory 130 (2010) 2701–2706

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

On 2-adic orders of some binomial sums

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ARTICLE INFO

Article history:
Received 21 December 2009
Revised 22 June 2010
Accepted 25 June 2010
Communicated by Ronald Graham

MSC:
primary 11B65
secondary 05A10, 05A19, 11A07, 11S99

Keywords:
2-adic order
Binomial sum

ABSTRACT

We prove that for any nonnegative integers n and r the binomial sum

$$\sum_{k=-n}^{n}\binom{2n}{n-k}k^{2r}$$

is divisible by $2^{2n-\min\{\alpha(n),\alpha(r)\}}$, where $\alpha(n)$ denotes the number of 1s in the binary expansion of n. This confirms a recent conjecture of Guo and Zeng [J. Number Theory 130 (2010) 172–186].

1. Introduction

In 1976 Shapiro [3] introduced the Catalan triangle $(\frac{k}{n}\binom{2n}{n-k})_{n\geqslant k\geqslant 1}$ and determined the sum of entries in the nth row; namely, he showed that

$$\sum_{k=1}^{n}k\binom{2n}{n-k}=\frac{n}{2}\binom{2n}{n}.$$

Let $n,r\in\mathbb{N}=\{0,1,2,\ldots\}$. Recently, Guo and Zeng [1] proved that

$$\frac{2}{n^2\binom{2n}{n}}\sum_{k=1}^{n}\binom{2n}{n-k}k^{2r+1}$$

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[1] The first author is supported by the National Natural Science Foundation for Youths in China (grant 10901078).
[2] The second author is supported by the National Natural Science Foundation (grant 10871087) and the Overseas Cooperation Fund (grant 10928101) of China.

0022-314X/$ – see front matter
doi:10.1016/j.jnt.2010.06.007

is an odd integer if $n, r \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$. They also conjectured that the binomial sum

$$F(n,r) = \sum_{k=-n}^{n} \binom{2n}{n-k} k^{2r} \tag{1.1}$$

is divisible by $2^{2n-\min\{\alpha(n),\alpha(r)\}}$, where $\alpha(n)$ denotes the number of 1s in the binary expansion of n. Note that if $n, r \in \mathbb{Z}^+$ then $F(n,r) = 2\sum_{k=1}^{n} \binom{2n}{n-k} k^{2r}$. Actually the conjecture was motivated by Guo and Zeng's following observations:

$$\sum_{k=1}^{n} \binom{2n}{n-k} k^2 = 2^{2n-2} n,$$

$$\sum_{k=1}^{n} \binom{2n}{n-k} k^4 = 2^{2n-3} n(3n-1),$$

$$\sum_{k=1}^{n} \binom{2n}{n-k} k^6 = 2^{2n-4} n\bigl(15n^2 - 15n + 4\bigr),$$

$$\sum_{k=1}^{n} \binom{2n}{n-k} k^8 = 2^{2n-5} n\bigl(105n^3 - 210n^2 + 147n - 34\bigr).$$

In this paper we shall confirm the sophisticated conjecture of Guo and Zeng.

For an integer n and a prime p, the p-adic order of n at p is given by

$$\nu_p(n) = \sup\{\nu \in \mathbb{N}\colon\ p^\nu \mid n\}.$$

Now we state our main result.

Theorem 1.1. *For any $n, r \in \mathbb{N}$ we have*

$$\nu_2\bigl(F(n,r)\bigr) \geqslant 2n - \min\bigl\{\alpha(n), \alpha(r)\bigr\}, \tag{1.2}$$

where $F(n,r)$ is given by (1.1).

Note that (1.2) can be split into two inequalities:

$$\nu_2\bigl(F(n,r)\bigr) \geqslant 2n - \alpha(n) \tag{1.3}$$

and

$$\nu_2\bigl(F(n,r)\bigr) \geqslant 2n - \alpha(r). \tag{1.4}$$

In Sections 2 and 3 we will show (1.3) and (1.4) respectively.

2. Proof of (1.3)

Let p be any prime. A useful theorem of Legendre (see, e.g., [2, pp. 22–24]) asserts that for any $n \in \mathbb{N}$ we have

$$\nu_p(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor = \frac{n-\alpha_p(n)}{p-1},$$

where $\alpha_p(n)$ is the sum of the digits of n in the expansion of n in base p. In particular, $\nu_2(n!) = n - \alpha(n)$ for all $n = 0, 1, 2, \ldots$.

Lemma 2.1.

(i) *For any* $n \in \mathbb{Z}^+$ *we have*

$$\nu_2(n) - 1 = \alpha(n-1) - \alpha(n). \tag{2.1}$$

(ii) *Let* $s > t \geqslant 0$ *be integers. Then*

$$\nu_2\left(\binom{s}{t}\right) \geqslant \alpha(t) - \alpha(s) + 1. \tag{2.2}$$

Proof. (i) In view of Legendre's theorem, for any positive integer n we have

$$\nu_2(n) = \nu_2(n!) - \nu_2\big((n-1)!\big) = n - \alpha(n) - \big(n - 1 - \alpha(n-1)\big) = \alpha(n-1) - \alpha(n) + 1.$$

This proves (2.1).

(ii) With the help of Legendre's theorem,

$$\begin{aligned}\nu_2\left(\binom{s}{t}\right) &= \nu_2(s!) - \nu_2(t!) - \nu_2\big((s-t)!\big)\\ &= s - \alpha(s) - \big(t - \alpha(t)\big) - \big(s - t - \alpha(s-t)\big)\\ &= \alpha(t) - \alpha(s) + \alpha(s-t)\\ &\geqslant \alpha(t) - \alpha(s) + 1 \quad (\text{since } s - t \geqslant 1).\end{aligned}$$

So (2.2) holds. □

Lemma 2.2. *For* $n, r \in \mathbb{Z}^+$ *we have*

$$F(n,r) = n^2 F(n, r-1) - 2n(2n-1)F(n-1, r-1). \tag{2.3}$$

Proof. Since

$$\big(n^2 - k^2\big)\binom{2n}{n-k} = 2n(2n-1)\binom{2n-2}{n-1-k},$$

we have

$$\sum_{k=-n}^{n}\binom{2n}{n-k}k^{2r}=n^2\sum_{k=-n}^{n}\binom{2n}{n-k}k^{2r-2}-2n(2n-1)\sum_{k=-n+1}^{n-1}\binom{2n-2}{n-1-k}k^{2r-2},$$

which gives (2.3). □

Proof of (1.3). We use induction on $n+r$. Clearly (1.3) holds trivially when $n=0$ or $r=0$.

Now let $n,r\in\mathbb{Z}^+$ and assume (1.3) for any smaller value of $n+r$. By (2.1), (2.3) and the induction hypothesis, we have

$$\begin{aligned}\nu_2\big(F(n,r)\big)&\geqslant\min\big\{\nu_2\big(n^2F(n,r-1)\big),\nu_2\big(2n(2n-1)F(n-1,r-1)\big)\big\}\\&=\min\big\{2\nu_2(n)+\nu_2\big(F(n,r-1)\big),1+\nu_2(n)+\nu_2\big(F(n-1,r-1)\big)\big\}\\&\geqslant\min\big\{2\nu_2(n)+2n-\alpha(n),1+\nu_2(n)+2(n-1)-\alpha(n-1)\big\}\\&=2n-\alpha(n).\end{aligned}$$

This concludes the induction step. □

3. Proof of (1.4)

Lemma 3.1. *For $n,r\in\mathbb{Z}^+$ we have*

$$\begin{aligned}F(n,r)=4F(n-1,r)&-\sum_{i=0}^{r-1}\binom{2r}{2i}F(n,i)-2(2n-1)\sum_{i=0}^{r-1}\binom{2r}{2i+1}F(n-1,i)\\&+n\sum_{i=0}^{r-1}\binom{2r}{2i+1}F(n,i)+2\sum_{i=0}^{r-1}\binom{2r}{2i}F(n-1,i).\end{aligned}\tag{3.1}$$

Proof. Let $n\in\mathbb{N}$ and $r\in\mathbb{Z}^+$. We want to prove (3.1) with n in it replaced by $n+1$.

Clearly

$$\begin{aligned}F(n,r)&=\sum_{k=-n-1}^{n-1}\binom{2n}{n-1-k}(k+1)^{2r}\\&=\sum_{k=-n-1}^{n}\left(\binom{2n+1}{n-k}-\binom{2n}{n-k}\right)\sum_{j=0}^{2r}\binom{2r}{j}k^j\\&=\sum_{j=0}^{2r}\binom{2r}{j}\sum_{k=-n-1}^{n}\binom{2n+1}{n-k}k^j-\sum_{i=0}^{r}\binom{2r}{2i}\sum_{k=-n}^{n}\binom{2n}{n-k}k^{2i};\end{aligned}\tag{3.2}$$

in the last step we use the fact that if j is odd then

$$\begin{aligned}\sum_{k=-n}^{n}\binom{2n}{n-k}k^j&=\frac12\left(\sum_{k=-n}^{n}\binom{2n}{n-k}k^j+\sum_{k=-n}^{n}\binom{2n}{n+k}(-k)^j\right)\\&=\frac12\sum_{k=-n}^{n}\binom{2n}{n-k}\big(k^j+(-k)^j\big)=0.\end{aligned}$$

When j is even, we have

$$2\sum_{k=-n-1}^{n}\binom{2n+1}{n-k}k^j=\sum_{k=-n-1}^{n+1}\left(\binom{2n+1}{n-k}+\binom{2n+1}{n+k}\right)k^j$$
$$=\sum_{k=-n-1}^{n+1}\binom{2n+2}{n+1-k}k^j=F\left(n+1,\frac{j}{2}\right). \tag{3.3}$$

If j is odd, then

$$(n+1)\sum_{k=-n-1}^{n}\binom{2n+1}{n-k}k^{j-1}+\sum_{k=-n-1}^{n}\binom{2n+1}{n-k}k^j$$
$$=\sum_{k=-n-1}^{n}(n+1+k)\binom{2n+1}{n-k}k^{j-1}=(2n+1)\sum_{k=-n}^{n}\binom{2n}{n-k}k^{j-1},$$

i.e.,

$$\sum_{k=-n-1}^{n}\binom{2n+1}{n-k}k^j=(2n+1)\sum_{k=-n}^{n}\binom{2n}{n-k}k^{j-1}-(n+1)\sum_{k=-n-1}^{n}\binom{2n+1}{n-k}k^{j-1}$$
$$=(2n+1)F\left(n,\frac{j-1}{2}\right)-\frac{n+1}{2}F\left(n+1,\frac{j-1}{2}\right), \tag{3.4}$$

where we use (3.3) in the last step. Combining (3.2)–(3.4), we get

$$F(n,r)=\frac{1}{2}\sum_{i=0}^{r}\binom{2r}{2i}F(n+1,i)+(2n+1)\sum_{i=0}^{r-1}\binom{2r}{2i+1}F(n,i)$$
$$-\frac{n+1}{2}\sum_{i=0}^{r-1}\binom{2r}{2i+1}F(n+1,i)-\sum_{i=0}^{r}\binom{2r}{2i}F(n,i),$$

which yields the desired result. $\square$

Proof of (1.4). We still use induction on $n+r$. There is nothing to do if $n=0$ or $r=0$. Assume that $n,r\geqslant 1$ and (1.4) holds for any smaller value of $n+r$. In view of Lemma 3.1, $\nu_2(F(n,r))$ is not smaller than the minimum of the following numbers:

$$2+\nu_2\big(F(n-1,r)\big),\qquad \min_{0\leqslant i<r}\nu_2\left(\binom{2r}{2i}F(n,i)\right),\qquad \min_{0\leqslant i<r}\nu_2\left(n\binom{2r}{2i+1}F(n,i)\right),$$
$$1+\min_{0\leqslant i<r}\nu_2\left(\binom{2r}{2i+1}F(n-1,i)\right),\qquad 1+\min_{0\leqslant i<r}\nu_2\left(\binom{2r}{2i}F(n-1,i)\right).$$

By the induction hypothesis and Lemma 2.1(ii), we have $\nu_2(F(n-1,r))\geqslant 2n-2-\alpha(r)$, and also

$$\nu_2\left(\binom{2r}{2i}F(n,i)\right) \geqslant 2n-\alpha(i)+\alpha(2i)-\alpha(2r)+1 = 2n-\alpha(r)+1,$$

$$\nu_2\left(\binom{2r}{2i+1}F(n-1,i)\right) \geqslant 2n-2-\alpha(i)+\alpha(2i+1)-\alpha(2r)+1 = 2n-\alpha(r),$$

$$\nu_2\left(n\binom{2r}{2i+1}F(n,i)\right) \geqslant 2n-\alpha(i)+\alpha(2i+1)-\alpha(2r)+1 = 2n-\alpha(r)+2,$$

and

$$\nu_2\left(\binom{2r}{2i}F(n-1,i)\right) \geqslant 2n-2-\alpha(i)+\alpha(2i)-\alpha(2r)+1 = 2n-\alpha(r)-1.$$

Thus (1.4) follows. □

Acknowledgment

The authors wish to thank the referees for their helpful comments.

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