On the perturbation bound in unitarily invariant norms for subunitary polar factors

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Abstract

Let $\mathscr{C}_r^{m \times n}$ be the set of $m \times n$ complex matrices with rank $r$, and let $A \in \mathscr{C}_r^{m \times n}$ and $\tilde{A} = A + E \in \mathscr{C}_r^{m \times n}$ have the generalized polar decompositions

$$A = QH \quad \text{and} \quad \tilde{A} = \tilde{Q}\tilde{H}.$$  

In this article, a new perturbation bound for subunitary polar factors in any unitarily invariant norm is given by

$$\|\tilde{Q} - Q\| \leq \frac{3}{\sigma_r + \tilde{\sigma}_r} \|E\|,$$

where $\sigma_r, \tilde{\sigma}_r$ are the smallest positive singular values of $A$ and $\tilde{A}$, respectively, which improves some existing perturbation bounds.

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1. Introduction

Let $\mathscr{C}_r^{m \times n}$ be the set of $m \times n$ complex matrices and $\mathscr{C}_r^{m \times n}$ be the set of $m \times n$ complex matrices with rank $r$. Without loss of generality we always assume that $m \geq n$. We denote by

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Let $A = U \Sigma V^*$ be the singular value decomposition (SVD) of $A$ and

$$H = V_1 \Sigma_1 V_1^*, \quad Q = U_1 V_1^*, \quad (1.1)$$

where $U = (U_1, U_2) \in \mathfrak{U}^{m \times m}$ and $V = (V_1, V_2) \in \mathfrak{U}^{n \times n}$ are unitary, $U_1 \in \mathfrak{U}^{m \times r}$, $V_1 \in \mathfrak{U}^{n \times r}$, $\Sigma_1 = \text{diag}(\sigma_1, \ldots, \sigma_r)$, $\sigma_i, i = 1, 2, \ldots, r$, are the singular values of $A$ with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$, and the superscript $*$ denotes the conjugate transpose. The generalized polar decomposition of the matrix $A$ is defined by

$$A = Q H. \quad (1.3)$$

The matrix $Q$ is called the (sub) unitary polar factor of $A$; $H$ is called the Hermitian positive (semi)definite factor. The decomposition (1.3) is unique if $R(Q^*) = R(H)$, where $R(\cdot)$ is the column space of a matrix (see [18]). From now on we always assume that the generalized polar decomposition satisfies this condition.

The perturbation bounds of subunitary polar factors have been studied by many authors, e.g., see Barrlund [1], Bhatia [2], Bhatia and Mukherjea [3], Chatelin and Gratton [4], Chen et al. [5], Li [10,11,12], Li and Sun [13–15], and Mathias [17]. Some applications for polar decompositions can be found in [7,9].

Let $A \in \mathfrak{U}^{m \times n}$ and $\tilde{A} = A + E \in \mathfrak{U}^{r \times n}$ have the generalized polar decompositions

$$A = Q H \quad \text{and} \quad \tilde{A} = \tilde{Q} \tilde{H}. \quad (1.4)$$

Let $\Delta Q = \tilde{Q} - Q$. In [11] the author showed the following bounds:

1. For $m = n = r$

$$\|\Delta Q\| \leq \frac{2}{\sigma_n + \tilde{\sigma}_n} \|E\|, \quad (1.5)$$

which improves the Mathias’ bound [17]

$$\|\Delta Q\| \leq - \frac{\|E\|}{\|E\|_2} \log \left(1 - \frac{\|E\|_2}{\sigma_n}\right)$$

under the assumption that $\|E\|_2 < \sigma_n$.

2. For $m > n = r$

$$\|\Delta Q\| \leq \left(\frac{2}{\sigma_n + \tilde{\sigma}_n} + \frac{1}{\max\{\sigma_n, \tilde{\sigma}_n\}}\right) \|E\| \quad (1.6)$$

and

$$\|\Delta Q\|_F \leq \sqrt{\left(\frac{2}{\sigma_n + \tilde{\sigma}_n}\right)^2 + \left(\frac{1}{\max\{\sigma_n, \tilde{\sigma}_n\}}\right)^2} \|E\|_F. \quad (1.7)$$

For any $r \leq \min\{m, n\}$ Li and Sun [13] showed that the bound (1.5) also holds for the Frobenius norm, i.e.,

$$\|\Delta Q\|_F \leq \frac{2}{\sigma_r + \tilde{\sigma}_r} \|E\|_F. \quad (1.8)$$

It is noted that the bound (1.8) is also a generalization of the corresponding bounds in [10,18]. Li and Sun [14] considered the unitarily invariant norm bound for subunitary polar factors and presented the following bounds:
\[ \| \Delta Q \| \leq \left( \frac{2}{\sigma_r + \tilde{\sigma}_r} + \frac{2}{\max\{\sigma_r, \tilde{\sigma}_r\}} \right) \| E \| \]  
(1.9)

and

\[ \| \Delta Q \|_2 \leq \sqrt{\left( \frac{2}{\sigma_r + \tilde{\sigma}_r} \right)^2 + \frac{2}{\max\{\sigma_r^2, \tilde{\sigma}_r^2\}} \| E \|_2} \]  
(1.10)

for \( r \leq n < m \).

Recently, they [15] obtained two bounds below:

\[ \| \Delta Q \| \leq \left( \frac{2}{\sigma_r + \tilde{\sigma}_r} + \frac{1}{\min\{\sigma_r, \tilde{\sigma}_r\}} \right) \| E \| \]  
(1.11)

and

\[ \| \Delta Q \|_2 \leq \delta \| E \|_2^2, \]  
(1.12)

where

\[ \delta = \frac{1}{2} \left[ \left( \frac{2}{\sigma_r + \tilde{\sigma}_r} \right)^2 + \frac{1}{\sigma_r^2} + \frac{1}{\tilde{\sigma}_r^2} + \sqrt{\left( \left( \frac{2}{\sigma_r + \tilde{\sigma}_r} \right)^2 + \frac{1}{\sigma_r^2} + \frac{1}{\tilde{\sigma}_r^2} \right)^2 - \frac{4}{\sigma_r^2 \tilde{\sigma}_r^2}} \right]. \]

It is noted that the bounds (1.11) and (1.12) are sharper than those in (1.9) and (1.10), respectively. Since \( A \) is perturbed to \( \tilde{A} \), \( \| E \| \) may be very small and \( \sigma_r \approx \tilde{\sigma}_r \). Asymptotically, the bound (1.11) improves the bound (1.9) by a factor 1.5 and

\[ \delta \approx \frac{2.618}{\sigma_r^2} < \frac{3}{\sigma_r^2} \approx \left( \frac{2}{\sigma_r + \tilde{\sigma}_r} \right)^2 + \frac{2}{\max\{\sigma_r^2, \tilde{\sigma}_r^2\}}. \]

Li [16] extended the \( F \)- and 2-norm bounds in (1.7) and (1.10) to more general norm – \( Q \)-norm.

Much effort has been made in order to improve the unitarily invariant norm bound of perturbation of subunitary polar factors. However, a number of examples show that the existing bounds (1.6), (1.9)–(1.12) can be further improved. By this motivation, in this paper the perturbation bounds for subunitary polar factors are further discussed. In particular, a unitarily invariant norm bound of perturbation of subunitary polar factors will be obtained, the new bound improves the corresponding one, our proof technique is simple and mainly from Davis and Kahan’s result [6].

2. Preliminaries

Let \( A, \tilde{A} \in \mathcal{C}^{m \times n}_r \) with the singular value decompositions

\[ A = U \Sigma V^* \quad \text{and} \quad \tilde{A} = \tilde{U} \tilde{\Sigma} \tilde{V}^*, \]  
(2.1)

where \( U = (U_1, U_2) \in C^{m \times m} \) and \( V = (V_1, V_2) \in C^{n \times n} \) are unitary, \( U_1 \in C^{m \times r}_r, V_1 \in C^{n \times r}_r, \tilde{U} = (\tilde{U}_1, \tilde{U}_2) \in C^{m \times m} \) and \( \tilde{V} = (\tilde{V}_1, \tilde{V}_2) \in C^{n \times n} \) are unitary, \( \tilde{U}_1 \in C^{m \times r}_r, \tilde{V}_1 \in C^{n \times r}_r, \)

\[ \Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{C}^{m \times n} \quad \text{and} \quad \tilde{\Sigma} = \begin{pmatrix} \tilde{\Sigma}_1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{C}^{m \times n}_r, \]

\( \Sigma_1 = \text{diag}(\sigma_1, \ldots, \sigma_r), \tilde{\Sigma}_1 = \text{diag}(\tilde{\sigma}_1, \ldots, \tilde{\sigma}_r), \sigma_1 \geq \cdots \geq \sigma_r > 0 \) and \( \tilde{\sigma}_1 \geq \cdots \geq \tilde{\sigma}_r > 0 \).

A unitarily invariant norm \( \| \cdot \| \) is called a \( Q \)-norm (e.g. see [2]) if there exists another unitarily invariant norm \( \| \cdot \|' \) such that \( \| Y \| = \frac{1}{2} \| Y^* Y \|' \), which is denoted by \( \| \cdot \|_Q \). It is noted that the Ky-Fan \( p \)-\( k \) norm is a \( Q \)-norm for \( p \geq 2 \); in fact,
\[ \|Y\|_{k;p} \equiv \left( \sum_{i=1}^{k} \sigma_i^p \right)^{1/p} = \|Y^*Y\|_{k;p/2}^{1/2} \]

for \( p \geq 2 \) and \( k = 1, \ldots, n \). It is easy to prove that both spectral norm and Frobenius norm are \( Q \)-norms.

Let \( S = \tilde{U}^*U \) and \( T = \tilde{V}^*V \) have the block form
\[ S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \in \mathbb{C}^{m \times m} \quad \text{and} \quad T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \in \mathbb{C}^{n \times n}, \tag{2.2} \]
where both \( S_{11} \) and \( T_{11} \) are \( r \times r \). Then \( S \) and \( T \) are unitary matrices. From [14] it is readily to obtain
\[ \|E\| = \left\| \begin{pmatrix} S_{11} \Sigma_1 - \tilde{\Sigma}_1 T_{11} & -\tilde{\Sigma}_1 T_{12} \\ S_{21} \tilde{\Sigma}_1 & 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} \tilde{\Sigma}_1 S_{11} - T_{11} \Sigma_1 & \tilde{\Sigma}_1 S_{12} \\ -T_{21} \Sigma_1 & 0 \end{pmatrix} \right\|. \tag{2.3} \]

Hence
\[ \|E\| \geq \max\{\|S_{11} \Sigma_1 - \tilde{\Sigma}_1 T_{11}\|, \|\tilde{\Sigma}_1 S_{11} - T_{11} \Sigma_1\|, \|S_{21} \Sigma_1\|, \|\tilde{\Sigma}_1 S_{12}\|, \|T_{21} \Sigma_1\|\}. \tag{2.4} \]

Similarly, we have
\[ \|\Delta Q\| = \left\| \begin{pmatrix} S_{11} - T_{11} & -T_{12} \\ S_{21} & 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} S_{11} - T_{11} & S_{12} \\ -T_{21} & 0 \end{pmatrix} \right\|. \tag{2.5} \]

From [14] we have
\[ \|S_{12}\| = \|S_{21}\| \quad \text{and} \quad \|T_{12}\| = \|T_{21}\|. \tag{2.6} \]
Particularly, if \( r = n \), then
\[ \|E\| = \left\| \begin{pmatrix} S_{11} \Sigma_1 - \tilde{\Sigma}_1 T \end{pmatrix} \right\| = \|\tilde{\Sigma}_1 S_{11} - T \Sigma_1 \| \quad \text{S}_1 S_{12} \| \tag{2.7} \]
and
\[ \|\Delta Q\| = \left\| \begin{pmatrix} S_{11} - T \end{pmatrix} \right\| = \|S_{11} - T \| \quad \text{S}_1 S_{12} \|. \tag{2.8} \]

### 3. The unitarily invariant norm bound

The following lemmas will be used in the sequel. The first two lemmas can be found in [6,16], respectively.

**Lemma 3.1.** Let \( B_1 \) and \( B_2 \) be two Hermitian matrices and let \( P \) be a complex matrix. Suppose that there are two disjoint intervals separated by a gap of width at least \( \eta \), where one interval contains the spectrum of \( B_1 \) and the other contains that of \( B_2 \). If \( \eta > 0 \), then there exists a unique solution \( X \) to the matrix equation \( B_1 X - X B_2 = P \) and , moreover
\[ \|X\| \leq \frac{1}{\eta} \|P\|. \]
Lemma 3.2. Let \( A \) have the block form
\[
A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.
\]
Then
\[
\| A \|_Q^2 \leq \| A_{11} \|_Q^2 + \| A_{12} \|_Q^2 + \| A_{21} \|_Q^2 + \| A_{22} \|_Q^2.
\]

Lemma 3.3. Let \( \tilde{M} \) and \( M \) be given by
\[
\tilde{M} = \begin{pmatrix} 0 & -\tilde{\sigma}T_{12} \\ \sigma S_{21} & 0 \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} 0 & -\Sigma_{11}T_{12} \\ S_{21}\Sigma_1 & 0 \end{pmatrix},
\]
where \( S_{ij} \) and \( T_{ij} \) are defined by (2.2). Then for any unitarily invariant norm \( \| \cdot \| \), \( \| \tilde{M} \| \leq \| M \| . \)

Proof. A simple calculation reveals
\[
\tilde{M} = \begin{pmatrix} \tilde{\sigma} \Sigma_1^{-1} & 0 \\ 0 & I \end{pmatrix} M \begin{pmatrix} \sigma \Sigma_1^{-1} & 0 \\ 0 & I \end{pmatrix}.
\]
Note that \( \| \tilde{\sigma} \Sigma_1^{-1} \|_2 = 1 \) and \( \| \sigma \Sigma_1^{-1} \|_2 = 1 \). Then the lemma follows immediately from the fact that \( \| BAC \| \leq \| B \| \| A \| \| C \| \) (e.g. see [8,19]). \( \square \)

Lemma 3.4. \( \| S_{21} \| \leq \max \{ \tilde{\sigma}, \sigma \} \| E \| \) and \( \| T_{12} \| \leq \max \{ \tilde{\sigma}, \sigma \} \| E \| \).

Proof. By (2.4) we have \( \| S_{21} \Sigma_1 \| \leq \| E \| \) and \( \| \Sigma_1 T_{12} \| \leq \| E \| \), which implies that \( \| S_{21} \| \leq \frac{1}{\sigma_r} \| E \| \) and \( \| S_{12} \| \leq \frac{1}{\sigma_r} \| E \| \). Then the first inequality of this lemma follows from (2.4). By an analogous argument, we can prove another inequality. \( \square \)

The following theorem is the main result in this paper.

Theorem 3.5. Let \( A \) and \( \tilde{A} \in \mathbb{C}^{m \times n} \) have the singular value decomposition (2.1).

(1) If \( r \leq n \leq m \), then for any unitarily invariant norm \( \| \cdot \| \) we have
\[
\| \Delta Q \| \leq \frac{3}{\sigma_r + \tilde{\sigma}_r} \| E \| .
\]
Particularly, for \( Q \)-norms we have
\[
\| \Delta Q \|_Q \leq \frac{1 + \sqrt{3}}{\sigma_r + \tilde{\sigma}_r} \| E \| .
\]

(2) If \( r = n < m \), then for any unitarily invariant norm \( \| \cdot \| \) we have
\[
\| \Delta Q \| \leq \frac{2}{\max \{ \sigma_n, \tilde{\sigma}_n \}} \| E \| ,
\]
moreover,
\[
\| \Delta Q \|_Q \leq \frac{1 + \sqrt{2}}{\sigma_n + \tilde{\sigma}_n} \| E \| .
\]
Proof. (1) First assume that \( r < n \leq m \). In this case, let
\[
X = \begin{pmatrix}
S_{11} & -T_{11} \\
S_{21} & T_{12}
\end{pmatrix},
\quad D_1 = \begin{pmatrix}
\Sigma_1 & 0 \\
0 & \tilde{\sigma}_r I
\end{pmatrix},
\quad D_2 = \begin{pmatrix}
\Sigma_1 & 0 \\
0 & \sigma_r I
\end{pmatrix}.
\]
A simple calculation gives
\[
XD_1 + D_2 X = \begin{pmatrix}
S_{11} \Sigma_1 - \tilde{\Sigma}_1 T_{11} & -\tilde{\Sigma}_1 T_{12} \\
\sigma_r S_{21} & 0
\end{pmatrix} + \begin{pmatrix}
\tilde{\Sigma}_1 S_{11} - T_{11} \Sigma_1 & -\tilde{\sigma}_r T_{12} \\
\sigma_r S_{21} & 0
\end{pmatrix}.
\]
It follows from Lemma 3.1 and (2.3) that
\[
\|X\| \leq \frac{1}{\sigma_r + \tilde{\sigma}_r} \left( \|E\| + \left\| \begin{pmatrix}
\tilde{\Sigma}_1 S_{11} - T_{11} \Sigma_1 & -\tilde{\sigma}_r T_{12} \\
\sigma_r S_{21} & 0
\end{pmatrix} \right\| \right). \quad (3.5)
\]
Clearly, we have
\[
\left\| \begin{pmatrix}
\tilde{\Sigma}_1 S_{11} - T_{11} \Sigma_1 & -\tilde{\sigma}_r T_{12} \\
\sigma_r S_{21} & 0
\end{pmatrix} \right\| \leq \left\| \begin{pmatrix}
\tilde{\Sigma}_1 S_{11} - T_{11} \Sigma_1 & 0 \\
\sigma_r S_{21} & 0
\end{pmatrix} \right\| + \left\| \begin{pmatrix}
0 & -\tilde{\sigma}_r T_{12} \\
\sigma_r S_{21} & 0
\end{pmatrix} \right\|. \quad (3.6)
\]
It follows from Lemma 3.3 and Theorem 3.3 of [19] that
\[
\left\| \begin{pmatrix}
0 & -\tilde{\sigma}_r T_{12} \\
\sigma_r S_{21} & 0
\end{pmatrix} \right\| \leq \|M\| \leq \|E\|, \quad (3.7)
\]
where \( M \) is given by Lemma 2.3. By (2.4), (3.6) and (3.7) we have
\[
\left\| \begin{pmatrix}
\tilde{\Sigma}_1 S_{11} - T_{11} \Sigma_1 & -\tilde{\sigma}_r T_{12} \\
\sigma_r S_{21} & 0
\end{pmatrix} \right\| \leq 2\|E\|,
\]
which together with (3.5) gives
\[
\|X\| \leq \frac{3}{\sigma_r + \tilde{\sigma}_r}\|E\|.
\]
By (2.5), \( \|X\| = \|\Delta Q\| \), which proves (3.1).

In the case that \( r = n \leq m \), by (1.5) it needs only to prove the case that \( n < m \). Taking
\[
X = \begin{pmatrix} S_{11} - T \\ S_{21} \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix}
\tilde{\Sigma}_1 & 0 \\
0 & \tilde{\sigma}_n I
\end{pmatrix},
\]
we have
\[
X \Sigma_1 + DX = \begin{pmatrix}
S_{11} \Sigma_1 - \tilde{\Sigma}_1 T \\
S_{21} \Sigma_1
\end{pmatrix} + \begin{pmatrix}
\tilde{\Sigma}_1 S_{11} - T \Sigma_1 \\
\sigma_r S_{21}
\end{pmatrix}.
\]
From Lemma 3.1 and (2.3) it follows that
\[
\|X\| \leq \frac{1}{\sigma_n + \tilde{\sigma}_n} \left( \|E\| + \left\| \begin{pmatrix}
\tilde{\Sigma}_1 S_{11} - T \Sigma_1 \\
\sigma_r S_{21}
\end{pmatrix} \right\| \right),
\]
which together with Lemma 3.4 and (2.4) yields the desired bound (3.1).

For the \( Q \)-norm, it follows from Lemmas 3.2 and 3.3 that
\[
\left\| \begin{pmatrix}
\tilde{\Sigma}_1 S_{11} - T_{11} \Sigma_1 & -\tilde{\sigma}_r T_{12} \\
\sigma_r S_{21} & 0
\end{pmatrix} \right\|_Q^2 \leq \|\tilde{\Sigma}_1 S_{11} - T_{11} \Sigma_1\|_Q^2 + \|\tilde{\sigma}_r T_{12}\|_Q^2 + \|\sigma_r S_{21}\|_Q^2
\leq 3\|E\|_Q^2,
\]
which together with (3.5) gives
\[ \|X\|_Q \leq \frac{1 + \sqrt{3}}{\sigma_r + \tilde{\sigma}_r} \|E\|_Q \]
and hence (3.2) holds.

(2) Let \( r = n < m \). Suppose that \( \tilde{\sigma}_n \leq \sigma_n \). If we take
\[ X = \begin{pmatrix} S_{11} - T \\ S_{21} \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} \tilde{\Sigma}_1 & 0 \\ 0 & 0 \end{pmatrix}, \]
then we have
\[ X\Sigma_1 + DX = \begin{pmatrix} S_{11} \Sigma_1 - \tilde{\Sigma}_1 T \\ S_{21} \Sigma_1 \end{pmatrix} + \begin{pmatrix} \tilde{\Sigma}_1 S_{11} - T \Sigma_1 \\ 0 \end{pmatrix}. \]
From Lemma 3.1 and (2.7) it follows that
\[ \|X\| \leq \frac{1}{\tilde{\sigma}_n} \left( \|E\| + \left\| \begin{pmatrix} \tilde{\Sigma}_1 S_{11} - T \Sigma_1 \\ 0 \end{pmatrix} \right\| \right) \]
from which one can deduce the bound (3.3).

The \( Q \)-norm bound (3.4) follows from the similar proof to that of (3.2). This completes the proof of the theorem. \( \square \)

**Remark 3.1.** Now we compare our bounds in Theorem 3.5 with existing bounds.

(1) Any unitarily invariant norm case.

If \( r < n \leq m \), the existing bounds are (1.9) and (1.11). Note that
\[ \frac{3}{\sigma_r + \tilde{\sigma}_r} < \frac{2}{\sigma_r + \tilde{\sigma}_r} + \frac{2}{\max\{\sigma_r, \tilde{\sigma}_r\}}, \]
\[ \frac{3}{\sigma_r + \tilde{\sigma}_r} < \frac{2}{\sigma_r + \tilde{\sigma}_r} + \frac{1}{\min\{\sigma_r, \tilde{\sigma}_r\}}. \]
Hence the bound (3.1) always improves the bounds (1.9) and (1.11).

If \( r = n \leq m \), the existing bounds are the Li’s bound (1.6) and the Mathias’ bounds [17]
\[ \|\tilde{Q} - Q\| \leq \frac{2\|E\|}{\|E\|_2} \log \left( 1 - \frac{\|E\|_2^2}{\sigma_n} \right) \]  \( (3.8) \)
and
\[ \|\tilde{Q} - Q\| \leq \max_{0 \leq t \leq 1} \left\{ \frac{2}{\sigma_n(t)} \right\} \|E\|, \]  \( (3.9) \)
where \( \sigma_n(t) = \sigma_n(A + tE) \), \( t \in [0, 1] \).

Since
\[ \frac{3}{\sigma_n + \tilde{\sigma}_n} < \frac{2}{\sigma_n + \tilde{\sigma}_n} + \frac{1}{\max\{\sigma_n, \tilde{\sigma}_n\}} \]
and
\[ \max\{\sigma_n, \tilde{\sigma}_n\} \leq \max_{0 \leq t \leq 1} \left\{ \frac{2}{\sigma_n(t)} \right\}, \]
the bounds in (3.1) and (3.3) are always sharper than those in (1.6) and (3.9), respectively. It is easy to see that the bound in (3.3) is sharper than the one in (3.8).
(2) The $Q$-norm case.

The existing bounds were given by Li [16] as follows

$$\|\Delta Q\|_Q \leq \sqrt{\left(\frac{2}{\sigma_r + \tilde{\sigma}_r}\right)^2 + \frac{2}{\max\{\sigma_r^2, \tilde{\sigma}_r^2\}}} \|E\|_Q, \quad r < n \leq m$$

(3.10)

and

$$\|\Delta Q\|_Q \leq \sqrt{\left(\frac{2}{\sigma_n + \tilde{\sigma}_n}\right)^2 + \frac{1}{\max\{\sigma_n^2, \tilde{\sigma}_n^2\}}} \|E\|_Q, \quad r = n \leq m.$$  

(3.11)

Let $r < n \leq m$ and $\max\{\sigma_r, \tilde{\sigma}_r\} = \sigma_r$. Then when $\tilde{\sigma}_r > (3^{\frac{3}{4}} - 1)\sigma_r \approx 0.316\sigma_r$, we have

$$\frac{1 + \sqrt{3}}{\sigma_r + \tilde{\sigma}_r} \leq \sqrt{\left(\frac{2}{\sigma_r + \tilde{\sigma}_r}\right)^2 + \frac{2}{\max\{\sigma_r^2, \tilde{\sigma}_r^2\}}}.$$

which implies that the bound in (3.2) is sharper than the one in (3.10). Since $A$ is perturbed to $\tilde{A}$, $\|E\|$ may be very small, which leads to $\sigma_r \approx \tilde{\sigma}_r$, and therefore $\tilde{\sigma}_r > (3^{\frac{3}{4}} - 1)\sigma_r$.

Let $r = n \leq m$. Then when $\tilde{\sigma}_n > \frac{2}{(\sqrt{2\sqrt{2}-1}+1)(\sqrt{2}+1)}\sigma_n \approx 0.352\sigma_n$, it is easy to see that the bound in (3.4) is better than the one in (3.11).

(3) The spectral norm case.

The existing bound is (1.12). Since the spectral norm is the $Q$-norm, it is easy to see that the bound in (3.2) is always sharper than the one in (1.12).

(4) The Frobenius norm case.

The bound (3.3) shows that the Sun and Chen’s F-norm bound [18]

$$\|\Delta Q\|_F \leq \frac{2}{\max\{\sigma_r, \tilde{\sigma}_r\}} \|E\|_F$$

(3.12)

can be extended to any unitarily invariant norm for $r = n \leq m$. It is also noted that the bound (3.12) holds for any unitarily invariant norm in the case that $\max\{\sigma_r, \tilde{\sigma}_r\} \leq 2 \min\{\sigma_r, \tilde{\sigma}_r\}$.

**Remark 3.2.** Let $A$ and $\tilde{A} \in \mathcal{M}_{r \times n}$. A large number of examples show that the following inequality

$$\|\Delta Q\| \leq \frac{2}{\sigma_r + \tilde{\sigma}_r} \|E\|$$

(3.13)

holds for any unitarily invariant norm and any rank $r$, which has been proved for the Frobenius norm (see (1.8)) or for $r = m = n$ (see (1.5)). However, it is very difficult to prove (3.13) for more general cases, which remains as an open problem.

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