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Tau-Function Constructions of the Recurrence Coefficients of Orthogonal Polynomials*

Yang Chen†

Department of Mathematics, Imperial College, 180 Queen's Gate, London SW7 2BZ,

Mourad E. H. Ismail

Department of Mathematics, University of South Florida, Tampa, Florida 33620

and

Walter Van Assche§

Department of Mathematics, Katholieke Universiteit Leuven Celestijnenlaan 200
BB-3001 Leuven, Belgium

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In this paper we compute the recurrence coefficients of orthogonal polynomials using τ-function techniques. It is shown that for polynomials orthogonal with respect to positive weight functions on a noncompact interval, the recurrence coefficient can be expressed as the change in the chemical potential which, for sufficiently large N is the second derivative of the free energy with respect to N, the particle number. We give three examples using this technique: Freud weights, Erdős weights, and weak exponential weights.

Key Words: Freud weights; recurrence relations; Erdős weights; weak exponential weights; recursion coefficients; zeros; asymptotics; Hankel determinants.

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† E-mail: y.chen@ic.ac.uk.
‡ E-mail: ismail@math.usf.edu.
§ E-mail: Walter.VanAssche@wis.kuleuven.ac.be.

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1. INTRODUCTION

The theory of random matrices, originally conceived to provide a description of the energy levels of the excited states of heavy nuclei, has in recent years found applications in a diverse area of theoretical physics, such as low-dimensional string theory, quantum transport in disordered electronic systems and quantum chaos. In the formulation of Wigner, a probability measure $P(M)dM$, where dM is Haar measure, is constructed for an ensemble of $N \times N$ random matrices. If $\{x_j; 1 \le j \le N\}$ is the eigenvalue set of M, then

$$dM = \prod_{1 \le j < k \le N} |x_j - x_k|^{\beta} \prod_{l=1}^{N} dx_l \, d[\text{group}],$$

where, according to a theorem of Dyson, β takes on the values 1, 2, and 4, for matrices with orthogonal, unitary, and symplectic symmetries, respectively. Here d[group] is the measure of the corresponding unitary group. The simplest choice for dM, which is invariant under a similarity transformation, is

$$P(M) \propto \exp[-\text{tr}\, v(M)],$$

where $v(x)$ is a function of x and for the purpose of this paper is taken to be real. The proportional constant obtained by normalizing the probability measure is

$$Z_N = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp[-\Phi(x_1, \dots, x_n)] \, dx_1 \cdots dx_N,$$

where,

$$\Phi(x_1, \dots, x_N) := -\beta \sum_{1 \le j < k \le N} \ln|x_j - x_k| + \sum_{1 \le k \le N} v(x_j).$$

If M is complex Hermitian, and therefore invariant under unitary transformation, then $\beta = 2$. This is the simplest case. With appropriate normalization, it can be shown that [17],

$$\exp[-\Phi(x_1, \dots, x_N)] \prod_{l=1}^{N} dx_l = \det_{1 \le j, k \le N} [K_N(x_j, x_k)] \prod_{l=1}^{N} dx_l,$$

where

$$K_N(x, y) := \sum_{j=0}^{N-1} \sqrt{v(x)}\, \hat{p}_j(x) \sqrt{v(y)}\, \hat{p}_j(y),$$

is the reproducing kernel of the following system of orthonormal polynomials, $\{\hat{p}_n(x)\}_{n \ge 0}$,

$$\int_{-\infty}^{\infty} w(x)\hat{p}_m(x)\hat{p}_n(x)\,dx = \delta_{m,n},$$

with the *positive* weight function,

$$W(x) := \exp[-v(x)], \qquad -\infty < x < \infty.$$

According to the standard theory [17], statistical quantities that measure correlations between the eigenvalues can be expressed in terms of the reproducing kernel. The best understood ensemble, also known as the Gaussian ensemble has $v(x) = x^2$. In the Gaussian unitary case ($\beta = 2$), the appropriate polynomials are the normalized Hermite polynomials. If we adopt a statistical physics point of view, Z_N becomes the partition function of a gas of N logarithmically repelling classical charged particles in one dimension held together by the confining potential $v(x)$. Stability requirement for the gas demands that $w(x)$ must decrease sufficiently quickly as x increases so that the partition function exists. The Hermitian potential, $v(x) = x^2$, is *strongly confining* in the sense that the associated classical moment problem is determinate.

In the course of investigating quantum transport in disordered systems we are led to a potential which diverges slowly, such that

$$\lim_{x \to \infty} v(x)/|x| = 0.$$

Note that in this case, although $w(x)$ decreases slowly, the partition function exists. Of particular interest is the case where the weight function is log normal, i.e., $v(x) = O([\ln x]^2)$. Here the classical moment problem is indeterminate. Technically, the Jacobi matrix, constructed from the recurrence coefficients of the orthonormal polynomials, is unbounded and does not have a unique self-adjoint extension. Because the reproducing kernel, K_N, which plays a central role in the theory, is (via the Christoffel–Darboux formula) expressed in terms of $\hat{p}_N$, $\hat{p}_{N-1}$, and the recurrence coefficients; it is therefore of interest to have accurate information on polynomials that are orthogonal with respect to weight functions that deviate from the Hermite weight. These weights as previously discussed are not only physically interesting, they are also of independent interest in the theory of approximation. We are led to investigate three classes of weight functions and the associated recurrence coefficients.

There are three classes of weight functions that attracted a lot of attention in theory of orthogonal polynomials on infinite intervals in the last 25 years. The first class is the class of Freud weights [13, 15, 22],

$$w_F(x;\alpha) := \exp(-|x|^{\alpha}), \qquad x \in (-\infty, \infty). \tag{1.1}$$

The other two are Erdős weights,

$$w_E(x;\alpha) := \exp(-\exp(|x|^\alpha)), \qquad x \in (-\infty, \infty),\ \alpha > 0, \tag{1.2}$$

and weak exponential weights [5],

$$w_{we}(x;\alpha) := \exp(-c\,|\ln x|^\alpha), \qquad x \in (0,\infty),\ \alpha > 0,\ c > 0. \tag{1.3}$$

Let $\{p_n^E(x;\alpha)\}$, $\{p_n^F(x;\alpha)\}$, $\{p_n^{we}(x;\alpha)\}$ be the polynomials orthogonal with respect to the Erdős, Freud, and weak exponential weight, respectively. Freud [9] conjectured the large N behavior of the largest zeros of $p_N^F(x;\alpha)$ and the limiting behavior of their recursion coefficients. These conjectures were confirmed and Lubinsky's papers [13, 15] provide an excellent up to date survey of the research in this area.

More recently the Erdős weights were studied in [11] and [14] and the weak exponential weights were studied in [16] and [5], although some special cases of them go back at least to Stieltjes' work in 1894, see Chihara [6, Chap. VI, Section 2]. Many of the q-orthogonal polynomials on infinite intervals are orthogonal with respect to weight functions which behave for large $|x|$ as a weak exponential weight $w_{we}(x;2)$. It is more convenient to rewrite (1.3) as

$$w_{we}(x;\alpha) := \exp\left(-\frac{|\ln x|^\alpha}{\alpha\beta^{\alpha-1}}\right), \qquad x \in (0,\infty),\ \alpha > 0,\ \beta > 0. \tag{1.4}$$

In Section 2 we outline the connection between Hankel determinants, the tau function, and orthogonal polynomials. We also state the method of Coulomb fluid approximation and we indicate how to use it in connection with asymptotics of zeros and recursion coefficients of orthogonal polynomials. Sections 3, 4 and 5 apply the material of Section 2 to the cases of Freud, Erdős, and weak exponential weights, respectively. In each case we carry out the details of the steps indicated in Section 2. In Section 4 we apply the Birkhoff–Trjitzinsky method and state the form of the strong asymptotics of the monic polynomials orthogonal with respect to an Erdős weight. The corresponding result for Freud polynomials is stated in Section 3 and is taken from [4].

The following lemma will be used repeatedly in later development and is of independent interest. It will be proved in Section 6.

LEMMA 1.1. *Let*

$$G(z) = \sum_{n=0}^{\infty} \frac{c_n z^n}{n!}, \tag{1.5}$$

with

$$\lim_{n\to\infty} \frac{c_n}{n^\alpha} = 1, \tag{1.6}$$

for some real number α. *Then*

$$\lim_{z \to +\infty} z^{-\alpha} G(z) e^{-z} = 1. \tag{1.7}$$

Our use of Lemma 1.1 will mostly consist of replacing quotients of gamma functions of large arguments by their polynomial limiting behavior. The idea of replacing a quotient of gamma functions by the first few terms in their asymptotic development can be used, for example, to determine the large z behavior of ${}_1F_1(a; c; z)$ and gives the result [8, (6.13.3)]. It can also be applied to other entire generalized hypergeometric functions of the type ${}_pF_p$. Lemma 1.1 (and some mild extensions of it) provide a derivation of the limiting behavior which is more elementary than the use of the Mellin–Barnes integral representations. This will be elaborated and explored in some detail in a future work on asymptotics of generalized hypergeometric functions of the type ${}_rF_s$ with $r \le s$.

2. THE τ-FUNCTION AND THE COULOMB FLUID ON **R**

The τ-function is a multiple integral representation for the solution of the (multitime) Toda Lattice,

$$\partial_{t_j} Q = \left[Q, (Q^j)_+ \right], \qquad j = 1, 2, 3, \ldots, \tag{2.1}$$

where,

$$t = (t_1, t_2, \ldots),$$

and Q is the tridiagonal (or Jacobi) matrix whose elements $Q_{m,n}$ are the recurrence coefficients of the three term recurrence relations of a monic t-dependent family of orthogonal polynomials $\{p_n(x,t)\}$,

$$xp_n(x,t) = p_{n+1}(x,t) + Q_{n,n}(t) p_n(x,t) + Q_{n-1,n}(t) p_{n-1}(x,t). \tag{2.2}$$

If A is a finite or an infinite matrix, A_+ is the matrix which results from A by replacing all the entries below the diagonal by zeros. In (2.2) x is the spectral parameter and the polynomials $\{p_k(x,t)\}$ satisfy the orthogonality relation,

$$\int_K \exp\left[-u(x) - \sum_{j=1}^{M} t_j x^j \right] p_m(x,t) p_n(x,t)\, dx = h_n(t)\, \delta_{m,n}, \tag{2.3}$$

where $K \subset \mathbf{R}$ and M is finite. Now the τ-function is given by

$$\tau_N(t) := \frac{Z_N(t)}{N!}, \tag{2.4}$$

where the partition function Z_N is

$$Z_N(t) := \int_{K^N} \exp\left[-\sum_{j=1}^{N} u(x_j) - \sum_{j=1}^{N}\sum_{l=1}^{M} t_l x_j^l\right] \times \prod_{1\le j<k\le N} (x_j - x_k)^2 \prod_{j=1}^{N} dx_j, \qquad M < \infty. \tag{2.5}$$

The quantity F_N is defined through

$$\exp(-F_N) := \tau_N(t) = \frac{Z_N(t)}{N!}. \tag{2.6}$$

We usually rename the recursion coefficients in (2.2) as

$$\beta_N(t) = Q_{N-1,N}(t), \qquad \alpha_N(t) = Q_{N,N}(t). \tag{2.7}$$

Note that the τ-function is equal to the Hankel determinant,

$$D_{N-1}(t) = \det(\mu_{j+k})_{j,k=0,1,\dots,N-1},$$

where,

$$\mu_n(t) = \int_K x^n \exp\left[-u(x) - \sum_{l=1}^{M} t_l x_j^l\right] dx$$

are the moments for the orthogonality weight for the polynomials $\{p_n(x,t)\}$ [21, Eq. (2.2.11) on p. 27].

From the orthogonality relation (2.3), (2.4), and (2.5) it follows that

$$\beta_N(t) = \frac{\tau_{N-1}(t)\tau_{N+1}(t)}{[\tau_N(t)]^2}. \tag{2.8}$$

Therefore,

$$\beta_N(t) = \exp[-(F_{N+1} + F_{N-1} - 2F_N)]. \tag{2.9}$$

Similarly we establish

$$\alpha_N(t) = \partial_{t_1} \ln\left[\frac{\tau_N(t)}{\tau_{N+1}(t)}\right] = \partial_{t_1}[F_{N+1}(t) - F_N(t)]. \qquad (2.10)$$

We also note that a theorem of Dyson [17] yields

$$Z_N(t) = \prod_{j=0}^{N-1} h_j(t), \qquad (2.11)$$

which corresponds to $h_n(t) = D_n(t)/D_{n-1}(t)$ [21, Eq. (2.2.15) on p. 28].

For polynomials orthogonal with respect to even weight functions (such as Freud weights and Erdős weights) it is clear that $\alpha_N(t) = 0$. Such polynomials are called symmetric. Thus for symmetric ‘‘unevolved’’ polynomials the only nonzero recursion coefficients are $\{\beta_n\}$. Therefore (2.9) at $t = 0$ is

$$\beta_N = \exp[-(F_{N+1} + F_{N-1} - 2F_N)]. \qquad (2.12)$$

Therefore we have established the limiting relation,

$$\beta_N = \exp\left(-\frac{\partial^2 F_N}{\partial N^2}\right)\left[1 - \frac{1}{12}\frac{\partial^4 F_N}{\partial N^4} + O\left(\frac{\partial^6 F_N}{\partial N^6}\right)\right]. \qquad (2.13)$$

We shall denote the distribution function of the zeros of the unevolved polynomials $\{p_n(x)\}$ by $\sigma(x)$. The Coulomb fluid method asserts that this function σ can be obtained from the following minimization problem,

$$\min_\sigma F[\sigma] \quad \text{subject to} \int_J \sigma(x)\, dx = N, \qquad (2.14)$$

where,

$$F[\sigma] = \int_J u(x)\sigma(x)\, dx - \int_J \int_J \sigma(x) \ln|x-y|\, \sigma(y)\, dy\, dx. \qquad (2.15)$$

The function σ satisfies the requirement $\sigma(x) \ge 0$ for $x \in J$ and is a solution to the integral equation,

$$A = u(x) - 2\int_J \sigma(y) \ln|x-y|\, dy, \qquad x \in J.$$

Here A known as the chemical potential (a constant for $x \in J$) is the Lagrange multiplier for the constraint $\int_J \sigma(x)\, dx = N$, $\exp[-u(x)] = w(x)$ is the weight function and N is the degree of polynomials orthonormal with respect to weight $w(x)$,

$$\int_K p_M(x) p_N(x) w(x)\, dx = \delta_{M,N}, \qquad (2.16)$$

where K is the interval of orthogonality. It is also assumed that we seek a solution for σ that is nonnegative in J. Observe that (2.14) is an extremal problem in logarithmic potential theory for the external field $u(x)/2N$ and the solution $\sigma(x)/N$ is a probability density with support J for the corresponding equilibrium measure, which describes the asymptotic distribution of the zeros [20, Appendix IX]. In this paper we shall focus our attention on cases for which K is the real line and where the external field is such that the set J is an interval $J = [e_L, e_R]$. Note that $\sigma(\cdot)$, being an approximation to the zero counting function, is positive over its support J. The minimizing function σ satisfies an equivalent singular integral equation,

$$u'(x) = 2P\int_{e_L}^{e_R} \frac{\sigma(y)}{x-y}\,dy, \qquad x \in (e_L, e_R). \tag{2.17}$$

The Coulomb fluid method was developed by Dyson [7] for orthogonal polynomials on the unit circle and has proved to be very accurate and effective in other problems concerning random matrices [2, 3]. In was used in [4, 5] to obtain asymptotics of the extreme zeros of orthogonal polynomials supported on infinite intervals. It may be appropriate here to quote Dyson's description of the Coulomb fluid method, [7, p. 158 (p. 382)]

> These assumptions ... can be summarized in the single statement that for large N the Coulomb gas obeys the laws of classical thermodynamics. The assumption ... means that the free energy density at any point being a function of the local density and temperature alone. To a physicist these assumptions are so hallowed by custom that they hardly require justification ...

In the examples in the subsequent sections J will always be $(-b, b)$ or $(0, b)$, because the zeros of the polynomials orthogonal with respect to $w_F(x;\alpha)$ or $w_E(x;\alpha)$ are symmetrically distributed around the origin while the zeros of the polynomials orthogonal with respect to $w_{we}(x;\alpha)$ are in $(0,\infty)$. The edge parameter b is determined by the normalization condition,

$$\int_J \sigma(x)\,dx = N. \tag{2.18}$$

The general solution of the singular integral equation (2.17) is given by

$$\sigma(x) = \sigma(x,b) = \frac{1}{2\pi^2}\sqrt{\frac{e_R-x}{x-e_L}}\;P\int_{e_L}^{e_R}\frac{u'(y)}{y-x}\sqrt{\frac{y-e_L}{e_R-y}}\,dy + \frac{C}{\sqrt{(x-e_L)(e_R-x)}}, \tag{2.19}$$

for $x \in (e_L, e_R)$ where C does not depend on x but may depend on b. If $\sigma(x)$ vanishes at $x = b$ then $C = 0$. In (2.19) P stands for Principal value. The function $\sigma(x)$ given by (2.19) is the potential theoretic approximation of

$$\sigma_N(x) := w(x) \sum_{n=0}^{N-1} [p_n(x)]^2,$$

and is expected to be valid for sufficiently large N. This technique was developed by Dyson [7] on certain random matrix ensembles in the 1960s and has recently found application in other matrix ensembles [17]. The extremal problem of logarithmic potential theory and its relation to orthogonal polynomials was explored by Rakhmanov, Gonchar, Mhaskar and Saff (see Lubinsky's surveys [13] and [15]). In this context the edge parameter b is known in the theory of orthogonal polynomials as the Mhaskar–Rakhmanov–Saff number. It is worth noting that this number appeared earlier in the linear theory of elasticity. It determines the points of contact between the elastic material and a rigid stamp [18].

Central to our approach are the concepts of chemical potential, interaction energy, and free energy. Recall the chemical potential A is

$$A = u(x) - 2\int_J \sigma(y) \ln|x-y|\, dy, \tag{2.20}$$

while the interaction energy F_{int} is

$$F_{\text{int}} = \tfrac{1}{2}\int_J \sigma(x,b) u(x)\, dx. \tag{2.21}$$

The chemical potential A is a Lagrange multiplier for the extremal problem (2.14) and (2.15), hence is independent of x for $x \in J$ but A may depend on b. Also the free energy $F(\sigma)$ is now F_N,

$$F_N = \int_J u(x)\sigma(x,b)\, dx - \int_J \int_J \sigma(x,b) \ln|x-y|\, \sigma(y,b)\, dy\, dx. \tag{2.22}$$

Observe that σ depends on x and b and its domain is the sector

$$S = \{(x,b) : x \in \mathscr{R}, b \in \mathscr{R}, x \in (-b,b), b > 0\}, \tag{2.23}$$

in the upper half plane.

In the potential theory framework the solution σ/N of the extremal problem (2.14) satisfies [20, Appendix IX],

$$\begin{aligned} U\left(x; \frac{\sigma}{N}\right) &= -\frac{u(x)}{2N} + F_\sigma, \qquad \text{quasi everywhere on } J, \\ U\left(x; \frac{\sigma}{N}\right) &\le -\frac{u(x)}{2N} + F_\sigma, \qquad x \notin J, \end{aligned} \tag{2.24}$$

where $U(x;\sigma/N) = \int_J \log|x-t|^{-1}(\sigma(t)/N)\,dt$ is the logarithmic potential of σ/N and F_σ is a constant (Robin's constant for the external field $u(x)/2N$). Quasi everywhere means that the property holds everywhere except for a set of capacity zero. Observe that (2.24) gives $2U(x;\sigma) + u(x) = 2NF_\sigma$ so that the chemical potential A is equal to $2NF_\sigma$, which is indeed a constant (quasi everywhere) for $x \in J$.

THEOREM 2.1. *Assume that $J = (-b, b)$ and σ is such that $\partial\sigma(x,b)/\partial b$ exists in S, and is such that $\ln|x-y|\,\partial\sigma(y,b)/\partial b$ as a function of y is integrable on $(-b,b)$ for every fixed $x \in (-b,b)$. Furthermore assume $\sigma(\pm b, b) = 0$. Then*

$$\frac{dA}{dN} = -2\ln\frac{b}{2}, \tag{2.25}$$

and

$$\beta_N = \frac{b^2}{4}\left[1 + O\left(\frac{d^2\ln b}{dN^2}\right)\right], \tag{2.26}$$

as $N \to \infty$ provided that

$$\frac{d^2\ln b}{dN^2} = o(1), \tag{2.27}$$

where N in (2.18) *and* (2.19) *is now treated as a continuous variable.*

Proof. Because $u(x)$ does not depend on b we differentiate (2.17) with respect to b and obtain

$$0 = P\int_{-b}^{b} \frac{\partial\sigma(y,b)}{\partial b}\,\frac{dy}{x-y}, \qquad x \in (-b,b),$$

whose solution is

$$\frac{\partial\sigma(y,b)}{\partial b} = \frac{c(b)}{\sqrt{b^2-y^2}}, \tag{2.28}$$

where $c(b)$ may depend on b but not on x. Now (2.20) gives

$$\frac{dA}{db} = -2c(b)\int_{-b}^{b}\frac{\ln|x-y|}{\sqrt{b^2-y^2}}\,dy,$$

and by letting $x \to 0^+$ we get

$$\begin{aligned}\frac{dA}{db} &= -2c(b)\int_0^1 \frac{\ln\left(b\sqrt{y}\right)}{\sqrt{1-y}}\, y^{-1/2}\, dy \\ &= -2c(b)\left[\pi \ln b + \frac{\partial}{\partial \nu}\int_0^1 y^{\nu}(1-y)^{-1/2}\, dy\bigg|_{\nu=1/2}\right] \\ &= -\pi c(b)\left[2\ln b + \psi\left(\tfrac{1}{2}\right) - \psi(1)\right],\end{aligned}$$

after the evaluation of a beta integral. From [8, (1.7.29)] and the fact that $\psi(1) = -\gamma$ we find

$$\psi\left(\tfrac{1}{2}\right) - \psi(1) = -2\ln 2. \tag{2.29}$$

Thus

$$\frac{dA}{db} = -\pi c(b) \ln \frac{b^2}{4}. \tag{2.30}$$

On the other hand differentiating (2.18) with respect to b gives

$$\begin{aligned}\frac{dN}{db} &= \int_{-b}^{b} \frac{\partial \sigma(x,b)}{\partial b}\, dx \\ &= c(b)\int_{-b}^{b} \frac{dx}{\sqrt{b^2 - x^2}},\end{aligned}$$

after using (2.28). Thus,

$$\frac{dN}{db} = \pi c(b). \tag{2.31}$$

This result combined with (2.30) establishes (2.25).

To prove (2.26) we differentiate (2.22) with respect to b and get

$$\begin{aligned}\frac{dF_N}{db} &= \int_{-b}^{b}\left[u(x) - 2\int_{-b}^{b} \sigma(y,b)\ln|x-y|\, dy\right]\frac{\partial \sigma(x,b)}{\partial b}\, dx \\ &= c(b)\int_{-b}^{b}\left[u(x) - 2\int_{-b}^{b} \sigma(y,b)\ln|x-y|\, dy\right]\frac{dx}{\sqrt{b^2-x^2}} \\ &= c(b)\, A\int_{-b}^{b} \frac{dx}{\sqrt{b^2-x^2}} = \pi c(b)\, A = A\,\frac{dN}{db}.\end{aligned}$$

Thus,

$$\frac{dF_N}{dN} = A, \tag{2.32}$$

and (2.26) follows from (2.25). ∎

From (2.20), (2.28) and the fact that A does not depend on x, and in view of (2.1) we get

$$F_N = \tfrac{1}{2}NA + F_{\text{int}} = \tfrac{1}{2}NA + \tfrac{1}{2}\int_{-b}^{b} \sigma(x)u(x)\,dx. \tag{2.33}$$

3. FREUD WEIGHTS

In the case of the Freud weight (1.1), a standard integration gives the density [4] $\sigma(x, \alpha)$,

$$\sigma(x,\alpha) = \frac{\alpha}{\pi}\,\frac{2^{1-\alpha}\Gamma(\alpha)}{[\Gamma(\alpha/2)]^2}\,b^{\alpha-2}\sqrt{b^2-x^2}\;{}_2F_1\!\left(1-\frac{\alpha}{2},1;\frac{3}{2};1-\frac{x}{b^2}\right). \tag{3.1}$$

The normalization condition, $\int_{-b}^{b}\sigma(x,\alpha)\,dx = N$, gives the edge parameter

$$b = \left[\frac{[\Gamma(\alpha/2)]^2 2^{\alpha-1}N}{\Gamma(\alpha)}\right]^{1/\alpha}. \tag{3.2}$$

We now proceed to compute the chemical potential A and the interaction energy F_{int}. We need the integral representation [19, (4.1.3)],

$$\begin{aligned}{}_3F_2(a,b,c;d,e;z) &= \frac{\Gamma(e)}{\Gamma(c)\Gamma(e-c)}\\ &\quad\times\int_0^1 t^{c-1}(1-t)^{e-c-1}\,{}_2F_1(a,b;d;zt)\,dt. \end{aligned} \tag{3.3}$$

From (2.19), (2.20), and (3.1) we obtain

$$\begin{aligned} A = -\frac{2^{2-\alpha}\alpha}{\pi}\,\frac{\Gamma(\alpha)}{[\Gamma(\alpha/2)]^2}\,b^{\alpha}\int_0^1\sqrt{\frac{t}{1-t}}\;{}_2F_1\!\left(1-\frac{\alpha}{2},1;\frac{3}{2};t\right)\\ \times\left[\ln b + \tfrac{1}{2}\ln(1-t)\right]dt. \end{aligned} \tag{3.4}$$

On the other hand (3.3) and Gauss's theorem,

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \qquad \Re(c-a-b) > 0, \tag{3.5}$$

[8, (2.1.14)] imply the evaluation,

$$\begin{aligned} \int_0^1 \sqrt{\frac{t}{1-t}}\, {}_2F_1\left(1-\frac{\alpha}{2},1;\frac{3}{2};t\right) dt &= \frac{\Gamma(3/2)\Gamma(1/2)}{\Gamma(2)}\, {}_3F_2\left(1-\frac{\alpha}{2},1,\frac{3}{2};\frac{3}{2},2;1\right) \\ &= \frac{\pi}{\alpha}. \end{aligned} \tag{3.6}$$

The power series defining the ${}_2F_1$ in (3.4) converges for all $t \in [0,1]$ if and only if $\alpha > 1$. If $\alpha \le 1$ we apply the Kummer transformation [8, (2.1.23)],

$${}_2F_1(a,b;c;z) = (1-z)^{c-a-b}\, {}_2F_1(c-a,c-b;c;z), \tag{3.7}$$

and conclude that the value of the integral in (3.6) is indeed π/α. In order to evaluate the integral in (3.4) we need to consider the integral,

$$I(\nu) := \int_0^1 t^{3/2-1}(1-t)^{\nu-1}\, {}_2F_1\left(1-\frac{\alpha}{2},1;\frac{3}{2};t\right) dt. \tag{3.8}$$

Using (3.3) we see that $I(\nu)$ is a multiple of a ${}_3F_2$ with a numerator parameter equal to a denominator parameter, hence reduces to a ${}_2F_1$. The result is

$$\begin{aligned} I(\nu) &= \frac{\Gamma(3/2)\Gamma(\nu)}{\Gamma(\nu+3/2)}\, {}_2F_1\left(1-\frac{\alpha}{2},1;\frac{3}{2}+\nu;1\right) \\ &= \frac{\Gamma(3/2)\Gamma(\nu)}{(\nu-1/2+\alpha/2)\Gamma(\nu+1/2)}, \end{aligned} \tag{3.9}$$

where in the last step we used the Gauss summation theorem (3.5). From this calculation we find

$$\left.\frac{\partial I(\nu)}{\partial \nu}\right|_{\nu=1/2} = -\frac{2\pi}{\alpha^2}(1+\alpha\ln 2). \tag{3.10}$$

Combining (3.4), (3.6), and (3.10) we establish

$$A = -\frac{2^{2-\alpha}\Gamma(\alpha)}{[\Gamma(\alpha/2)]^2}\, b^{\alpha}\left[\ln\frac{b}{2}-\frac{1}{\alpha}\right]. \tag{3.11}$$

We next evaluate the integral (2.21) which gives the interaction energy. The relationships (2.21) and (3.1) imply

$$F_{\text{int}} = \frac{2^{-\alpha}\alpha}{\pi} \frac{\Gamma(\alpha)}{[\Gamma(\alpha/2)]^2} b^{2\alpha}$$

$$\times \int_0^1 t^{3/2-1}(1-t)^{(\alpha+1)/2-1} {}_2F_1\left(1-\frac{\alpha}{2}, 1; \frac{3}{2}; t\right) dt$$

$$= \frac{2^{\alpha-3}\alpha N^2}{\sqrt{\pi}} \frac{\Gamma^2(\alpha/2)\Gamma((\alpha+1)/2)}{\Gamma(\alpha)\Gamma(2+\alpha/2)} {}_3F_2\left(1-\frac{\alpha}{2}, 1, \frac{3}{2}; \frac{3}{2}, 2+\frac{\alpha}{2}; 1\right),$$

where we used (3.3). Therefore,

$$F_{\text{int}} = \frac{N^2}{2\alpha}. \tag{3.12}$$

We again used the Gauss summation theorem (3.5). Using (2.22), (3.10), and (3.11) we obtain

$$F_N = \frac{N^2}{2\alpha} - \frac{N^2}{\alpha} \ln\left[\frac{N[\Gamma(\alpha/2)]^2}{2\Gamma(\alpha)e}\right]. \tag{3.13}$$

Thus,

$$\frac{\partial^2}{\partial N^2} F_N = -\frac{2}{\alpha} \ln\left[N \frac{\Gamma(\alpha/2)^2}{2\Gamma(\alpha)}\right]. \tag{3.14}$$

Therefore (2.13) and (3.14) yield

$$\beta_N = \left[\frac{N[\Gamma(\alpha/2)]^2}{2\Gamma(\alpha)}\right]^{2/\alpha} \left[1 + \frac{2}{\alpha} N^{-2}(1+o(1))\right]. \tag{3.15}$$

The Birkhoff–Trjitzinsky [1, 23] method can be used to determine the strong asymptotics of $p_N^F(x)$ from the knowledge of the asymptotics of β_Ns in (3.15). This was carried out in [4] assuming (3.15). The result for $\alpha \geq 1$ is

$$\frac{p_N^F(x)}{(N!)^{1/\alpha}} = O\left(c_1^N \cos\left(c_2 x N^{1-1/\alpha} + \frac{\pi}{2} N + \phi(x) + \epsilon_N\right)\right), \tag{3.16}$$

as $N \to \infty$ and x fixed, where,

$$c_1 = \left[\frac{\Gamma^2(\alpha/2)}{2\Gamma(\alpha)} \right]^{1/\alpha}, \qquad c_2 = \frac{1}{2(1-\alpha)} \left[\frac{2\Gamma(\alpha)}{\Gamma^2(\alpha/2)} \right]^{1/\alpha}, \quad (3.17)$$

and ϕ is a phase factor independent of N, $\epsilon_N = o(N^{1-1/a})$ and the O term contains a function of x.

4. ERDŐS WEIGHTS

We follow the plan of Section 3 and we evaluate approximately the integrals giving the chemical potential and the internal and the free energy functionals. The weight function now is given by (1.2). Set

$$\varrho(t) := \frac{\sigma(\sqrt{t})}{2\sqrt{t}}. \qquad (4.1)$$

In this case σ of (2.18) is

$$\begin{aligned} \varrho(s) &= \frac{\alpha}{4\pi^2} \sqrt{\frac{b^2-s}{s}}\; P \int_0^{b^2} \sqrt{\frac{t}{b^2-t}}\, \exp[t^{\alpha/2}] t^{\alpha/2-1} \frac{dt}{t-s} \\ &= \frac{\alpha}{4\pi^2} \sqrt{\frac{b^2-s}{s}} \sum_{k=0}^{\infty} \frac{1}{k!} P \int_0^{b^2} t^{(\alpha(k+1)-1)/2} (b^2-t)^{-1/2} \frac{dt}{t-s} \\ &= \frac{\alpha}{4\pi^2} \sqrt{\frac{b^2-s}{s}} \sum_{k=0}^{\infty} \frac{(-1)}{k!} b^{\alpha(k+1)-2} B\left(-\frac{1}{2}, \frac{\alpha(k+1)+1}{2}\right) \\ &\quad \times {}_2F_1\left(1 - \frac{\alpha(k+1)}{2}, 1; \frac{3}{2}; 1 - \frac{s}{b^2}\right). \end{aligned} \qquad (4.2)$$

Therefore σ is given by

$$\begin{aligned} \sigma(x) &= \frac{\alpha b^{\alpha-2}}{\pi^{3/2}} \sqrt{b^2-x^2} \sum_{k=0}^{\infty} \frac{\Gamma([\alpha(k+1)+1]/2)}{\Gamma(\alpha(k+1)/2)} \frac{b^{\alpha k}}{k!} \\ &\quad \times {}_2F_1\left(1 - \frac{\alpha(k+1)}{2}, 1; \frac{3}{2}; 1 - \frac{x^2}{b^2}\right). \end{aligned} \qquad (4.3)$$

The ${}_2F_1$ in the preceding representation in the hypergeometric series together with its analytic continuations. For example, the direct power series is valid near $x = \pm b$. Near $x = 0$ the series converges if $\alpha > 1$. If this latter condition is not satisfied we must use an appropriate analytic continuation of the ${}_2F_1$ function. Thus as $x \to b$ we see that

$$\sigma(x) \sim G(b)\sqrt{b - x}, \tag{4.4}$$

and

$$G(b) = \frac{\alpha\sqrt{2}\, b^{\alpha - 3/2}}{\pi^{3/2}} \sum_{k=0}^{\infty} \frac{\Gamma([\alpha(k+1)+1]/2)}{k!\,\Gamma(\alpha(k+1)/2)} b^{\alpha k}. \tag{4.5}$$

Recall that

$$\lim_{k\to\infty} \frac{k^{c-a}\Gamma(k+a)}{\Gamma(k+c)} = 1, \tag{4.6}$$

[8, (1.1.4)]. Because we are interested in the behavior of $G(b)$ for large b we can replace the quotient of gamma functions in (4.5) by $(\alpha(k+1)/2)^{1/2}$ then through the use of Lemma 1.1 we can approximate $G(b)$ as

$$G(b) = \frac{\alpha\sqrt{2}\, b^{\alpha - 3/2}}{\pi^{3/2}} \left[\sum_{k=0}^{\infty} \frac{(k+1)^{1/2}}{k!} \sqrt{\frac{\alpha}{2}}\, b^{\alpha k} \right] [1 + O(b^{-\alpha})]. \tag{4.7}$$

Thus by Lemma 1.1,

$$G(b) = \frac{\alpha^{3/2} b^{(3(\alpha-1)/2)}}{\pi^{3/2}} \exp(b^{\alpha})[1 + O(b^{-\alpha})]. \tag{4.8}$$

We now proceed with the determination of the largest zero b of an Erdős polynomial through the side condition $\int_{-b}^{b} \sigma(x)\,dx = N$. Therefore,

$$\begin{aligned}
\int_{-b}^{b} \sigma(x)\,dx &= \frac{2\alpha b^{\alpha}}{\pi^{3/2}} \sum_{n=0}^{\infty} \frac{\Gamma[(\alpha(n+1)+1)/2]\, b^{\alpha n}}{\Gamma[\alpha(n+1)/2]\, n!} \\
&\quad \times \int_0^1 (1-x^2)^{1/2}\, {}_2F_1\left(1 - \frac{\alpha(n+1)}{2}, 1; \frac{3}{2}; 1 - x^2\right) dx \\
&= \frac{\alpha b^{\alpha}}{\pi^{3/2}} \sum_{n=0}^{\infty} \frac{\Gamma[(\alpha(n+1)+1)/2]\, b^{\alpha n}}{\Gamma[\alpha(n+1)/2]\, n!} \\
&\quad \times \int_0^1 y^{1/2}(1-y)^{-1/2}\, {}_2F_1\left(1 - \frac{\alpha(n+1)}{2}, 1; \frac{3}{2}; y\right) dy \\
&= \frac{\alpha b^{\alpha}}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\Gamma[(\alpha(n+1)+1)/2]\, b^{\alpha n}}{\Gamma[\alpha(n+1)/2]\, n!} \\
&\quad \times {}_3F_2\left(1 - \frac{\alpha(n+1)}{2}, 1, \frac{3}{2}; \frac{3}{2}, 2; 1\right).
\end{aligned} \tag{4.9}$$

The last ${}_3F_2$ is a ${}_2F_1$ of argument 1 and can be summed by Gauss's theorem (3.5). The result is

$$N = \int_{-b}^{b} \sigma(x)\, dx = \frac{\alpha b^{\alpha}}{2\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\Gamma[(\alpha(n+1)+1)/2]\, b^{\alpha n}}{\Gamma[1+\alpha(n+1)/2]\, n!}.$$

Here again we take advantage of

$$\frac{\Gamma[(\alpha(n+1)+1)/2]}{\Gamma[1+\alpha(n+1)/2]} = \left(\frac{\alpha(n+1)}{2}\right)^{-1/2}\left[1+O\left(\frac{1}{n}\right)\right],$$

and apply Lemma 1.1. This leads to the asymptotic transcendental equation,

$$N = \sqrt{\frac{\alpha b^{\alpha}}{2\pi}}\, \exp(b^{\alpha})[1+O(b^{-\alpha})]. \tag{4.10}$$

The value of b given by the Coulomb fluid method seems to be an upper bound for the largest zero. To find a better approximation to the largest zero we set

$$c = \int_{a}^{b} \sigma(x)\, dx, \tag{4.11}$$

for some $c, c \in (0, 1]$, and we determine a asymptotically. In the earlier papers of Chen and Ismail, [4, 5], c was chosen as 1, but this choice may over compensate and gives a smaller a then need be, so we just assume c is a constant in $(0, 1)$. From (4.4) and (4.11) we get

$$a = b - \left(\frac{3c}{2G(b)}\right)^{2/3}[1+O(b^{-\alpha})], \qquad c \in (0,1),$$

which after some simplification becomes

$$a = b\left[1 - \frac{1}{2}\left(\frac{3\pi c}{\alpha N \ln N}\right)^{2/3}\left[1+O\left(\frac{1}{\ln N}\right)\right]\right]. \tag{4.12}$$

This is the form conjectured by Lubinsky [14, p. 14], which was subsequently proved in [12, Corollary 1.3 on p. 205]. Observe that the latter reference gives the correct order $O((N \log N)^{-2/3})$ whereas our result in addition gives the correct sign in the second order term. We now summarize our findings so far in the form of a theorem.

THEOREM 4.1. *The largest zero $X^E_{N,N}(\alpha)$ of an Erdős polynomial $p^E_N(x;\alpha)$ satisfies the limiting relationship,*

$$X^E_{N,N}(\alpha) = b_N\left[1 - \frac{1}{2}\left(\frac{3\pi c}{\alpha N \ln N}\right)^{2/3}\left[1 + O\left(\frac{1}{\ln N}\right)\right]\right], \tag{4.13}$$

where b_N is the solution to the transcendental equation,

$$N = \sqrt{\frac{\alpha b_N^\alpha}{2\pi}}\ \exp(b_N^\alpha), \tag{4.14}$$

and $c \in (0,1)$.

We feel that $(3\pi c)^{2/3}/2$ in (4.13) is very likely to be $6^{-1/3} i_1$, i_1 being the smallest positive zero of the Airy function. This amounts to taking $c = 0.758671$.

It is worth noting that

$$b_N^\alpha = \ln\left(N\sqrt{\frac{2\pi}{\alpha \ln N}}\right) - \frac{1}{2}\ln\ln\left(N\sqrt{\frac{2\pi}{\alpha \ln N}}\right). \tag{4.15}$$

To compute the interaction energy we proceed as follows,

$$\begin{aligned}
F_{\text{int}} &= \frac{1}{2}\int_{-b}^{b} \sigma(x) u(x)\,dx \\
&= \int_0^b \sigma(x) u(x)\,dx \\
&= \frac{\alpha b^{\alpha-2}}{2\pi^{3/2}} \int_0^{b^2} \sqrt{\frac{b^2 - x}{x}} \sum_{k=0}^{\infty} \frac{\Gamma([\alpha(k+1)+1]/2)}{\Gamma(\alpha(k+1)/2)} \frac{b^{\alpha k}}{k!} \\
&\quad \times {}_2F_1\left(1 - \frac{\alpha(k+1)}{2}, 1; \frac{3}{2}; 1 - \frac{x}{b^2}\right)\exp(x^{\alpha/2})\,dx \\
&= \frac{\alpha b^{\alpha}}{2\pi^{3/2}} \int_0^{1} \sqrt{\frac{x}{1-x}} \sum_{j,k=0}^{\infty} \frac{\Gamma([\alpha(k+1)+1]/2)}{\Gamma(\alpha(k+1)/2)} \frac{b^{\alpha k}}{k!} \\
&\quad \times \frac{b^{j\alpha}}{j!}\, {}_2F_1\left(1 - \frac{\alpha(k+1)}{2}, 1; \frac{3}{2}; x\right)(1-x)^{j\alpha/2}\,dx \\
&= \frac{b^{\alpha}}{2\pi} \sum_{j,k=0}^{\infty} \frac{\Gamma([\alpha(k+1)+1]/2)}{\Gamma(\alpha(k+1)/2)} \frac{b^{\alpha(j+k)}}{k!\,j!} \\
&\quad \times \frac{\Gamma((\alpha j+1)/2)}{\Gamma(1+\alpha j/2)(j+k+1)}.
\end{aligned} \tag{4.16}$$

At this stage we write $1/(j+k+1)$ as $\int_0^\infty e^{-y(j+k+1)}\,dy$ and we obtain

$$F_{\text{int}} = \frac{b^\alpha}{2\pi}\int_0^\infty \sum_{j,k=0}^\infty \frac{\Gamma([\alpha(k+1)+1]/2)}{\Gamma(\alpha(k+1)/2)} \frac{b^{\alpha(j+k)}}{k!\,j!} \times \frac{\Gamma((\alpha j+1)/2)}{\Gamma(1+\alpha j/2)} e^{-y(j+k)}e^{-y}\,dy.$$

The double sum now factors as a product of two single sums each of which can be estimated for large b^α through Lemma 1.1. We can then interchange the limit and the integral and we arrive at the approximation,

$$F_{\text{int}} = \frac{b^\alpha}{2\pi}\left[\int_0^\infty \exp(-y+2b^\alpha e^{-y})\,dy\right][1+O(b^{-\alpha})]. \tag{4.17}$$

This establishes the following result.

THEOREM 4.2. *The interaction energy for the Erdős weight $w_E(x;\alpha)$ is given by*

$$F_{\text{int}} = \frac{1}{4\pi}\exp(2b^\alpha)[1+O(b^{-\alpha})]. \tag{4.18}$$

The next step is to compute the chemical potential A. From (2.19), (4.3) and the fact that A is independent of x, so we may set $x=0$ in (2.20), it follows that

$$\begin{aligned} A &= 1 - 2\ln b\int_{-b}^{b}\sigma(x)\,dx - 4b\int_0^1\sigma(yb)\ln y\,dy \\ &= 1 - 2N\ln b - 4b\int_0^1 \sigma(yb)\ln y\,dy \\ &= 1 - 2N\ln b - \frac{\alpha b^\alpha}{\pi^{3/2}}\left[\frac{\partial}{\partial s}\int_0^1 y^{1/2}(1-y)^{s-1/2} \times \sum_{k=0}^\infty \frac{\Gamma([\alpha(k+1)+1]/2)}{\Gamma(\alpha(k+1)/2)} \times \frac{b^{\alpha k}}{k!}\,{}_2F_1\left(1-\frac{\alpha(k+1)}{2},1;\frac{3}{2};y\right)dy\right]_{s=0} \\ &= 1 - 2N\ln b + \frac{2b^\alpha}{\alpha\sqrt{\pi}}\left[\sum_{k=0}^\infty \frac{\Gamma([\alpha(k+1)+1]/2)}{\Gamma(\alpha(k+1)/2)}\frac{b^{\alpha k}}{k!} \times \frac{1}{(k+1)^2}[1+\alpha((k+1)\ln 2)]\right][1+O(b^{-\alpha})]. \end{aligned}$$

The foregoing steps parallel the corresponding calculations of A in Section 3. Now replace $\Gamma([\alpha(k+1)+1]/2)/\Gamma(\alpha(k+1)/2)$ by $\sqrt{\alpha(k+1)/2}$ and apply Lemma 1.1. The result is

$$A = 1 - 2N \ln b + (\ln 2)\sqrt{\frac{2\alpha}{\pi}}\, b^{\alpha/2} \exp(b^{\alpha})\left[1 + O(b^{-\alpha})\right]. \quad (4.19)$$

In summary we were led to the following theorem.

THEOREM 4.3. *The chemical potential of an Erdős weight $w_E(x;\alpha)$ is given by*

$$A = 1 - 2N \ln b + 2(\ln 2) N\left[1 + O(b^{-\alpha})\right]. \quad (4.20)$$

Proof. Use the relation of b^{α} and N in (4.10) to write the chemical potential in (4.19) in the form (4.20). ∎

Now that we established the asymptotics of both the interaction energy and the chemical potential we proceed with the computation of the recursion coefficients. The total free energy F_N as given by (2.22) is

$$F_N = \frac{N}{2} - N^2 \ln b + \left[\frac{N^2}{2\alpha b^{\alpha}} - \frac{1}{4\pi} + N^2 \ln 2\right]\left[1 + O(b^{-\alpha})\right]. \quad (4.21)$$

THEOREM 4.4. *The recursion coefficients in the monic recurrence relation are asymptotically given by*

$$\beta_N = \frac{b_N^2}{4}\left[1 + O(b_N^{-\alpha})\right], \quad (4.22)$$

where b_N is as in (4.14).

Proof. Apply (2.13) and observe that

$$\frac{\partial b^{\alpha}}{\partial N} = \frac{2b^{\alpha}}{N(1 + 2b^{\alpha})}.$$

One can easily verify that the contribution of the terms containing the term $O(b^{-\alpha})$ to $\partial^2 F_N/\partial N^2$ is smaller than the main terms. Indeed,

$$\frac{\partial^2 F_N}{\partial N^2} = 2\ln 2 - \frac{2}{\alpha} \ln b^{\alpha} + O(b^{-\alpha}).$$

Furthermore,

$$\frac{\partial^4 F_N}{\partial N^4} = O(b^{-\alpha}).$$

This establishes (4.22). ∎

We next apply the Birkhoff–Trjitzinsky method, [23] and determine the strong asymptotics of $p_N^E(x)$ for large N and fixed x. In order to do so we first apply the Birkhoff–Trjitzinsky theory to solutions of

$$xy_n = y_{n+1} + \beta_n y_n, \tag{4.23}$$

where,

$$\beta_n = \tfrac{1}{4}(\ln n)^{2/\alpha}\left[1 - \frac{2}{\alpha}\,\frac{\ln\ln n}{\ln n} + \cdots\right]. \tag{4.24}$$

We shall restrict ourselves to the case $\alpha > 1$. We try a solution of (4.23) of the type,

$$y_N = 2^{-N}(\pm i)^N \exp\bigl(\lambda N \ln\ln N + \mu N^{\gamma}(\ln N)^{\delta}\bigr). \tag{4.25}$$

The presence of the factors 2^{-N} and $(\pm i)^N$ in (4.25) is more or less clear, so we proceed and substitute (4.25) into (4.23). After some tedious calculations we find that

$$\begin{aligned}2x = (\pm i)\exp&\left[\lambda\ln\ln N + \frac{\lambda(N+1/2)}{N\ln N} + \mu N^{\gamma-1}(\ln N)^{\delta}\left(\gamma + \frac{\delta}{\ln N}\right)\right]\\ &+(\mp i)\left[1 - \frac{2}{\alpha}\,\frac{\ln\ln N}{\ln N}\right]\exp\left[\left(\frac{2}{\alpha} - \lambda\right)\ln\ln N - \frac{\lambda(N-1/2)}{N\ln N}\right.\\ &\qquad\left.-\mu N^{\gamma-1}(\ln N)^{\delta}\left(\gamma + \frac{\delta}{\ln N}\right)\right].\end{aligned}$$

This suggests that we choose

$$\lambda = \frac{1}{\alpha}. \tag{4.26}$$

The result is

$$\begin{aligned}2x = (\pm i)&(\ln N)^{1/\alpha}\\ &\times\exp\left[\frac{1}{\alpha\ln N} + \frac{1/\alpha}{2N\ln N} + \mu N^{\gamma-1}(\ln N)^{\delta}\left(\gamma + \frac{\delta}{\ln N}\right)\right]\\ &\mp i(\ln N)^{1/\alpha}\left[1 - \frac{2}{\alpha}\,\frac{\ln\ln N}{\ln N}\right]\\ &\quad\times\exp\left[-\frac{1}{\alpha\ln N} + \frac{1/\alpha}{2N\ln N} - \mu N^{\gamma-1}(\ln N)^{\delta}\left(\gamma + \frac{\delta}{\ln N}\right)\right].\end{aligned}$$

This calculation points to the choices,

$$\gamma = 1, \qquad \delta = -\frac{1}{\alpha}, \quad \text{for } \alpha > 1. \tag{4.27}$$

Hence,

$$\begin{aligned} 2x &= \pm i(\ln N)^{1/\alpha}\left[1 - \frac{\mu}{\alpha}(\ln N)^{-1/\alpha} + \cdots\right] \\ &\quad \mp i(\ln N)^{1/\alpha}\left[1 + \frac{\mu}{\alpha}(\ln N)^{-1/\alpha} + \cdots\right] \\ &= \mp\frac{2i\mu}{\alpha}. \end{aligned}$$

Therefore,

$$\mu = \pm i\alpha x, \tag{4.28}$$

and we proved that for $x \neq 0$ the difference equation (4.23) has two linear independent solutions of the form (4.25) with λ, μ, and; γ and δ given by (4.26), (4.28), and (4.27), respectively. This proves that for x real the monic polynomials orthogonal with respect to an Erdős weight satisfy

$$p_N(x;\alpha) = O\left((\ln N)^{N/\alpha}\cos\left(N\frac{\pi}{2} + \frac{\alpha x N}{(\ln N)^{1/\alpha}} + \phi(x) + \epsilon_N\right)\right), \tag{4.29}$$

with a possibly x-dependent phase $\phi(x)$ and $\epsilon_N = o(N(\ln N)^{-1/\alpha}$.

5. WEAK EXPONENTIAL WEIGHTS

In this section we shall only consider the cases

$$\alpha = m = \text{positive even integers}. \tag{5.1}$$

In these cases σ is given by [5],

$$\sigma(x) = \sigma(x;m) = \frac{1}{2\pi^2\beta^{m-1}}\sqrt{\frac{b-x}{x}}\; P\int_0^b \frac{(\ln y)^{m-1}}{\sqrt{y(b-y)}}\,\frac{dy}{y-x}. \tag{5.2}$$

The sequence $\{\sigma(x;m)\colon m = 1, 2, \ldots\}$ can now be formally defined for all positive integers m by the right-hand side of (5.2) and this would make

$\sigma(x;1)=0$. Chen and Ismail [5] established the generating function,

$$\sum_{m=1}^{\infty}\sigma(x;m)\frac{\beta^{m-1}t^{m-1}}{(m-1)!}$$
$$=\frac{b^{t-1}}{\pi^{3/2}}\frac{\Gamma(t+1/2)}{\Gamma(t+1)}\sqrt{\frac{b-x}{x}}\,{}_2F_1\left(1-t,1;\frac{3}{2};1-\frac{x}{b}\right). \quad (5.3)$$

Because we will use a generating function technique we will denote the corresponding chemical potential of (2.20) by A_m, that is

$$A_m=\frac{(\ln x)^m}{m\beta^{m-1}}-2\int_0^b\sigma(y;m)\ln|x-y|\,dy. \quad (5.4)$$

Consider the generating function,

$$A(t)=\sum_{m=1}^{\infty}A_n\frac{\beta^{m-1}t^{m-1}}{(m-1)!}. \quad (5.5)$$

Now, (5.3), (5.5), and the Kummer transformation (3.7) give for $t>0$ the relationship,

$$A(t)=\frac{x^t-1}{t}-2\frac{b^t}{\pi^{3/2}}\frac{t\Gamma(t+1/2)}{\Gamma(t+1)}\int_0^1 y^{t-1}(1-y)^{1/2}\ln|x-by|$$
$$\times{}_2F_1\left(t+\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2};1-y\right)dy. \quad (5.6)$$

The power series expansion of the ${}_2F_1$ on the right-hand side of (5.6) converges for all $y\in[0,1]$ for $t\in(0,1/2)$, which we shall now assume. Because A_m, hence $A(t)$ does not depend on x and we let x tend to 0^+ in (5.6). After a change of variables we obtain

$$A(t)=-\frac{1}{t}-2\frac{b^t}{\pi^{3/2}}\frac{t\Gamma(t+1/2)}{\Gamma(t+1)}$$
$$\times\int_0^1[\ln b+\ln(1-y)](1-y)^{t-1}y^{1/2}$$
$$\times{}_2F_1\left(t+\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2};y\right)dy$$
$$=-\frac{1}{t}-2\frac{b^t}{\pi^{3/2}}\frac{t\Gamma(t+1/2)}{\Gamma(t+1)}$$
$$\times\left[\ln b\,\frac{\Gamma(3/2)\Gamma(t)}{\Gamma(t+3/2)}\,{}_2F_1\left(t+\tfrac{1}{2},\tfrac{1}{2};t+\tfrac{3}{2};1\right)\right.$$
$$\left.+\frac{\partial}{\partial\nu}\left(\frac{\Gamma(3/2)\Gamma(t+\nu)}{\Gamma(t+\nu+3/2)}\,{}_2F_1\left(t+\tfrac{1}{2},\tfrac{1}{2};t+\nu+\tfrac{3}{2};1\right)\right)\Bigg|_{\nu=0}\right],$$

where we used the integral representation (3.3). Now Gauss's theorem (3.5) and the fact,

$$\frac{1}{\Gamma(1/2)} \frac{\partial}{\partial \nu} \frac{\Gamma(\nu + 1/2)}{(t+\nu)\Gamma(\nu+1)}\bigg|_{\nu=0} = -t^{-2} + \left[\psi\left(\tfrac{1}{2}\right) - \psi(1)\right] t^{-1},$$

enables us to express $A(t)$ in the form,

$$A(t) = -\frac{1}{t} - \frac{b^t}{\pi^{1/2}} \frac{\Gamma(t+1/2)}{\Gamma(t+1)} \left[\ln b + \psi\left(\tfrac{1}{2}\right) - \psi(1) - t^{-1}\right]. \quad (5.7)$$

From the analytic structure of the generating function (5.7) it is clear that dA_m/db exists and the sequence $\{dA_m/db\}$ has the generating function dA/db. It is straight forward to derive

$$\sum_{m=1}^{\infty} \frac{dA_m}{db} \frac{\beta^{m-1} t^{m-1}}{(m-1)!} = -\frac{t\Gamma(t+1/2)}{\Gamma(t+1)} \frac{b^{t-1}}{\sqrt{\pi}} \left[\ln b + \psi\left(\tfrac{1}{2}\right) - \psi(1)\right]. \quad (5.8)$$

On the other hand the normalizing condition in (2.14) is

$$N_m = \frac{1}{2\pi\beta^{m-1}} \int_0^b \frac{(\ln bt)^{m-1}}{\sqrt{t(1-t)}}\, dt, \quad (5.9)$$

which gives the generating function for the sequence $\{N_m\}$,

$$N(t) = \sum_{m=1}^{\infty} N_m \frac{\beta^{m-1} t^{m-1}}{(m-1)!} = \frac{b^t}{2\sqrt{\pi}} \frac{\Gamma(t+1/2)}{\Gamma(t+1)}. \quad (5.10)$$

Therefore,

$$\sum_{m=1}^{\infty} \frac{\beta^{m-1} t^{m-1}}{(m-1)!} \frac{dN_m}{db} = \frac{tb^{t-1}}{2\sqrt{\pi}} \frac{\Gamma(t+1/2)}{\Gamma(t+1)}. \quad (5.11)$$

Now (5.8) and (5.11) give

$$\sum_{m=1}^{\infty} \frac{dA_m}{db} \frac{\beta^{m-1} t^{m-1}}{(m-1)!} = -2\left[\ln b + \psi\left(\tfrac{1}{2}\right) - \psi(1)\right] \sum_{m=1}^{\infty} \frac{\beta^{m-1} t^{m-1}}{(m-1)!} \frac{dN_m}{db}. \tag{5.12}$$

Equating coefficients of t^{m-1} in (5.12) and making use of (2.29) lead to

$$-\frac{dA_m}{dN} = -\frac{dA_m}{dN_m} = \ln \frac{b^2}{16}. \tag{5.13}$$

Thus,

$$\beta_N \sim \frac{b^2}{4} \sim \frac{\exp\left(N\beta^{m-1}\right)^{2/(m-1)}}{16}. \tag{5.14}$$

It is worth noting that from (5.10) it follows that

$$N = \beta^{1-m} (\ln b)^{m-1} \left(1 + \frac{1}{\ln b} + o \frac{1}{\ln b}\right). \tag{5.15}$$

6. PROOF OF LEMMA 1.1

We now give a proof of Lemma 1.1.

Proof. First take $\alpha \leq 0$, in which case we put $\alpha = -\beta$. Observe that $\Gamma(n+1)/\Gamma(n+\beta+1) \approx n^{-\beta}$ so that this ratio behaves like n^{α}. For this ratio we have

$$\begin{aligned}
G_1(z) &= \sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+\beta+1)} \frac{z^n}{n!} \\
&= \frac{1}{\Gamma(\beta)} \sum_{n=0}^{\infty} B(n+1, \beta) \frac{z^n}{n!} \\
&= \frac{1}{\Gamma(\beta)} \sum_{n=0}^{\infty} \int_0^1 t^n (1-t)^{\beta-1} \frac{z^n}{n!}\, dt \\
&= \frac{1}{\Gamma(\beta)} \int_0^1 e^{tz} (1-t)^{\beta-1}\, dt,
\end{aligned}$$

where the interchanging of sum and integral is justified because all the terms are positive for $z > 0$. Taking $tz = x$ gives

$$G_1(z) = \frac{1}{\Gamma(\beta)} z^{-\beta} \int_0^z e^x (z-x)^{\beta-1}\, dx,$$

and another substitution $z - x = y$ gives

$$G_1(z) = \frac{1}{\Gamma(\beta)} z^{-\beta} e^z \int_0^z e^{-y} y^{\beta-1}\, dy,$$

from which obviously follows that

$$\lim_{z \to +\infty} z^{\beta} G_1(z) e^{-z} = 1.$$

From this we can prove the result for $\alpha \le 0$ and any sequence c_n satisfying $c_n / n^{\alpha} \to 1$, because then

$$c_n^* = \frac{c_n \Gamma(n+\beta+1)}{\Gamma(n+1)} \to 1, \quad \text{as } n \to \infty,$$

and

$$G(z) = \sum_{n=0}^{\infty} c_n^* \frac{\Gamma(n+1)}{\Gamma(n+\beta+1)} \frac{z^n}{n!}.$$

The ratio $G(z)/G_1(z)$ is a transformation of the sequence c_n^* of the form $\sum_{n=0}^{\infty} c_n^* a_n(z)$ with $a_n(z) = z^n / [\Gamma(n+\beta+1) G_1(z)]$, for which $\lim_{z \to \infty} a_n(z) = 0$ and $a_n(z) > 0$ for every integer n, and $\sum_{n=0}^{\infty} a_n(z) = 1$. Hence by the Toeplitz–Silverman theorem for regular transformations [10],

$$\lim_{z \to +\infty} \frac{G_1(z)}{G_1(z)} = 1,$$

which combined with the asymptotic behavior of $G_1(z)$ obtained earlier gives

$$\lim_{z \to +\infty} z^{-\alpha} G(z) e^{-z} = 1.$$

Now consider the case when $\alpha > 0$. There exists an integer m such that $m < \alpha \le m+1$, which gives $\alpha - m - 1 \le 0$. Write

$$G(z) = \sum_{n=0}^{m} c_n \frac{z^n}{n!} + \sum_{n=m+1}^{\infty} c_n \frac{z^n}{n!} = P_m(z) + z^{m+1} G^*(z).$$

Here $G^*(z)$ is a series of the form considered in the lemma for the sequence

$$c_n^* = \frac{c_{n+m+1}}{(n+m+1)\cdots(n+1)}.$$

Clearly $c_n^*/n^{\alpha-m-1} \to 1$ as $n \to \infty$, and because $\alpha - m - 1 \leq 0$ we can use the result we already proved to conclude that $\lim_{z\to+\infty} z^{m+1-\alpha}G^*(z)e^{-z} = 1$. Finally, because $P_m(z)$ is a polynomial, this gives

$$\begin{aligned}\lim_{z\to+\infty} z^{-\alpha}G(z)e^{-z} &= \lim_{z\to+\infty} z^{-\alpha}P_m(z)e^{-z} + \lim_{z\to+\infty} z^{m+1-\alpha}G^*(z)e^{-z} \\ &= 1,\end{aligned}$$

which is what we wanted to prove. ∎

REFERENCES

1. G. D. Birkhoff and W. J. Trjitzinsky, Analytic theory of singular difference equations, *Acta Math.* **60** (1933), 1–89.
2. Y. Chen and K. J. Eriksen, Gap formation probability of the α-Ensemble, *Int. J. Mod. Phys.* **B9** (1995), 1205–1225.
3. Y. Chen and S. M. Manning, Some eigenvalue distribution functions of the Laguerre ensembles, *J. Phys.* **A29** (1996), 1–19.
4. Y. Chen and M. E. H. Ismail, Hermitean matrix ensembles and orthogonal polynomials, *Stud. Appl. Math.* **100** (1998), 33–52.
5. Y. Chen and M. E. H. Ismail, Asymptotics of the largest zeros of some orthogonal polynomials, preprint, 1996.
6. T. S. Chihara, "An Introduction to Orthogonal Polynomials," Gordon & Breach, New York, 1978.
7. F. J. Dyson, Statistical theory of energy levels of complex systems. II, *J. Math. Phys.* **3** (1962), 157–165, reprinted *in* "Selected Papers of Freeman Dyson with Commentary" (F. Dyson, ed.), Am. Math. Soc., Procidence, RI, 1996, pp. 381–389.
8. A. Erdélyi, W. Magnus, F. Oberhettinger, and G. F. Tricomi, "Higher Transcendental Functions," Vol. 1, McGraw-Hill, New York, 1953.
9. G. Freud, On the coefficients in the recursion formula of orthogonal polynomials, *Proc. Roy. Irish Acad.* **76** (1976), 1–6.
10. G. H. Hardy, "Divergent Series," Oxford Univ. Press, London, 1956.
11. A. Knopfmacher and D. S. Lubinsky, Analogues of Freud's conjectures for Erdős type weights and related polynomial approximation problems, *in* "Approximation Theory, Tampa" (E. B. Saff, ed.), pp. 21–69, Lecture Notes in Mathematics, Vol. 1287, Springer-Verlag, Berlin/New York, 1987.
12. A. L. Levin, D. S. Lubinsky, and T. Z. Mthembu, Christoffel functions and orthogonal polynomials for Erdős weights on $(-\infty, \infty)$, *Rend. Mat. (7)* **14** (1994), 199–289.
13. D. S. Lubinsky, A survey of general orthogonal polynomials for weights on finite and infinite intervals, *Acta Appl. Math.* **10** (1987), 237–296.

14. D. S. Lubinsky, ‘‘Strong Asymptotics for External Errors and Polynomials Associated with Erdős-Type Weights,’’ Pitman Research Notes in Mathematics, Vol. 202, Longman, Harlow, 1989.
15. D. S. Lubinsky, An update on orthogonal polynomials and weighted approximation on the real line, *Acta Appl. Math.* **33** (1993), 121–164.
16. D. S. Lubinsky and A. Sharif, On the largest zeros of orthogonal polynomials for certain weights, *Math. Comp.* **41** (1983), 199–202.
17. M. L. Mehta, ‘‘Random Matrices,’’ 2nd ed., Academic Press, New York, 1991.
18. N. I. Muskhelishvili, ‘‘Some Basic Problems of the Mathematical Theory of Elasticity,’’ 3rd ed., Noordhoff, Groningen, 1953.
19. L. J. Slater, ‘‘Generalized Hypergeometric Functions,’’ Cambridge Univ. Press, Cambridge, U.K., 1966.
20. H. Stahl and V. Totik, ‘‘General Orthogonal Polynomials,’’ Encyclopedia of Mathematics and its Applications, Vol. 43, Cambridge Univ. Press, Cambridge, U.K., 1992.
21. G. Szegŏ, ‘‘Orthogonal Polynomials,’’ Amer. Math. Soc. Colloq. Publ., Vol. 23, 4th ed., Am. Math. Soc., Providence, RI, 1975.
22. W. Van Assche, ‘‘Asymptotics for Orthogonal Polynomials,’’ Lecture Notes in Mathematics, Vol. 1265, Springer-Verlag, Berlin, 1987.
23. J. Wimp, ‘‘Computing with Recurrence Relations,’’ Pitman, London, 1984.