JOURNAL OF ALGEBRA 48, 305–320 (1977)

Subprojective Lattices and Projective Geometry

GEORGE MARKOWSKY

Computer Sciences Department, IBM T. J. Watson Research Center, Yorktown Heights, New York 10598

AND

MARIO PETRICH

Mathematics Department, Pennsylvania State University, University Park, Pennsylvania 16802

Communicated by S. MacLane

Received February 17, 1976

The class of lattices we are interested in (subprojective lattices), can be gotten by taking the MacNeille completions of the class of complemented, modular, atomic lattices. McLaughlin showed that subprojective lattices can be represented as the lattices of W-closed subspaces of a vector space U in duality with a vector space W. In this paper, we give a characterization of subprojective lattices in terms of atoms and dual atoms, by means of an incidence space satisfying self-dual axioms. In the finite-dimensional case, a subprojective lattice is projective, and hence our self-dual axioms characterize finite-dimensional projective spaces in terms of points and hyperplanes. No numerical parameters appear explicitly in these axioms. For each subprojective lattice L with at least three elements, we define a *projective envelope* $\mathscr{P}(L)$ for it. $\mathscr{P}(L)$ is a projective lattice and there is a natural inf-preserving injection of L into $\mathscr{P}(L)$. This injection has other important properties which we take as the definition of a *geometric map*. In the course of studying geometric maps, we obtain a lattice theoretic proof of Mackey's result that the join of a U-closed subspace of V and a finite-dimensional subspace is U-closed, where (U, V) form a dual pair of vector spaces over a division ring. Furthermore, we show that if L is a subprojective lattice, P a projective lattice, and $\psi: L \to P$ a geometric map, then P is isomorphic to the projective envelope $\mathscr{P}(L)$ of L. The paper presents many other properties of subprojective lattices. It concludes with a characterization of subprojective lattices which are also projective.

1. INTRODUCTION AND SUMMARY

The class of lattices we are interested in, here called subprojective, was introduced by McLaughlin [11], under the name of C-lattices, as lattices which are MacNeille completions of complemented, modular, atomic lattices. By definition, they are complete, atomistic, dual atomistic, and satisfy the

ISSN 0021-8693

double covering condition. McLaughlin [11] also proved that they can be represented as direct products of irreducible subprojective lattices, and that the latter, except for low dimensions, can in turn be isomorphically represented as the lattices of all W-closed subspaces of a vector space U in duality with a vector space W. Since the lattice of all subspaces of a vector space is a projective lattice (by definition, the lattice of all subspaces of a projective space) McLaughlin's result represents a connection between subprojective and projective lattices, and thus between subprojective lattices and projective geometry. McLaughlin noted in [10] that the entire subprojective lattice can be obtained as a MacNeille completion of the partially ordered set which consists of atoms and dual atoms of the lattice. An analogous statement for a larger class of lattices was established by Markowsky [9].

Since the definition of a subprojective lattice is meet–join symmetric, a characterization of subprojective lattices in terms of atoms and dual atoms must be possible by means of an incidence space satisfying self-dual axioms. In the finite-dimensional case, a subprojective lattice is projective, and hence such self-dual axioms would characterize a finite-dimensional projective space in terms of points and hyperplanes. For an n-dimensional projective space (or geometry) such a set of axioms has been devised by Esser [3], and for finite projective spaces by Dembowsky and Wagner [2], Kantor [4], and Lüneburg [6]. All of these sets of axioms contain at least one axiom with numerical parameters, so that an infinite-dimensional analog cannot be deduced from them.

Several charactzrizations and properties of subprojective lattices were established by Petrich [12, 13]. Direct decompositions of certain classes of lattices, which include subprojective lattices, were studied by Markowsky [9].

This paper divides naturally into several parts as follows. Section 2 contains most of the general definitions needed throughout the paper. In Section 3, we prove one of the principal results of this paper, viz., a theorem which establishes a strong relationship between subprojective lattices and a class of incidence spaces, here called subprojective. Various results concerning direct decompositions of subprojective lattices and spaces form the content of Section 4. A special meet-embedding, here called a geometric map, of a subprojective lattice into a projective lattice is discussed in Section 5. Section 6 contains further properties of geometric maps and a construction of the image lattice under a geometric map. Finally, in Section 7, we prove the uniqueness of the projective envelope of a subprojective lattice and find necessary and sufficient conditions on a subprojective lattice to be projective.

2. Definitions

We summarize here basic notions needed throughout the paper. The basic reference for lattices is Birkhoff [1] and for projective geometry is Lenz [5].

2.1. Let L be a lattice. The *least* (resp. *greatest*) element of L is denoted by 0 (resp. 1), if it exists. Let $x, y \in L$.

(a) x *covers* y (in notation $x \succ y$) if $x > y$ and $x > z > y$ for no $z \in L$. An element which covers 0 (if it exists) is an *atom*, and element which covers an atom is a *hyperatom*. Dually, an element covered by 1 (if it exists) is a *dual atom*, an element covered by a dual atom is a *dual hyperatom*. The set of all atoms (resp. dual atoms) of L is denoted by A_L (resp. D_L).

(b) If 0 (resp. 1) exists, then the *height* (resp. *deficiency*) of x is the supremum of the lengths of all chains between x and 0 (resp. 1) if this supremum exists and is finite. In particular, the height of 0 is zero, of atoms is one, etc., and dually for 1, dual atoms, etc. If the height of 1 exists, it is the *dimension* of L, otherwise the dimension of L is *infinite*.

(c) A nonempty subset D of L is *join dense* if every element of L is the supremum of some subset of D; L is *atomistic* if A_L is join dense in L. Meet dense and dual atomistic are defined dually.

(d) L satisfies the *double covering condition* if for any x, y, L,

$$x \succ x \wedge y \qquad \text{if and only if} \quad x \vee y \succ y.$$

2.2. A complete, atomistic, dual atomistic lattice with a least two elements, satisfying the double covering condition, is a *subprojective lattice*.

Remark. Note that in a finite lattice the double covering condition implies modularity (see [1, Theorem 16, p. 41]). Thus any finite-length subinterval of a subprojective lattice is modular. It also follows that the elements of finite height (finite deficiency) form a modular lattice since any three of them are contained in a finite-length subinterval.

In the terminology of McLaughlin [11], these are precisely the C-lattices, and in that of Maeda and Maeda [8], the complete DAC-lattices.

2.3. An *incidence space*, I, is an ordered triple $(A, D; |)$, where A and D are nonempty sets and $|$ is a binary relation between them.

(a) For any $X \subseteq A$, we write

$$X^* = \{d \in D \mid \text{for all } a \in X,\ a \mid d\},$$

and for any $Y \subseteq D$, we write

$$Y^+ = \{a \in A \mid \text{for all } d \in Y,\ a \mid d\}.$$

For any $a \in A$ (resp. $d \in D$), we write a^* (resp. d^+) instead of $\{a\}^*$ (resp. $\{d\}^+$).

Note that $(*, +)$ form a Galois connection between the power sets of A and D (see [1, p. 124] for the pertinent definitions and properties).

(b) By *the lattice of closed sets of I* we mean the set

$$\mathscr{C}(I) = \{B^+ \mid B \subseteq D\}$$

ordered by set inclusion. It follows from the properties of Galois connections that $\mathscr{C}(I)$ is a complete lattice with meet being intersection.

(c) Let $I = (A, D; |)$ and $I' = (A', D'; |')$ be incidence spaces. An ordered pair (f, g) of bijections $f\colon A \to A'$ and $g\colon D \to D'$ is an *isomorphism* of I into I' if for all $a \in A$, $d \in D$,

$$a \mid d \qquad \text{if and only if} \quad f(a) \mid f(d).$$

In such a case, we say that I and I' are *isomorphic* and write $I \cong I'$.

2.4. Let L be a subprojective lattice. The *incidence space* $\mathscr{I}(L)$ of L is the triple $(A_L, D_L; |)$ where for any $a \in A_L$, $d \in D_L$,

$$a \mid b \qquad \text{if and only if} \quad a \leqslant b.$$

2.5. An incidence space $(A, D; |)$ is a *subprojective space* if it satisfies the following axioms:

S_0: for all $p \in A$, $p^* \neq \varnothing^*$;

let $p, q \in A$, $p \neq q$;

S_1: $p^* \not\subseteq q^*$;

S_2: for all $d \in D$ there exists $a \in A$ such that $a^* \supseteq \{p, q\}^* \cup \{d\}$;

S_3: if $r, s \in A$, $r \neq s$, and $\{r, s\}^* \supseteq \{p, q\}^*$, then $\{r, s\}^* = \{p, q\}^*$;

and their duals:

S'_0: for all $h \in D$, $h^+ \neq \varnothing^+$;

let $h, k \in D$, $h \neq k$;

S'_1: $h^+ \not\subseteq k^+$;

S'_2: for all $a \in A$ there exists $d \in D$ such that $d^+ \supseteq \{h, k\}^+ \cup \{a\}$;

S'_3: if $c, d \in D$, $c \neq d$ and $\{c, d\}^+ \supseteq \{h, k\}^+$, then $\{c, d\}^+ = \{h, k\}^+$.

Remark. If $A(D)$ has more than one element $S_0(S'_0)$ is implied by $S_1(S'_1)$. Also note that when $A(D)$ contains two or more elements, $S_1(S'_1)$ implies that $D(A)$ does also. Thus whenever A or D contains more than one element we do not bother to verify axioms S_0 or S'_0.

2.6. If A and B are sets, we write $A \backslash B = \{a \in A \mid a \notin B\}$.

3. Subprojective Lattices and Subprojective Spaces

We are now ready for the first main result of the paper. To start with, we have the following simple statement.

3.1 Lemma. *Let L be a subprojective lattice. Then for all $X \subseteq A_L$, we have*

$$X^* = \left\{d \in D_L \,\middle|\, d \geqslant \bigvee X\right\} = X^{*+*},$$

$$\bigvee X = \bigwedge X^* = \bigwedge X^{*+*} = \bigvee X^{*+};$$

dually, for all $Y \subseteq D_L$, we have

$$Y^+ = \left\{a \in A_L \,\middle|\, a \leqslant \bigwedge Y\right\} = Y^{+*+},$$

$$\bigwedge Y = \bigvee Y^+ = \bigvee Y^{+*+} = \bigwedge Y^{+*}.$$

Proof. By definition

$$X^* = \{d \in D_L \mid d \geqslant a \text{ for all } a \in X\}.$$

It is clear that $d \geqslant a$ for all $a \in X$ if and only if $d \geqslant \vee X$. Dual atomisticity of L implies that $\vee X = \wedge X^*$. The equality $X^* = X^{*+*}$ follows from the basic properties of Galois connections (see [1, p. 124]). The statements concerning Y follow by duality. ∎

3.2. Correspondence Theorem. If L is a subprojective lattice, then $\mathscr{I}(L)$ is a subprojective space and $\mathscr{C}(\mathscr{I}(L)) \cong L$. Conversely, if I is a subprojective space, then $\mathscr{C}(I)$ is a subprojective lattice and $\mathscr{I}(\mathscr{C}(I)) \cong I$.

Proof. There is one subprojective lattice L_0 for which $|A_{L_0}| = 1$ or $|D_{L_0}| = 1$. It is the two element lattice in which $|A_{L_0}| = 1$ or $|D_{L_0}| = 1$. The only subprojective space, I_0, for which $|A| = 1$ or $|D| = 1$ is the one in which $|A| = |D| = 1$ and $| = \varnothing$. The reader can easily verify that $\mathscr{I}(L_0) = I_0$ and $\mathscr{C}(I_0) = L_0$. Thus we need to verify the theorem only for subprojective lattices with $|A_L| \geqslant 2 \leqslant |D_L|$ and subprojective spaces with $|A| \geqslant 2 \leqslant |D|$. By the remark in Section 2.5 we can ignore axioms S_0 and S'_0 in the rest of the proof.

Let $p, q \in A_L$, $p \neq q$. Since L is dual atomistic, there exists $d \in D_L$ such that $p \leqslant d$ and $q \nleqslant d$. Hence $d \in p^* \backslash q^*$ so that $p^* \nsubseteq q^*$. This establishes S_1.

Let $d \in D_L$ be such that $d \notin p^* \cup q^*$ where p, q are as above. Then $d \ngeqslant p \vee q$ and thus $d \vee (p \vee q) = 1$ since d is a dual atom. Hence $d \vee (p \vee q) \succ d$ which by the double covering condition implies that $p \vee q \succ d \wedge (p \vee q)$. Since $p \neq q$, we have that $p \vee q$ is a hyperatom by the double covering condition.

Consequently, $a = d \wedge (p \vee q)$ is an atom. It is clear that $a^* \supseteq \{p, q\}^* \cup \{d\}$. This proves S_2.

Next let $p, q, r, s \in A_L$ be such that $r \neq s$ and $\{r, s\}^* \supseteq \{p, q\}^*$. First S_1 implies that $p \neq q$. By the above lemma, we obtain

$$r \vee s = \bigwedge \{r, s\}^* \leqslant \bigwedge \{p, q\}^* = p \vee q.$$

By the double covering condition, we have $r, s \prec r \vee s$ and $p, q \prec p \vee q$. Since $r \neq s$, we may assume that $p \neq r$. Again by the double covering condition, we obtain $r, p \prec p \vee r \leqslant (r \vee s) \vee (p \vee q) = p \vee q$. Further, $p \vee r = p \vee q$ since $p \vee q \succ p$. It follows that $r \prec r \vee s = p \vee r$ and thus $p \vee q = p \vee r = r \vee s$. Consequently, $\{r, s\}^* = \{p, q\}^*$, establishing S_3.

Axioms S'_1, S'_2 and S'_3 follow by duality. Therefore $\mathscr{I}(L)$ is a subprojective space.

We define a function f by

$$f(Y^+) = \bigvee Y^+ \qquad (Y^+ \in \mathscr{C}(\mathscr{I}(L))).$$

It is clear that $f\colon \mathscr{C}(\mathscr{I}(L)) \to L$ and is isotone. Let $x \in L$. Then $x = \wedge Y_x$, where $Y_x = \{y \in D_L \mid x \leqslant y\}$, since L is dual atomistic. By the above lemma, we have $\wedge Y_x = \vee Y_x^+ = f(Y_x^+)$ so that f is surjective. Let $Y_1^+, Y_2^+ \in \mathscr{C}(\mathscr{I}(L))$ be such that $f(Y_1^+) \leqslant f(Y_2^+)$. Again by the lemma, we obtain

$$\bigwedge Y_1 = \bigvee Y_1^+ = f(Y_1^+) \leqslant f(Y_2^+) = \bigvee Y_2^+ = \bigwedge Y_2,$$

so that

$$Y_1^+ = \left\{a \in A_L \,\middle|\, a \leqslant \bigwedge Y_1\right\} \subseteq \left\{a \in A_L \,\middle|\, a \leqslant \bigwedge Y_2\right\} = Y_2^+.$$

It follows that f is injective, since $f(a) = f(b)$ implies that $a \leqslant b$ and $b \leqslant a$. Hence f is an isotone bijection with an isotone inverse and is thus a lattice isomorphism.

Conversely, let $I = (B, C; |)$ be a subprojective space. Let $D = \{c^+ \mid c \in C\}$. We claim that D is the set of all dual atoms of $\mathscr{C}(I)$. Clearly $\varnothing^+ = B$ is the greatest element of $\mathscr{C}(I)$. Let $c \in C$ and $Y \subseteq C$ and assume that $c^+ \subseteq Y^+ \subsetneq B$. Since $Y^+ \neq B$, we have $Y \neq \varnothing$. If $k \in Y$, then $k^+ \supseteq Y^+ \supseteq c^+$, which contradicts axiom S'_1. Consequently, every element of D is a dual atom of $\mathscr{C}(I)$. Since for any $Y \subseteq C$, we have $Y^+ = \bigcap_{k \in Y} k^+$, it follows that every dual atom of $\mathscr{C}(I)$ is of the form c^+ for some $c \in C$ and that $\mathscr{C}(I)$ is dual atomistic.

We assert next that $p^{*+} = \{p\}$ for all $p \in B$. It is clear that $p \in p^{*+}$. If $q \in p^{*+}$, then $q^* \supseteq p^{*+*} = p^*$ by the above lemma, which by axiom S_1 implies that $q = p$. Consequently, letting $A = \{\{p\} \mid p \in B\}$ satisfies $A \subseteq \mathscr{C}(I)$. It now follows easily that A is the set of all atoms of $\mathscr{C}(I)$ and that $\mathscr{C}(I)$ is atomistic.

In order to prove that $\mathscr{C}(I)$ satisfies the double covering condition, we need some preparation. We first claim that if $p, q \in B$ and $h, k \in C$ are such that

$p \notin h^+ \cup k^+$ and $q \in h^+ \backslash k^+$, then there exists $c \in C$ such that $q \notin c^+$ and $c^+ \supseteq \{h, k\}^+ \cup \{p\}$. Indeed, axiom S'_2 guarantees the existence of $c \in C$ such that $c^+ \supseteq \{h, k\}^+ \cup \{p\}$. Observe that $\{c, h\}^+ = c^+ \cap h^+ \supseteq \{h, k\}^+$. By axiom S'_3, we then have $\{c, h\}^+ = \{h, k\}^+$. Since $q \in h^+ \backslash k^+$, it follows that $q \notin \{c, h\}^+$, and thus $q \notin c^+$.

Given $a \in \mathscr{C}(I)$ and $\{p\} \in A$ such that $\{p\} \nleqslant a$, we show next that $a \prec a \vee \{p\}$ in $\mathscr{C}(I)$. By contradiction, assume that $a \not\prec a \vee \{p\}$. Since $\mathscr{C}(I)$ is atomistic, there exists $\{q\} \in A$ such that $a < a \vee \{q\} < a \vee \{p\}$. Dual atomisticity of $\mathscr{C}(I)$ provides $h^+, k^+ \in D$ such that $a \subseteq h^+$, k^+, $a \vee \{q\} \subseteq h^+$, $a \vee \{q\} \nsubseteq k^+$, and $a \vee \{p\} \nsubseteq h^+$. Hence $p \notin h^+ \cup k^+$ and $q \in h^+ \backslash k^+$, and by the above, there exists $c \in C$ such that $c^+ \supseteq \{h, k\}^+ \cup \{p\}$ and $q \notin c^+$. But then $c^+ \supseteq h^+ \cap k^+ \supseteq a$ and $p \in c^+$ which implies that $c^+ \supseteq a \vee \{p\} \supseteq a \vee \{q\}$, contradicting the fact that $q \notin c^+$.

Now let $x \wedge y \prec x$. There exists $\{p\} \in A$ such that $(x \wedge y) \vee \{p\} = x$. Note that $\{p\} \nleqslant y$. Thus $y \prec y \vee \{p\} = y \vee (x \wedge y) \vee \{p\} = y \vee x$.

In order to prove the second half of the double covering condition, we need a preliminary result. Given $a \in \mathscr{C}(I)$ and $h^+ \in D$ such that $a \nleqslant h^+$, we show that $a \succ a \wedge h^+$. Suppose the conclusion is false. Since $\mathscr{C}(I)$ is dual atomistic, there exists $k^+ \in D$ such that $a > a \wedge k^+ > a \wedge h^+$. Atomisticity of $\mathscr{C}(I)$ provides $\{p\}, \{q\} \in A$ such that $\{q\} \leqslant a$, $\{q\} \nleqslant a \wedge k^+$, $\{p\} \leqslant a \wedge k^+$, and $\{p\} \nleqslant a \wedge h^+$. It follows that, $q \notin k^+ \cup h^+$ and $p \in k^+ \backslash h^+$. Thus $h \notin p^* \cup q^*$ and $k \in p^* \backslash q^*$. By duality, a statement proved above yields $b \in B$, such that $b^* \supseteq \{p, q\}^* \cup \{h\}$ and $k \notin b^*$. Note that $h \in b^*$ implies that $b \in h^+$. Since $\mathscr{C}(I)$ is dual atomistic, we have $a = \bigwedge \{c^+ \in D \mid a \leqslant c^+\}$. If $c^+ \geqslant a$, then $\{p, q\} \subseteq c^+$, and thus $c \in \{p, q\}^* \subseteq b^*$, so that $c^+ \geqslant \{b\}$. This proves that $a \geqslant \{b\}$. But $b \in h^+$ implies that $k^+ \geqslant a \wedge k^+ \geqslant a \wedge h^+ \geqslant \{b\}$, contradicting the fact that $k \notin b^+$. Consequently, $a \succ a \wedge h^+$.

Now suppose that $y \prec x \vee y$. There exists $h^+ \in D$ such that $(x \vee y) \wedge h^+ = y$. Note that $x \nleqslant h^+$. It follows that $x \succ x \wedge h^+ = x \wedge (x \vee y) \wedge h^+ = x \wedge y$.

This completes the proof of the double covering condition in $\mathscr{C}(I)$. We have observed before this theorem that $\mathscr{C}(I)$ is complete. Therefore $\mathscr{C}(I)$ is a subprojective lattice. It is straightforward to verify that $\mathscr{I}(\mathscr{C}(I)) \simeq I$. ∎

A concept somewhat more general than that of a subprojective space was used by Markowsky [9] to describe a large class of lattices. If we consider an incidence space I as being a partially ordered set with the order given by the incidence relation in an obvious way, then $\mathscr{C}(I)$ corresponds to taking the MacNeille completion of I. It is easy to see that isomorphisms of incidence spaces I and I' induce isomorphisms of $\mathscr{C}(I)$ and $\mathscr{C}(I')$, and conversely. In fact, we may consider the categories of subprojective spaces (resp. lattices) and their isomorphisms. The above theorem and these statements about isomorphisms can be used to prove that these two categories are equivalent.

4. Direct Decomposition

It is easy to show that a nonempty direct product of subprojective lattices is a subprojective lattice. Conversely, it is also easy to show directly that if a product of lattices is a subprojective lattice, then each of the factors is itself a subprojective lattice. We outline here the basic facts about direct decompositions of subprojective lattices and spaces.

A subprojective lattice is directly irreducible if and only if it satisfies Fano's condition (i.e., every hyperatom covers at least three atoms); this is equivalent to subdirect irreducibility. For details, see [13, Proposition 8.4], where it is also proved that every subprojective lattice is uniquely a direct product of directly irreducible subprojective lattices. A more general result can be found in [9, Theorem 15]. We show below that direct decomposition of subprojective lattices corresponds to the following concepts.

4.1 Definition. Let $I_\alpha = (A_\alpha, D_\alpha ; |_\alpha)$ be a nonempty family of incidence spaces, with $\alpha \in \Delta$. By a *direct sum* of the family I_α (to be denoted by $\sum_{\alpha\in\Delta} I_\alpha$) we mean the incidence space $(\dot\cup A_\alpha, \dot\cup D_\alpha ; |)$, where $\dot\cup A_\alpha$ and $\dot\cup D_\alpha$ are disjoint unions over Δ, and $a \mid d$ if

$$\begin{aligned}&\text{either } a \in A_\alpha,\ d \in D_\alpha \text{ for some } \alpha \in \Delta \text{ and } a \mid_\alpha d,\\ &\text{or } a \in A_\alpha,\ d \in D_\beta \text{ with } \alpha \neq \beta.\end{aligned}$$

The following result is a special case of [9, Theorem 14]; we may thus omit the proof.

4.2 Proposition. *If $\{L_\alpha\}_{\alpha\in\Delta}$ is a nonempty family of subprojective lattices, then $I(\prod_{\alpha\in\Delta} L_\alpha) \cong \sum_{\alpha\in\Delta} I(L_\alpha)$. If $\{I_\alpha\}_{\alpha\in\Delta}$ is a nonempty family of subprojective spaces, then $\mathscr{C}(\sum_{\alpha\in\Delta} I_\alpha) \cong \prod_{\alpha\in\Delta} \mathscr{C}(I_\alpha)$.* ∎

It is natural to introduce the following concept for incidence spaces.

4.3 Definition. An incidence space I is *irreducible* if I cannot be nontrivially written as a direct sum of incidence spaces.

In view of this definition, Theorem 3.2, Proposition 4.2, and the remarks preceding Definition 4.1 imply the following statement.

4.4 Proposition. (a) *A direct sum of a nonempty family of subprojective spaces is a subprojective space.*

(b) *Every subprojective space is uniquely a direct sum of a nonempty family of irreducible subprojective spaces.*

(c) *A subprojective lattice L is directly irreducible if and only if $\mathscr{I}(L)$ is irreducible. Conversely, a subprojective space I is irreducible if and only if $\mathscr{C}(I)$ is directly irreducible.* ∎

We now devise an equivalent of Fano's condition for subprojective spaces.

4.5 PROPOSITION. *A subprojective space $I = (A, D; |)$ is irreducible if and only if for any $p, q \in A$, there exists $d \in D$ such that $p, q \nmid d$.*

Proof. From Proposition 4.4, we know that I is irreducible if and only if $\mathscr{C}(I)$ is directly irreducible, equivalently $\mathscr{C}(I)$ satisfies Fano's condition.

For the direct part, let $h \in \mathscr{C}(I)$ be a hyperatom. From the proof of Theorem 3.2, we know that $h = \{p\} \vee \{q\}$ for some $p, q \in A$ with $p \neq q$. Let $d \in D$ be such that $p, q \nmid d$. We have seen in the proof of Theorem 3.2 that d^+ is a dual atom of $\mathscr{C}(I)$. Since $\mathscr{C}(I)$ is a subprojective lattice, we have $h \succ h \wedge d^+ = \{a\}$ for some $a \in A$. Now $p, q \notin d^+$ and $a \notin \{p, q\}$ so that h covers at least three atoms.

Conversely, suppose that $\mathscr{C}(I)$ satisfies Fano's condition. Let $p, q \in A$ and $h = \{p\} \vee \{q\}$ be a hyperatom of $\mathscr{C}(I)$. Let $a \in A$ be such that $a \notin \{p, q\}$ and $h \succ \{a\}$. By dual atomisticity of $\mathscr{C}(I)$, there exists $d \in D$ such that $d^+ \ngeqslant h$ but $d^+ \geqslant \{a\}$. By the double covering condition, we have $h \wedge d^+ = \{a\}$, whence $p, q \notin d^+$ and thus $p, q \nmid d$. ∎

5. EMBEDDING A SUBPROJECTIVE LATTICE INTO A PROJECTIVE LATTICE

We assume that the reader is familiar with the definitions of a projective space, point, line, subspace of a projective space, hyperplane, etc. A basic reference is Lenz [5]. The principal result here is the embedding mentioned in the title of the section. For this we need some preparation.

5.1 *Notation.* Let L be a subprojective lattice with at least three elements. Let $\mathscr{P}(L) = (A_L, H_L; |)$ be the incidence space defined as follows:

(a) A_L is the set of all atoms of L,

(b) H_L is the set of all hyperatoms of L,

(c) $a \mid h$ if and only if $a \leqslant h$.

5.2 PROPOSITION. *If L is a subprojective lattice with at least three elements, then $\mathscr{P}(L)$ is a projective space.*

Proof. Since L is atomistic and has at least three elements, it must have at least two atoms. In view of the double covering condition, L must have at least one hyperatom. Consequently $A_L \neq \varnothing \neq H_L$ and $\mathscr{P}(L)$ is an incidence space.

In the usual terminology, the elements of A_L are points and the elements of H_L are lines of the space $\mathscr{P}(L)$. Since L is atomistic, every line is incident with at least two points (i.e., every hyperatom covers at least two atoms). By the double covering condition, any two distinct points are incident with a unique line.

It remains to verify Pasch's axiom. Let l_1, l_2, l_3, l be distinct lines intersecting in the points $p_3 = l_1 \wedge l_2$, $p_2 = l_1 \wedge l_3$, $p_1 = l_2 \wedge l_3$, $p = l_1 \wedge l$, $q = l_2 \wedge l$, such that neither p nor q is equal to any of the points p_1, p_2, p_3. We must show that $l_3 \wedge l$ is a point. In L, we have $l_1 = p_2 \vee p_3$, $l_2 = p_1 \vee p_3$, $l_3 = p_1 \vee p_2$.

Let $m = p_1 \vee p_2 \vee p_3$. Then m covers l_1, l_2, and l_3 by the double covering condition. Note that $l_1 = p \vee p_3$ and $l_2 = q \vee p_3$. Hence $m = p \vee q \vee p_3 = l \vee p_3$ and thus $m \succ l$. Since $l_3 \vee l = m$ and $m \succ l$, it follows that $l_3 \succ l_3 \wedge l$, that is, $l_3 \wedge l$ is a point. ∎

We are now able to introduce the following concept.

5.3 Definition. With the notation of 5.1, $\mathscr{P}(L)$ is the *projective space associated with L*. The (projective) lattice of all (projective) subspaces of $\mathscr{P}(L)$ is the *projective envelope of L*, which we denote by $\tilde{\mathscr{P}}(L)$.

The last part of the above definition finds its justification in the next theorem. For it, as well as for later reference, it is convenient to have the following terminology.

5.4 Definition. Let L and K be subprojective lattices. A map $\psi: L \to K$ is *geometric* if it satisfies the following conditions:

(a) ψ is injective and inf-preserving;

(b) if $a, b \in A_K$, $a \neq b$, then there exists $d \in D_L$ such that $\psi(d) \geqslant a$ but $\psi(d) \ngeqslant b$;

(c) if $c, d \in D_L$ and $e \in D_K$, are such that $e \geqslant \psi(c) \wedge \psi(d)$, then $e \in \psi(D_L)$.

We are finally ready for the main result of this section. Observe that every projective lattice is subprojective.

5.5 Representation Theorem. *Let L be a subprojective lattice. The mapping ψ defined by*

$$\psi(x) = \{a \in A_L \mid a \leqslant x\} \qquad (x \in L)$$

is a geometric mapping of L into $\tilde{\mathscr{P}}(L)$.

Proof. We note first that $\psi(x)$ is a subspace of $\mathscr{P}(L)$ for any $x \in L$. For if $p, q \in \psi(x)$, then $p, q \leqslant x$ and thus $p \vee q \leqslant x$, where $p \vee q$ is the line incident with p and q if $p \neq q$. Hence ψ maps L into $\tilde{\mathscr{P}}(L)$.

(a) Since L is atomistic, for any $x \in L$, we have $x = \bigvee \psi(x)$, and thus ψ is injective. For any $S \subseteq L$, we have

$$\psi\left(\bigwedge S\right) = \bigcap_{x \in S} \psi(x) = \bigwedge \psi(S)$$

which shows that ψ preserves arbitrary infs.

(b) Let $\{a\}, \{b\} \in A_{\bar{\mathscr{P}}(L)}$, $\{a\} \neq \{b\}$. Then $a, b \in A_L$ and $a \neq b$, which by dual atomisticity of L implies the existence of $d \in D_L$, such that, $d \geqslant a$ but $d \not\geqslant b$. In view of part (a), we must have $\psi(d) \geqslant \{a\}$ but $\psi(d) \not\geqslant \{b\}$, since $0 = \psi(0) = \psi(d \wedge b) = \psi(d) \wedge \psi(b)$.

(c) First let $d \in D_L$; we show that $\psi(d)$ is a hyperplane, that is, a maximal proper subspace of $\mathscr{P}(L)$.

Since ψ is injective, we have $\psi(d) \neq \psi(1) = 1_{\bar{\mathscr{P}}(L)} = A_L$, that is, $\psi(d)$ is a proper subspace of $\mathscr{P}(L)$. In order to show maximality of $\psi(d)$, we let $p \in A_L \backslash \psi(d)$, and we must show that the subspace of $\mathscr{P}(L)$ generated by $\psi(d)$ and p equals A_L. Indeed, let $q \in (A_L \backslash \psi(d)) \backslash \{p\}$. Then $d \vee (p \vee q) = 1$, and by the double covering condition, we have that $p \vee q \succ (p \vee q) \wedge d$, and thus $(p \vee q) \wedge d \in A_L$. Consequently, q is contained in the subspace of $\mathscr{P}(L)$ generated by $\psi(d)$ and p.

Now let $c, d \in D_L$ and $e \in D_{\bar{\mathscr{P}}(L)}$ such that $e \geqslant \psi(c) \wedge \psi(d)$. We have just proved that $\psi(c), \psi(d) \in D_{\bar{\mathscr{P}}(L)}$. Hence e covers $\psi(c) \cap \psi(d)$ if $c \neq d$. If $c = d$, we have $e = \psi(c)$ and there is nothing to prove. Hence assume that $c \neq d$. Let $p \in e \backslash (\psi(c) \cap \psi(d))$ so that $e = (\psi(c) \cap \psi(d)) \vee \{p\}$. Let $e' = (c \wedge d) \vee p$; then $e' \in D_L$. By the above, $\psi(e')$ is a hyperplane of $\mathscr{P}(L)$, and contains $\psi(c) \cap \psi(d)$ and p. But then $\psi(e') = e$ which shows that $e \in \psi(D_L)$. ∎

Part (a) in the proof of the theorem follows from [8, (27.16)]. We can summarize the above theorem by saying that: every subprojective lattice L admits a geometric mapping into its projective envelope $\bar{\mathscr{P}}(L)$. The essential uniqueness of $\bar{\mathscr{P}}(L)$, proved in Section 7, further justifies the terminology "projective envelope." The associated projective space $\mathscr{P}(L)$ provides a link between subprojective lattices and projective geometry. Since the projective lattice $\bar{\mathscr{P}}(L)$ is a more familiar object than the subprojective lattice L, it is of particular interest to establish the precise relationship between L and $\bar{\mathscr{P}}(L)$.

6. Properties of a Geometric Map

Our aim here is to establish a number of properties of geometric maps. We start with a new concept and an auxiliary result of independent interest.

6.1 Definition. Let L be a subprojective lattice. A nonempty subset S

of D_L is a *bundle* if for any $a, b \in S$ and $c \in D_L$, $c \geqslant a \wedge b$ implies that $c \in S$. For a bundle S, let $L_S = \{\wedge T \mid T \subseteq S\}$. L_S is naturally ordered by inclusion.

It is easy to see that L_S is a complete lattice, but is in general not a sublattice of L. We have seen in the proof of Theorem 5.5 that $\psi(D_L)$ is a bundle.

6.2 Lemma. *Let S be a bundle in a subprojective lattice L. If $m \in L_S$ and $p \in A_L$, then $m \vee_L p \in L_S$.*

Proof. If $p \leqslant m$, the result is trivial. Hence assume that $p \nleqslant m$. By the double covering condition, we have $m \prec m \vee p$. Let $N = \{d \in S \mid m \vee p \leqslant d\}$ and $n = \wedge N$. It is clear that $m < m \vee p \leqslant n$.

In order to prove the lemma, it suffices to show that $m \prec n$, for in such a case, $m \vee p = n \in L_S$. By contradiction, assume that $m \nprec n$. Let $M \subseteq S$ be such that $m = \wedge M$. Since $p \nleqslant m$, there exists $d \in M$ such that $m \leqslant d$ but $p \nleqslant d$. Now $1 = d \vee p = d \vee n \succ d$, hence by the double covering condition, we have $m \leqslant n \wedge d \prec n$. By hypothesis $m \nprec n$, so that $m < n \wedge d$. By atomisticity, there is $q \in A_L$ such that $q \leqslant n \wedge d$ but $q \nleqslant m$. Since $q \nleqslant m = \wedge M$, there exists $h \in M$ such that $q \nleqslant h$. We consider two cases.

Case 1. $p \leqslant h$. Then $h \in N$, which implies that $q \leqslant n \wedge d \leqslant n = \wedge N \leqslant h$, contradicting the fact that $q \nleqslant h$.

Case 2. $p \nleqslant h$. Let $c = d \wedge h$. Note that $m \leqslant c$ but $p, q \nleqslant c$. Let $k = c \vee p$; then $k \in D_L$. Since $d, h \in S$ and S is a bundle, also $k \in S$. It is clear that $m \vee p \leqslant k$, and thus $k \in N$. Assume that $q \leqslant k$. Then $d = c \vee q \leqslant k$ which implies that $d = k$ since $d, k \in D_L$. This contradicts the fact that $p \nleqslant d$ and $p \leqslant k$. Consequently, $q \nleqslant k$; but $q \leqslant n = \wedge N \leqslant k$, a contradiction. ∎

A simple inductive argument shows that p in the above lemma may be taken to be of finite height. As a consequence of this lemma, we obtain a result due to Mackey [7] that the join of a U-closed subspace of V and a finite-dimensional subspace is U-closed, where (U, V) form a dual pair of vector spaces over a division ring. Indeed, it suffices to verify that the hypotheses of the lemma are satisfied under these circumstances.

6.3 *Notation.* In any subprojective lattice L, we denote by $H_{n,L}$ (resp. $D_{n,L}$) the set of all elements of height (resp. deficiency) n.

We are now ready for the general properties of geometric maps.

6.4 Theorem. *Let L and L' be subprojective lattices and $\psi: L \to L'$ be a geometric map. Then the following statements hold, with x and y arbitrary elements of L.*

(d) $\psi(x) \leqslant \psi(y)$ *implies* $x \leqslant y$.

(e) $x \prec y$ *implies* $\psi(x) \prec \psi(y)$.

(f) $\psi(H_{n,L}) = H_{n,L'}$, $\psi(D_{n,L}) \subseteq D_{n,L'}$ *for all integers* $n \geqslant 0$.

(g) $\psi(x \vee a) = \psi(x) \vee \psi(a)$ *for all* $a \in A_L$.

Proof. (d) Suppose that $\psi(x) \leqslant \psi(y)$. Then $\psi(x) = \psi(x) \wedge \psi(y) = \psi(x \wedge y)$ which implies that $x = x \wedge y$ since ψ is injective. Consequently, $x \leqslant y$.

We prove next that $\psi(D_L) \subseteq D_{L'}$. Since ψ is inf-preserving, we have $\psi(1) = 1$. Let $d \in D_L$. Then $\psi(d) < 1$ by injectivity of ψ. By dual atomisticity of L', there exists $e \in D_{L'}$ such that $e \geqslant \psi(d)$. Hence $e \geqslant \psi(d) \cap \psi(d)$ which implies that $e \in \psi(D_L)$ since ψ is geometric, say $e = \psi(c)$. Thus $\psi(d) \leqslant \psi(c)$ which by part (d) implies that $d \leqslant c$. Since $c, d \in D_L$, we must have $c = d$ and hence $\psi(d) \in D_{L'}$. As a consequence, we have that $\psi(D_L)$ is a bundle in L'.

(e) Let $x \prec y$ in L. Then $\psi(x) < \psi(y)$ since ψ is injective and isotone. Assume that $\psi(x) \not\prec \psi(y)$. Atomisticity of L' provides $q \in A_{L'}$, such that $\psi(x) < \psi(x) \vee q < \psi(y)$. Since $\psi(D_L)$ is a bundle in L', we may apply Lemma 6.2 to obtain $\psi(x) \vee q \in L_{\psi(D_L)} = \psi(L)$. Consequently, there exists $t \in L$ such that $\psi(x) < \psi(t) < \psi(y)$. But then $x < t < y$, by (d), which contradicts the hypothesis that $x \prec y$. Therefore $\psi(x) \prec \psi(y)$.

(f) If L or L' has only one atom, then both L and L' are isomorphic. Otherwise, we show that $\psi(0) = \psi(\wedge D_L) = \wedge \psi(D_L)$. By part (b) of the definition of a geometric map, $a \leqslant \wedge \psi(D_L)$ for no $a \in A_{L'}$, which by atomisticity of L' yields that $\wedge \psi(D_L) = 0$. Consequently, $\psi(0) = 0$. A simple inductive argument now shows that $\psi(H_{n,L}) \subseteq H_{n,L'}$ in view of (e), and $\psi(H_{n,L}) \supseteq H_{n,L'}$ in view of Lemma 6.2. It follows that $\psi(H_{n,L}) = H_{n,L'}$ for all $n \geqslant 0$.

We have seen above that $\psi(1) = \psi(1)$. In view of part (e), a simple inductive argument shows that $\psi(D_{n,L}) \subseteq D_{n,L'}$ for all $n \geqslant 0$.

(g) Let $x \in L$ and $a \in A_L$. If $a \leqslant x$, there is nothing to prove. Suppose $a \nleqslant x$. Since L is subprojective, $x \prec x \vee a$, and thus $\psi(x) \prec \psi(x \vee a)$ by part (e). In view of part (f), we have that $\psi(a) \in A_{L'}$. Part (d) implies that $\psi(a) \nleqslant \psi(x)$. As in the proof of Theorem 3.2, it follows that $\psi(x) \prec \psi(x) \vee \psi(a) \leqslant \psi(x \vee a)$. Finally $\psi(x) \prec \psi(x \vee a)$ now implies that $\psi(x \vee a) = \psi(x) \vee \psi(a)$. ∎

The next result provides a construction of certain subprojective lattices as subsystems of a given subprojective lattice L. On the one hand, this characterizes images of subprojective lattices under geometric maps into L, and on the other hand, if L is a projective lattice, gives a construction of subprojective lattices in terms of projective lattices, in view of Theorem 5.5. Recall the definition and notation in Definition 6.1.

6.5 Construction Theorem. *Let L be a subprojective lattice and S be a bundle in L satisfying*

(b′) *if $a, b \in A_L$, $a \neq b$, then there exists $d \in S$ such that $a \leqslant d$ but $b \nleqslant d$. Then L_S is a subprojective lattice.*

Proof. It is clear that L_S is complete. If L has only one atom, then $S = \{0\}$ and $L_S = L$. Hence we may assume that L has at least two atoms, and thus at least one hyperatom, in view of (b′). Furthermore, (b′) implies that $A_L \subseteq L_S$. Consequently, L_S is atomistic. It is obvious that L_S is dual atomistic with $D_{L_S} = S$. Note that in all cases, L_S has at least two elements. It remains to establish the double covering condition.

If $m \in L_S$ and $d \in S$ are such that $m \nleqslant d$, then $m \wedge d \prec m$ in L, since L is subprojective and thus $m \wedge d \prec m$ in L_S as well. The proof of Theorem 3.2 now establishes one-half of the double covering condition. The proof of Theorem 3.2 also shows that in order to prove that $x \wedge y \prec x$ implies $y \prec x \vee y$, it suffices to show that for $m \in L_S$ and $a \in A_{L_S} = A_L$, we have $m \prec m \vee_{L_S} a$. By Lemma 6.2, we have $m \vee_{L_S} a = m \vee_L a \succ m$, which completes the proof. ∎

7. Geometric Maps into Projective Lattices

We first establish essential uniqueness of the projective envelope of a subprojective lattice. Then we turn to the conditions on a subprojective lattice to be projective.

7.1 Uniqueness Theorem. *Let L be a subprojective lattice, P and P' be projective lattices, and $\psi: L \to P$, $\psi': L \to P'$ be geometric maps. Then there exists a unique isomorphism θ of P onto P' such that $\theta\mathscr{I} = \mathscr{I}'$.*

Proof. By part (f) of Theorem 6.4, we have $\psi(H_{n,L}) = H_{n,P}$ and $\psi'(H_{n,L}) = H_{n,P'}$ for all $n \geqslant 0$.

We define θ: $P \to P'$ by:

$$\theta(x) = \psi'(\psi^{-1}(x)) \qquad \text{for all } x \in H_{n,P}\text{, else}$$
$$\theta(x) = \sup\{\theta(p) \mid p \in A_p \text{ and } p \leqslant x\}.$$

Observe that, any isomorphism f: $P \to P'$, for which $f\psi = \psi'$, must agree with θ on $H_{n,P}$. Since f is sup-preserving, $f = \theta$. Thus it remains to show that θ is an isomorphism.

Clearly θ preserves all relations between atoms and hyperatoms. Since P and P' are projective lattices (lattices of subspaces determined by the geometry of atoms and hyperatoms), it follows easily that θ is an isomorphism. ∎

Recall that a lattice L is *compactly atomistic* if for any $S \subseteq A_L$ and $a \in A_L$, if $a \leqslant \bigvee S$, then $a \leqslant \bigvee F$ for some finite subset F of S. For convenience, we call the geometric map constructed in Theorem 5.5 *natural*.

7.2 Theorem. *The following conditions on a subprojective lattice L are equivalent.*

(a) *L is projective.*

(b) *L is compactly atomistic.*

(c) *The natural geometric map for L is surjective.*

(d) *The natural geometric map for L is* sup-*preserving.*

Proof. (a) $\Rightarrow$ (b). Let $S \subseteq A_L$ and $a \in A_L$ be such that $a \leqslant \bigvee S$. For any $T \leqslant A_L$, $\bigvee T$ can be identified with the subspace of $\mathscr{P}(L)$ generated by T. Let $F = \{\bigvee T \mid T \text{ is a finite subset of } S\}$. Then F is clearly directed, which can be used to verify that $\cup F$ is a subspace of $\mathscr{P}(L)$ containing S. It follows that $a \leqslant \bigvee S = \cup F$ and hence $a \in \bigvee T$ for some finite subset T of S.

(b) $\Rightarrow$ (c). We show first that the natural geometric map ψ preserves joins of atoms. An inductive argument, using part (g) of Theorem 6.4, shows that ψ preserves finite joins of atoms. Now let $S \subseteq A_L$ and assume that $\bigvee \psi(S) < \psi(\bigvee S)$. There exists $a \in A_{\bar{\mathscr{P}}(L)} = A_L$ such that $a \leqslant \psi(\bigvee S)$ but $a \nleqslant \bigvee \psi(\bigvee S)$. It follows that $a \leqslant \bigvee S$. Since L is compactly atomistic, $a \leqslant \bigvee T$ for a finite subset T of S. Hence $a \in \psi(\bigvee T) = \bigvee \psi(T) \leqslant \bigvee \psi(S)$ since ψ preserves finite joins of atoms. But this contradicts the fact that $a \nleqslant \bigvee \psi(S)$. Consequently ψ preserves joins of atoms. Now let $W \in \bar{\mathscr{P}}(L)$. Then $W = \bigvee S$ for some $S \subseteq A_L$, and $\psi(\bigvee_L S) = \bigvee_{\bar{\mathscr{P}}(L)} S = W$, proving that ψ is surjective.

(c) $\Rightarrow$ (d). This is trivial since the hypothesis implies that ψ is an isomorphism of L onto $\bar{\mathscr{P}}(L)$.

(d) $\Rightarrow$ (a). It follows at once that the natural geometric map is surjective, since it is surjective on atoms. Thus L and $\bar{\mathscr{P}}(L)$ are isomorphic. ∎

The implication (b) $\Rightarrow$ (c) above follows from [8, Theorem 15.5]. Some of the remaining statements also follow from statements in [8]. A further characterization of the class of projective lattices within the class of subprojective lattices is given in [13, 25.3] in terms of a maximality condition.

As a consequence of the above theorem, we have that every finite-dimensional subprojective lattice is projective. For this case, Theorem 3.2 provides a self-dual system of axioms for a finite-dimensional projective space..

Further characterizations of subprojective lattices, in terms of "pairs of dual projective spaces," can be found in [13, Chapt. V].

References

1. G. Birkhoff, "Lattice Theory," Amer. Math. Soc. Coll. Publ. Vol. 25, 3rd. ed., Providence, R.I., 1967.
2. P. Dembowski and A. Wagner, Some characterizations of finite projective spaces, *Arch. Math.* **11** (1960), 465–469.
3. M. Esser, Self-dual postulates for n-dimensional geometry, *Duke Math. J.* **18** (1951), 475–479.

4. W. M. KANTOR, Characterizations of Finite Projective and Affine Spaces, *Canad. J. Math.* **21** (1969), 64–75.
5. H. LENZ, "Vorlesungen über Projektive Geometrie," Akad. Verlagsgesellashcaft, Geest und Portig, K. G., Leipzig, 1965.
6. H. LÜNEBURG, Zentrale Automorphismen von λ-Räumen, *Arch. Math.* **12** (1961), 134–145.
7. G. W. MACKEY, On infinite-dimensional linear spaces, *Trans. Amer. Math. Soc.* **37** (1945), 155–207.
8. F. MAEDA AND S. MAEDA, "Theory of Symmetric Lattices," Springer, New York, 1970.
9. G. MARKOWSKY, The factorization and representation of lattices, *Trans. Amer. Math. Soc.* **203** (1975), 185–200.
10. J. E. MCLAUGHLIN, Structure theorems for relatively complemented lattices, *Pacific J. Math.* **3** (1953), 197–208.
11. J. E. MCLAUGHLIN, The normal completion of a complemented modular point lattice, *in* "Proceedings of the Symposium Pure Mathematics," Amer. Math. Soc., Vol. II, pp. 78–80 Providence, R.I., 1961.
12. M. PETRICH, Jacobson-type theorems for lattices, *Algebra Univ.* **2** (1972), 224–233.
13. M. PETRICH, "Categories of Algebraic Systems," Lecture Notes in Mathematics, 553, Springer–Verlag, Berlin/New York, 1976.