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# Space–time noncommutativity from particle mechanics

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## Abstract

We exploit the reparametrization symmetry of a relativistic free particle to impose a gauge condition which upon quantization implies space–time noncommutativity. We show that there is an algebraic map from this gauge back to the standard ‘commuting’ gauge. Therefore the Poisson algebra, and the resulting quantum theory, are identical in the two gauges. The only difference is in the interpretation of space–time coordinates. The procedure is repeated for the case of a coupling with a constant electromagnetic field, where the reparametrization symmetry is preserved. For more arbitrary interactions, we show that standard dynamical system can be rendered noncommutative in space and time by a simple change of variables.

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## 1. Introduction

Issues concerning the loss of unitarity have been raised in the context of field theories with space–time noncommutativity, despite the work of Doplicher, Fredenhagen and Roberts [1] to the contrary. In this regard, it might be useful to examine space–time noncommutativity in a simpler setting. In the context of quantum mechanics, space–time noncommutativity can be introduced in a trivial manner. Say that  $\mathbf{x}^i$  and  $\mathbf{p}_i$  are the position and momentum operators for a particle satisfying

$$[\mathbf{x}^i, \mathbf{p}_j] = i\delta_{ij}, \quad (1.1)$$

and evolution in some variable  $\tau$  is generated by Hamiltonian  $\mathbf{H}$ . We usually call  $\tau$  the ‘time’. Alternatively,

there have been attempts to make the time, like the spatial coordinate, be associated with a quantum operator [2]. This allows for the exotic possibility of having the space and time coordinates be noncommuting. A trivial way to achieve this is to declare the ‘time operator’ to be

$$\mathbf{x}^0 = \tau - \theta^{0i} \mathbf{p}_i, \quad (1.2)$$

where  $\theta^{0i}$  are constants. When  $\theta^{0i} \rightarrow 0$  one recovers the commutative time, while for  $\theta^{0i} \neq 0$ ,

$$[\mathbf{x}^0, \mathbf{x}^i] = i\theta^{0i}. \quad (1.3)$$

Similar redefinitions have been done to introduce noncommutativity among only spatial coordinates [3, 4]. Balachandran et al. [5] have developed a quantum theory based on commutation relations (1.3).

In this Letter we show that in theories with time reparametrization symmetry, space–time noncommu-

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tativity is simply a gauge choice. We consider familiar examples in particle mechanics. In Section 2 we re-examine the relativistic free particle. The action is reparametrization invariant with respect to the parameter labeling the position along the world line. By choosing a nonstandard gauge condition we can obtain Dirac brackets corresponding to the classical analogue of (1.3). The situation resembles the derivation of spatial noncommutativity for a charged particle in a strong magnetic field [6]. As the classical physics cannot depend on the gauge choice, this theory should be equivalent to the theory expressed in the standard gauge, where the parameter is identified with the time coordinate. This equivalence can be made explicit by displaying a simple algebraic map between the two theories. The time component of it is given by (1.2). Introducing interactions will in general spoil the reparametrization symmetry present for the free particle. An exceptional case is the coupling to an electromagnetic background. We consider the case of a constant electromagnetic background in Section 3. As before we show that there is a gauge condition which leads to (1.3) upon quantization. Also as before, the noncommuting space–time coordinates can be obtained by applying a coordinate transformation from the standard gauge.

In both of the above mentioned examples the only difference between the different gauges is what one chooses to call the ‘time’. In (1.2)  $\mathbf{x}^0$  and  $\tau$  represent a ‘noncommutative’ and ‘commutative time’, respectively. Furthermore, time as measured by  $\mathbf{x}^0$  or  $\tau$  runs at the same rate (at least classically). This is evident for the free particle, using (1.2), since the momentum is conserved, and hence  $d\mathbf{x}^0/d\tau = 1$ . It is also true in the case of interactions with a constant electromagnetic field provided one interprets  $\mathbf{p}_i$  in (1.2) as the conserved momenta. On the other hand,  $d\mathbf{x}^0/d\tau \neq 1$  for arbitrary interactions, which we briefly consider in Section 4. Furthermore, one has the possibility of  $d\mathbf{x}^0/d\tau < 0$  implying a time reversal upon mapping ‘time’  $\tau$  to ‘time’  $\mathbf{x}^0$  using (1.2).

## 2. Free particle

We start with the standard reparametrization invariant action for a relativistic free particle in  $d + 1$  dimen-

sions

$$S_0 = -m \int d\tau \sqrt{-\dot{x}^2}, \quad (2.1)$$

with  $x^\mu$ ,  $\mu = 0, 1, \dots, d$  being the space–time coordinate, the dot denoting differentiation with respect to the affine time  $\tau$ , and metric  $\eta = \text{diag}(-1, 1, \dots, 1)$ . From the equations of motion, the momenta

$$p_\mu = \frac{m\dot{x}_\mu}{\sqrt{-\dot{x}^2}} \quad (2.2)$$

are conserved. In the gauge invariant formulation of the theory, they are canonically conjugate to the space–time coordinates,

$$\{x^\mu, p_\nu\} = \delta_\nu^\mu, \quad \{x^\mu, x^\nu\} = \{p_\mu, p_\nu\} = 0, \quad (2.3)$$

and are subject to the mass shell condition

$$\phi_1 = p^2 + m^2 \approx 0, \quad (2.4)$$

where  $\approx$  indicates equality in weak sense.  $\phi_1$  generates gauge motion on the phase space associated with reparametrizations of the parameter  $\tau$ . The Poincaré symmetry is generated by  $p_\mu$  and  $j_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu$ .

The gauge symmetry can be fixed by imposing a gauge condition. The standard choice identifies the time coordinate  $x^0$  with the parameter  $\tau$ . We instead impose the following constraint:

$$\phi_2 = x^0 + \theta^{0i} p_i - \tau \approx 0, \quad i = 1, 2, \dots, d, \quad (2.5)$$

$\theta^{0i}$  being constants. The constraints (2.4) and (2.5) form a second class set with

$$\{\phi_1, \phi_2\} = 2p_0 \quad (2.6)$$

and resulting Dirac brackets [7]

$$\begin{aligned} \{A, B\}_{\text{DB}} &= \{A, B\} \\ &+ \frac{1}{\{\phi_1, \phi_2\}} \\ &\times (\{A, \phi_1\}\{\phi_2, B\} - \{A, \phi_2\}\{\phi_1, B\}). \end{aligned} \quad (2.7)$$

The Dirac bracket of the spatial coordinates  $x^i$  with the ‘time’  $x^0$  is

$$\{x^0, x^i\}_{\text{DB}} = \theta^{0i}, \quad (2.8)$$

leading to commutation relations (1.3) upon quantization. The remaining nonvanishing Dirac brackets are

$$\{x^i, x^j\}_{\text{DB}} = \frac{1}{p_0}(\theta^{0i} p_j - \theta^{0j} p_i), \quad (2.9)$$

$$\{x^i, p_0\}_{\text{DB}} = \frac{p_i}{p_0}, \quad (2.10)$$

$$\{x^i, p_j\}_{\text{DB}} = \delta_{ij}, \quad (2.11)$$

(2.10) and (2.11) are the same as in the standard gauge, while (2.9) implies nontrivial commutation relations among spatial coordinates upon quantization. Although  $x^0$  gets promoted to a noncommuting operator upon quantization, we can still regard  $\tau$  as a c-number in the quantum theory. Upon imposing  $\phi_2 = 0$  strongly,  $x^0 + \theta^{0i} p_i$  gets identified with the parameter  $\tau$ . By definition  $\phi_2$  has zero Dirac bracket with all phase space variables, and then so does  $x^0 + \theta^{0i} p_i$ . It then is in the center of the Poisson algebra, and consequently a c-number in the corresponding quantum algebra.

The reparametrization symmetry means that the Hamiltonian for the system is weakly zero, i.e., a linear combination of constraints  $\lambda_a \phi_a$ ,  $a = 1, 2$ , and so the evolution of any function  $A$  on phase space is given by

$$\dot{A} \approx \frac{\partial A}{\partial \tau} + \lambda_a \{A, \phi_a\}, \quad (2.12)$$

where the dot is a total  $\tau$  derivative. Imposing that the constraints are preserved in time, i.e.,  $\dot{\phi}_a \approx 0$ , fixes the Lagrange multipliers to be

$$\lambda_1 = \frac{1}{2p_0} \frac{\partial \phi_2}{\partial \tau}, \quad \lambda_2 = 0. \quad (2.13)$$

Then if  $x^0$  and  $p_i$  are presumed to have no explicit  $\tau$  dependence, substitution into (2.12) gives

$$\dot{A} \approx \frac{\partial A}{\partial \tau} - \frac{1}{2p_0} \{A, \phi_1\}. \quad (2.14)$$

Although (2.14) correctly reproduces the dynamics, since it is formulated in terms of Poisson brackets rather than Dirac brackets, it is not evident how to write it on the reduced phase space in the form of Hamilton's equations, and consequently the quantum dynamics in terms of Heisenberg's equations.

Alternatively, one can write Hamilton's equations using Dirac brackets. In this approach the Hamiltonian is not a priori determined. Furthermore, in order to have  $\dot{\phi}_a \approx 0$  it becomes necessary for some of the

original phase space variables to have an explicit  $\tau$  dependence. In familiar examples no such  $\tau$  dependent variables span the reduced phase space, as in the case of the free particle in the  $x^0 = \tau$  gauge, where the reduced phase space is coordinatized by  $x^i$  and  $p_i$ . On the other hand, the time coordinate  $x^0$  gets an explicit  $\tau$  dependence from the gauge condition either  $x^0 = \tau$  or (2.5). In addition, in the case of the gauge (2.5), it is desirable that  $x^0$  is not eliminated from the reduced phase space since we wish to recover (1.3) upon quantization. This is accomplished by using (2.5) to instead eliminate one of the momenta, and so the resulting reduced phase space gets an explicit  $\tau$  dependence. More generally an explicit  $\tau$  dependence may be induced in all of the original phase space variables using this approach, as we illustrate in Section 3.

Concerning the free particle in the  $x^0 = \tau$  gauge it is usual to choose

$$H = \sqrt{p_i p_i + m^2}, \quad (2.15)$$

for the Hamiltonian, generating evolution in the parameter  $\tau$ . The dynamics follows from

$$\dot{A} = \frac{\partial A}{\partial \tau} + \{A, H\}_{\text{DB}}. \quad (2.16)$$

The same choice can be made for the gauge (2.5). To recover the correct equations of motion one assumes that  $x^i$  and  $p_i$  have no explicit  $\tau$  dependence, in either gauge. As stated above, the same is not true for 'time' coordinate  $x^0$ . This follows from the demand that  $\dot{\phi}_2 = \partial \phi_2 / \partial \tau = 0$ , and consequently

$$\frac{\partial x^0}{\partial \tau} = 1. \quad (2.17)$$

As  $\{x^0, H\}_{\text{DB}} = 0$ , it also follows that  $\dot{x}^0 = 1$ , and as a result the commutative and noncommutative clock, as measured by  $\tau$  and  $x^0$ , respectively, run at the same rate.

After the gauge fixing, a one parameter family of Lorentz generators can be constructed

$$\begin{aligned} \tilde{J}_{ij} &= x_i p_j - x_j p_i + \alpha p_0 (\theta^{0i} p_j - \theta^{0j} p_i), \\ \tilde{J}_{0i} &= -x^0 p_i - x_i p_0 - \alpha \theta^{0i} p_0^2 - \alpha \theta^{0j} p_j p_i, \end{aligned} \quad (2.18)$$

$\alpha$  being the parameter. They satisfy as usual

$$\{\tilde{J}_{\mu\nu}, p_\lambda\}_{\text{DB}} = \eta_{\mu\lambda} p_\nu - \eta_{\nu\lambda} p_\mu, \quad (2.19)$$

$$\{\tilde{J}_{\mu\nu}, \tilde{J}_{\lambda\rho}\}_{\text{DB}} = \eta_{\mu\lambda}\tilde{J}_{\nu\rho} - \eta_{\nu\lambda}\tilde{J}_{\mu\rho} - \eta_{\mu\rho}\tilde{J}_{\nu\lambda} + \eta_{\nu\rho}\tilde{J}_{\mu\lambda}. \quad (2.20)$$

From (2.19) the momenta transform covariantly. For infinitesimal Lorentz transformations,

$$\delta_\omega p_\mu = \frac{1}{2}\omega^{\lambda\rho}\{p_\mu, \tilde{J}_{\lambda\rho}\}_{\text{DB}} = -\omega_{\mu\rho}p^\rho. \quad (2.21)$$

Lorentz transformations involve a change of gauge, and for that reason transformations of the space–time coordinates are more subtle [7].  $\phi_2$  is not invariant under Lorentz transformations. On the other hand,  $\phi_2$ , being in the center of the algebra, has zero Dirac bracket with the Lorentz generators  $\tilde{J}_{\mu\nu}$ . Therefore Lorentz transformations cannot in general be obtained by simply taking Dirac brackets with  $\tilde{J}_{\mu\nu}$  as in (2.21).

As a result of the gauge condition (2.5) we obtained the nontrivial Dirac brackets (2.8) and (2.9) implying space–time noncommutativity, as opposed to the trivial result for the standard gauge. However, as was shown in [3,4] a simple change of variables can remove the noncommutativity. In this case the change is

$$x^i \rightarrow q^i = x^i + \theta^{0i} p_0, \quad (2.22)$$

$$x^0 \rightarrow q^0 = x^0 + \theta^{0i} p_i = \tau, \quad (2.23)$$

up to canonical transformations (2.22) removes the space–space noncommutativity implied by (2.9), while (2.23) removes the space–time noncommutativity implied by (2.8). Eq. (2.23) also means that the coordinates  $q^\mu$  satisfy the standard gauge  $q^0 = \tau$  condition, and it agrees with (1.2). The only remaining nonzero brackets are

$$\{q^i, p_0\}_{\text{DB}} = \frac{p_i}{p_0}, \quad \{q^i, p_j\}_{\text{DB}} = \delta_{ij}, \quad (2.24)$$

which agrees with the Dirac brackets of the standard gauge. The free particle Hamiltonian is of course unaffected by the coordinate change. So the only difference between the two gauges is the interpretation of the space–time coordinates appearing in the free particle action. Both gauges give rise to an identical Poisson structure and dynamics (if we choose  $H$  to be the same in both gauges), and thus lead to identical quantum systems. Concerning the Lorentz generators, if one sets  $\alpha$  in (2.18) equal to one they have the usual form

$$\tilde{J}_{\mu\nu} = q_\mu p_\nu - q_\nu p_\mu, \quad \alpha = 1. \quad (2.25)$$

As shown in [7], Lorentz transformations of the space–time coordinates  $q^\mu$  can be written in a simple form:

$$\delta_\omega q^\mu = \frac{1}{2}\omega^{\lambda\rho}\{q^\mu, \tilde{J}_{\lambda\rho}\}_{\text{DB}} - \dot{q}^\mu \delta\tau = -\omega^{\mu\nu} q_\nu. \quad (2.26)$$

The subtraction is necessary because the change of gauge generated by Lorentz transformations corresponds to a shift  $\delta\tau$  in  $\tau$ . The analogous time derivative term is absent in the transformation of momentum (2.21) by the equations of motion. By putting  $\mu = 0$  in (2.26),  $\dot{q}^0 \delta\tau = \omega^{0\mu} q_\mu$ , while for  $\mu = i$  we then get

$$\frac{1}{2}\omega^{\lambda\rho}\{q^i, \tilde{J}_{\lambda\rho}\}_{\text{DB}} = \left(\frac{\dot{q}^i}{\dot{q}^0}\omega^{0\mu} - \omega^{i\mu}\right)q_\mu, \quad (2.27)$$

which is identically satisfied after using the equations of motion.

### 3. Constant electromagnetic field

Interactions with an electromagnetic background do not spoil the time reparametrization symmetry which was present for the relativistic free particle. In this case a gauge condition can be imposed which again leads to space–time noncommutativity upon quantization. Here we specialize to a constant electromagnetic field. The interaction term to be added to  $S_0$  is then

$$S_F = -\frac{1}{2} \int d\tau F_{\mu\nu} x^\mu \dot{x}^\nu, \quad (3.1)$$

where  $F_{\mu\nu}$  is a constant field strength tensor. The usual equations of motion

$$\dot{p}_\mu = -F_{\mu\nu} \dot{x}^\nu, \quad (3.2)$$

where  $p_\mu$  are given in (2.2), follow from varying  $x^\mu$  in the combined action  $S = S_0 + S_F$ . They state that

$$P_\mu = p_\mu + F_{\mu\nu} x^\nu \quad (3.3)$$

are constants of the motion and therefore can be used to label the trajectories. For the example of two space–time dimensions, where there is only a constant electric field  $F_{01} = E$ , solutions take the form

$$\begin{aligned} x^0 &= \frac{1}{E}(-P_1 \pm m \sinh \gamma(\tau)), \\ x^1 &= \frac{1}{E}(P_0 \pm m \cosh \gamma(\tau)), \end{aligned} \quad (3.4)$$

where  $\gamma(\tau)$  is arbitrary.

The reparametrization symmetry again leads to the mass shell constraint (2.4), only the momenta  $p_\mu$  appearing there are not the canonical momenta. Instead the Poisson brackets (2.3) are replaced by

$$\begin{aligned} \{x^\mu, p_\nu\} &= \delta_\nu^\mu, & \{x^\mu, x^\nu\} &= 0, \\ \{p_\mu, p_\nu\} &= -F_{\mu\nu}. \end{aligned} \tag{3.5}$$

$p_\mu$  do not have zero Poisson bracket with the constraint (2.4), and thus are not gauge invariant. Nor are they the conserved momenta  $P_\mu$ , which are related to  $p_\mu$  by (3.3). Since  $\{P_\mu, p_\nu\} = 0$ , it follows that the conserved momenta are gauge invariant observables. On the other hand, canonical momenta  $\pi_\mu$  are constructed as follows

$$\pi_\mu = p_\mu + \frac{1}{2}F_{\mu\nu}x^\nu, \tag{3.6}$$

and together with  $j_{\mu\nu} = x_\mu\pi_\nu - x_\nu\pi_\mu$  generate the Poincaré group. However for nonvanishing fields the generators are not gauge invariant observables.

In two space–time dimensions, a central extension  $\widetilde{ISO}(1, 1)$  of the Poincaré algebra can be constructed [8]. Moreover, its generators are gauge invariant. The translation generators are  $P_\mu$ , and they have a central extension:

$$\{P_\mu, P_\nu\} = E\epsilon_{\mu\nu}. \tag{3.7}$$

A gauge invariant boost generator is

$$K = \frac{E}{2}x^2 - \epsilon_{\mu\nu}x^\mu P^\nu, \tag{3.8}$$

and it leads to the usual transformation properties for  $P_\mu$  and  $x^\mu$ :

$$\{P_\mu, K\} = \epsilon_{\mu\nu}P^\nu, \tag{3.9}$$

$$\{x^\mu, K\} = \epsilon^{\mu\nu}x_\nu. \tag{3.10}$$

From (3.9) and (3.10) it follows that  $\{p_\mu, K\} = \epsilon_{\mu\nu}p^\nu$ , and hence that  $K$  is gauge invariant.  $\widetilde{ISO}(1, 1)$  has the Casimir

$$C = P^2 - 2EK = (P_\mu - E\epsilon_{\mu\nu}x^\nu)^2, \tag{3.11}$$

which from the mass shell constraint (2.4) equals  $-m^2$ . We can therefore more generally add to the boost generator a term proportional to the Casimir,

preserving the Poisson brackets (3.7) and (3.9):

$$K \rightarrow K^{(\alpha)} = K + \frac{\alpha}{2E}C = \frac{\alpha}{2E}P^2 + (\alpha - 1)K, \tag{3.12}$$

obtaining a one parameter family of  $\widetilde{ISO}(1, 1)$  algebras. Their generators are gauge invariant, and are distinguished by the Casimir, which has the value

$$C^{(\alpha)} = P^2 - 2EK^{(\alpha)} \approx (\alpha - 1)m^2, \tag{3.13}$$

after using the mass shell constraint (2.4). However only for  $\alpha = 0$ , does  $K^{(\alpha)}$  induce the standard Lorentz boost on space–time coordinates  $x^\mu$  following from (3.10).<sup>1</sup>

Next consider the gauge fixing. We are again interested in a nonstandard gauge condition leading to the Dirac brackets (2.8) and (2.9), and so implying nontrivial commutation relations for the space–time coordinates upon quantization. This is accomplished for

$$\phi_2 = x^0 + \theta^{0i}P_i - \tau \approx 0, \tag{3.14}$$

$\theta^{0i}$  again being constants and  $P_i$  being the gauge invariant momenta. It reduces to the previous gauge condition (2.5) for vanishing fields. The Poisson bracket between constraints  $\phi_1$  and  $\phi_2$  is again given by (2.6). So we recover the previous Dirac brackets (2.8) and (2.9) between space–time coordinates  $x^\mu$ , and commutation relations (1.3) upon quantization. The remaining Dirac brackets contain the interaction with the constant field tensor. The nonvanishing Dirac brackets are

$$\{x^i, p_0\}_{\text{DB}} = N^i{}_j \frac{P_j}{p_0}, \tag{3.15}$$

<sup>1</sup> For the special case  $\alpha = 1$ , the boost has the simple form  $K^{(1)} = \frac{1}{2E}P^2$  and we can define a new pair of gauge invariant space–time coordinates  $X^\mu$  which are just the dual of  $P_\mu$ ,

$$X^\mu = \frac{1}{E}\epsilon^{\mu\nu}P_\nu.$$

From

$$\{X^\mu, P_\nu\} = \delta_\nu^\mu, \quad \{X^\mu, K\} = \epsilon^{\mu\nu}X_\nu$$

they undergo the usual two-dimensional Poincaré transformations. Like  $P_\mu$ , they have nonvanishing Poisson brackets among themselves,  $\{X^\mu, X^\nu\} = -E^{-1}\epsilon^{\mu\nu}$ , and since they are reparametrization invariant merely serve to label the orbits.

$$\{x^i, p_j\}_{\text{DB}} = N^i_j - \theta^{0i} F_{jk} \frac{p^k}{p_0}, \quad (3.16)$$

$$\{p_0, p_i\}_{\text{DB}} = F_{ij} \frac{p_j}{p_0}, \quad (3.17)$$

$$\{p_i, p_j\}_{\text{DB}} = -F_{ij}, \quad (3.18)$$

where  $N^i_j = \delta_{ij} - \theta^{0i} F_{0j}$ . It then follows that

$$\{P_0, P_i\}_{\text{DB}} = F_{0i}. \quad (3.19)$$

For the dynamics we again write the Hamilton equations using Dirac brackets as in (2.16). Now we get that all the space–time coordinates have an explicit  $\tau$  dependence. A convenient choice for the Hamiltonian is  $P^0$ , since it is the conserved energy. So setting  $\phi_1$  strongly equal to zero,

$$H = \sqrt{p_i p_i + m^2} - F_{0i} x^i. \quad (3.20)$$

Since all  $P_\mu$  should be constants of the motion, from (3.19) we need that

$$\frac{\partial P_0}{\partial \tau} = 0, \quad \frac{\partial P_i}{\partial \tau} = -F_{0i}. \quad (3.21)$$

Additional requirements on partial derivatives come from demanding that  $\dot{\phi}_a = \partial \phi_a / \partial \tau = 0$ ,  $a = 1, 2$ , the dot again denoting a total  $\tau$  derivative. They lead to

$$p^\mu \frac{\partial p_\mu}{\partial \tau} = 0, \quad (3.22)$$

$$\frac{\partial x^0}{\partial \tau} = 1 + F_{0k} \theta^{0k}. \quad (3.23)$$

A solution consistent with (3.21)–(3.23) is

$$\begin{aligned} \frac{\partial x^i}{\partial \tau} &= -F_{0k} \theta^{0k} \frac{p_i}{p_0}, \\ \frac{\partial p_\mu}{\partial \tau} &= F_{0k} \theta^{0k} F_{\mu\nu} \frac{p^\nu}{p_0}, \end{aligned} \quad (3.24)$$

and so all the phase space variables  $x^\mu$  and  $p_\mu$  have explicit  $\tau$  dependence when the scalar product of  $\theta^{0i}$  with the electric field  $F_{0i}$  is not zero. The resulting Hamilton equations of motion are

$$\begin{aligned} \dot{x}^\mu &= \{x^\mu, H\}_{\text{DB}} + \frac{\partial x^\mu}{\partial \tau} = -\frac{p^\mu}{p_0}, \\ \dot{p}_\mu &= \{p_\mu, H\}_{\text{DB}} + \frac{\partial p_\mu}{\partial \tau} = F_{\mu\nu} \frac{p^\nu}{p_0}, \end{aligned} \quad (3.25)$$

which agrees with (3.2). As in the free case,  $\dot{x}^0 = 1$ , and the commutative and noncommutative clock, as

measured by  $\tau$  and  $x^0$ , respectively, run at the same rate.

Assuming  $[N^i_j]$  to be a nonsingular matrix ( $F_{0k} \theta^{0k} \neq 1$ ), the noncommutativity of the space–time coordinates following from (2.8) and (2.9) can again be removed by a trivial coordinate transformation. It now takes the form

$$\begin{aligned} x^i &\rightarrow q^i = N^{-1i}_j [x^j + \theta^{0j} p_0], \\ x^0 &\rightarrow q^0 = x^0 + \theta^{0i} P_i = \tau, \end{aligned} \quad (3.26)$$

up to canonical transformations. The coordinates  $q^\mu$  once again satisfy the standard gauge condition, and have its associated Dirac brackets

$$\{q^i, p_0\}_{\text{DB}} = \frac{p_i}{p_0}, \quad (3.27)$$

$$\{q^i, p_j\}_{\text{DB}} = \delta_{ij}, \quad (3.28)$$

along with  $\{q^\mu, q^\nu\}_{\text{DB}} = \{q^0, p_\nu\}_{\text{DB}} = 0$ , (3.17) and (3.18). Conversely, we can start with the standard gauge, and obtain the gauge (3.14) by applying the inverse of transformation (3.26),

$$\begin{aligned} x^0 &= \frac{q^0 - \theta^{0i} (p_i + F_{ij} q^j)}{1 - \theta^{0k} F_{0k}}, \\ x^i &= N^i_j q^j - \theta^{0i} p_0. \end{aligned} \quad (3.29)$$

So once again both gauges give rise to the same Poisson structure and resulting quantum commutation relations. Concerning the dynamics, the natural Hamiltonian in the standard gauge would be (3.20) with noncommuting coordinates  $x^i$  replaced by commuting ones  $q^i$ :

$$H_0 = \sqrt{p_i p_i + m^2} - F_{0i} q^i. \quad (3.30)$$

It now represents the conserved energy, and yields the same equations of motion as (3.25). (Now  $q^i$  and  $p_\mu$  have no explicit  $\tau$  dependence.)

#### 4. Other interactions

For arbitrary interactions there is no longer, in general, a conserved momenta. The latter was used previously in writing the gauge condition (3.14), and it led to the simple commutation relations (1.3) between the space and time coordinates. It also implied that the commutative and noncommutative clock, as measured by  $\tau$  and  $x^0$ , respectively, run at the same rate, i.e.,

$d\mathbf{x}^0/d\tau = 1$ . For most other systems, examples of which are considered below, these results get altered. Moreover, one can even have  $d\mathbf{x}^0/d\tau < 0$  implying time reversal in transforming from time  $\tau$  to time  $\mathbf{x}^0$ .

#### 4.1. Coupling to an arbitrary electromagnetic field

The first example is the case of a relativistic particle coupled to an arbitrary electromagnetic field. As before the action is reparametrization invariant. Here we replace (3.1) by

$$S_F = - \int d\tau A_\mu(x)\dot{x}^\mu, \tag{4.1}$$

with the resulting equations of motion (3.2), where  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$  is not in general constant. The mass shell constraint (2.4) and Poisson brackets (3.5) once again follow. If for the gauge constraint one takes (2.5), then (2.6) gets replaced by

$$\{\phi_1, \phi_2\} = 2(p_0 + \theta^{0i} F_{i\mu} p^\mu), \tag{4.2}$$

leading to a rather complicated Dirac bracket between the space and time coordinates

$$\{x^0, x^i\}_{\text{DB}} = \frac{\theta^{0i}}{1 + \theta^{0i} F_{i\mu} \frac{p^\mu}{p_0}}, \tag{4.3}$$

as opposed to the result obtained previously (2.8). Moreover, demanding that  $\dot{\phi}_2 = 0$  now gives the complicated result

$$\dot{x}^0 = \frac{1 + \theta^{0i} F_{ij} \dot{x}^j}{1 + \theta^{0k} F_{0k}}, \tag{4.4}$$

as opposed to  $\dot{x}_0 = 1$ .

An alternative approach is to start with the standard gauge  $\phi_2 = q^0 - \tau \approx 0$  (here we denote the space–time coordinates by  $q^\mu$ ), and simply define a noncommutative time, using for example (1.2). The nonvanishing Dirac brackets in the standard gauge are again given by (3.17), (3.18), (3.27) and (3.28). The dynamics in the standard gauge is recovered for the Hamiltonian

$$H_0 = \sqrt{p_i p_i + m^2} + A_0(q), \tag{4.5}$$

along with

$$\frac{\partial p_i}{\partial \tau} = \partial_0 A_i, \quad \frac{\partial p_0}{\partial \tau} = \partial_0 A_i \frac{p^i}{p_0}, \quad \frac{\partial q^0}{\partial \tau} = 1, \tag{4.6}$$

which is consistent with the conditions  $\dot{\phi}_a = \phi_a/\partial\tau = 0$ ,  $a = 1, 2$ . Now define  $x^0 = q^0 - \theta^{0i} p_i$  to obtain the familiar Dirac brackets

$$\{x^0, q^i\}_{\text{DB}} = \theta^{0i}. \tag{4.7}$$

A feature shared with the previous approach is that  $\dot{x}^0 \neq 1$ . Now

$$\dot{x}^0 = 1 - \theta^{0i} F_{0i} + \theta^{0i} F_{ij} \dot{q}^j. \tag{4.8}$$

Since this approach differs from the previous one only by a gauge choice, the dynamics in the two cases must be identical. The difference between the two approaches is in how the time variable  $x^0$  is defined. For both definitions  $\dot{x}^0 \neq 1$ , and even allows for the possibility of time reversal in going from time as measured by  $\tau$  to time as measured by  $x^0$ .

#### 4.2. Conservative system

In all the previous examples, a noncommutative time resulted either from a gauge choice or by a redefinition of coordinates. In sections two and three these approaches were equivalent, while in the above example one ends up with different definitions of the noncommutative time  $x^0$ . In systems with no time reparametrization symmetry, one can adapt the second approach. So once again by defining (1.2) and assuming the commutation relations (1.1), the result (1.3) follows. Applying this to a nonrelativistic conservative system described by Hamiltonian

$$H_0 = \frac{p_i^2}{2m} + V(q^i), \tag{4.9}$$

one gets

$$\dot{x}^0 = 1 + \theta^{0i} \frac{\partial V}{\partial q^i}, \tag{4.10}$$

where  $H_0$  generates evolution in  $\tau$ . If there are trajectories for which  $1 + \theta^{0i} \frac{\partial V}{\partial q^i} < 0$ , we then get a time reversal upon applying (1.2).

In the above we looked at replacing the commuting time with its noncommuting counterpart, using (1.2). One can instead make the analogous replacement of the spatial coordinate. For the free particle this corresponded to the inverse of (2.22), or

$$q^i \rightarrow x^i = q^i - \theta^{0i} H. \tag{4.11}$$

One can try repeating this for an interacting system, with  $H$  representing the resulting Hamiltonian for the system generating evolution in some new time variable, which we denote by  $\tau'$ . The generalizations of (2.9) and (2.11) are then

$$\begin{aligned} \{x^i, x^j\} &= \theta^{0i} \frac{dx^j}{d\tau'} - \theta^{0j} \frac{dx^i}{d\tau'}, \\ \{x^i, p_j\} &= \delta_{ij} + \theta^{0i} \frac{dp_j}{d\tau'}. \end{aligned} \quad (4.12)$$

So starting from the nonrelativistic conservative Hamiltonian (4.9), we would get

$$H = \frac{p_i^2}{2m} + V(x^i), \quad (4.13)$$

upon making the replacement (4.11). The Hamilton equations of motion resulting from (4.12) and (4.13) can be written

$$\begin{aligned} \left(1 + \theta^{0j} \frac{\partial V}{\partial x^j}\right) \frac{dx^i}{d\tau'} &= \frac{p_i}{m}, \\ \left(1 + \theta^{0j} \frac{\partial V}{\partial x^j}\right) \frac{dp_i}{d\tau'} &= -\frac{\partial V}{\partial x^i}. \end{aligned} \quad (4.14)$$

Provided  $1 + \theta^{0j} \frac{\partial V}{\partial x^j} > 0$ , the associated classical trajectories are identical to those generated from the standard Hamiltonian (4.9) after again performing a reparametrization

$$\frac{d\tau'}{d\tau} = 1 + \theta^{0i} \frac{\partial V}{\partial x^i}. \quad (4.15)$$

We thus arrive at the same Jacobian factor as in (4.10). Unlike in the previous paragraph, here both ‘times’ are associated with c-numbers. As before, if there are trajectories for which  $1 + \theta^{0i} \frac{\partial V}{\partial x^i} < 0$ , we then get a time reversal upon going from  $\tau$  to  $\tau'$ .

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