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A note on the spatial behavior for the generalized Tricomi equation ☆

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ABSTRACT

This short note is devoted to the study of the spatial decay estimates for the solutions of the generalized Tricomi equation. The relevance of this kind of study is that we obtain the decay for an equation which can be elliptic, parabolic and hyperbolic depending on the different points of the region. This equation is relevant in the study of fluids as well as for the anti-plane deformations of prestressed functionally graded linear elastic solids.

1. Introduction

The spatial evolution with distance from the end for solutions of equations is relevant to the study of Saint-Venant's principle in continuum mechanics. In fact, the study of edge effects for several thermomechanical situations has deserved much attention in the last three decades. Several decay estimates for elliptic [1], parabolic [2,3], and hyperbolic equations [4,5] and combinations of them [6,7] have been obtained. In these contributions the authors obtain growth/decay estimates for the solutions. However, there are few contributions in the case that the character of the equation changes with the points of the region [8]. Our contribution in this short note is addressed in this direction. That is, we will obtain spatial estimates in case that the character of the equation depends on the point. One known relevant example of this kind of equation is the Tricomi equation

$$yu_{xx} + u_{yy} = 0. \tag{1.1}$$

In this paper we will study the spatial behavior of solutions of the generalized Tricomi equation in a semi-infinite strip R with cross-section $[b_1, b_2]$, where $b_1 < 0 < b_2$. The finite end segment of the strip is contained in the line $x = 0$. We note that when y is positive Eq. (1.1) is *elliptic* and *hyperbolic* when $y < 0$. The x-axis is usually referred to as the *parabolic line*.

Tricomi's equation plays a relevant role in the study of transonic flow. It is a mixture of hyperbolic and elliptic equations in the regions where the sign of y changes. As the simplest equation satisfying this property, it proposes an interesting mathematical example of the transition from subsonic to supersonic speeds in aerodynamics. The *mysteries* of transonic flow lead to an particular situation in which the formulation of correct problems for this equation furnishes the best guide to an understanding of what should be expected of the corresponding physical phenomena. As the same time Tricomi's equation can be obtained in the study of anti-plane shear deformations for prestressed non-homogeneous neither isotropic elastic solids (see [9] for a recent contribution on the spatial stability of prestressed elastic solids). It is worth recalling that functionally graded materials are also an aspect under intensive investigation.

☆ To obtain our results the conservation laws have been used. Conserved quantities as a tool to obtain qualitative estimates for the solutions of PDEs was suggested to me by Professor R.J. Knops many years ago. This idea has been a point of reference in many of my works throughout the years. It is a pleasure to dedicate him this work in his 80th birthday with my big grateful and my highest admiration. The work is supported by the project "Ecuaciones en Derivadas Parciales en Termomecánica. Teoría y Aplicaciones" (MTM2009-08150) of the Spanish Ministry of Science and Technology. I also thank the referees their useful comments.

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In the next section we recall the equations with which we will work. In Section 3 we obtain a spatial decay estimate for the solutions of the generalized Tricomi equation. Further comments concerning nonexistence for a nonlinear version of the Tricomi equation are proposed in the last section.

2. Basic equations

We propose a problem for the generalized Tricomi differential equation

$$K(y)u_{xx} + u_{yy} = 0, \tag{2.1}$$

in the semi-infinite strip R, with a coefficient $K(y)$ which can be positive for several values of y and negative or null for other values. We adjoin to this equation the boundary conditions

$$u(x, b_1) = u(x, b_2) = 0, \quad x \geq 0, \tag{2.2}$$

$$u(0, y) = f(y), \quad y \in [b_1, b_2], \tag{2.3}$$

and the asymptotic conditions

$$u(x, y), u_x(x, y), u_y(x, y) \to 0 \quad \text{uniformly in } y \in [b_1, b_2], \text{ as } x \to \infty. \tag{2.4}$$

It is worth noting that for suitable choices of the data of the problem it is possible to find solutions which are different from zero. For instance let us consider the functions

$$M_1(y) = 1 + \sum_{k=1}^{\infty} \frac{(-y)^{3k}}{(2.3)(5.6)\cdots((3k-1)(3k))}, \qquad M_2(y) = -y + \sum_{k=1}^{\infty} \frac{(-y)^{3k+1}}{(3.4)(6.7)\cdots((3k)(3k+1))}.$$

They are solutions of the backward in time version of Airy's equation $M''(y) = -yM(y)$. It is known that when $y < 0$, these two functions agree in a point y_0; meanwhile when $y > 0$ they agree in a sequence of points $y_1 < y_2 < y_3 < \cdots$. If we consider $b_1 = y_0$ and $b_2 = y_n$, $n \geq 1$, the function $u(x, y) = \exp(-x)(M_1(y) - M_2(y))$ satisfies our problem when $K(y) = y$ and $f(y) = M_1(y) - M_2(y)$.

To analyze our problem we first consider the function

$$F(z) = \frac{1}{2} \int_{L(z)} K(y)u^2 dy, \tag{2.5}$$

where $L(z) = \{(x, y) \in R, x = z\}$. We have that

$$F(z + h) - F(z) = \int_{R(z+h,z)} K(y)uu_x dy, \tag{2.6}$$

where $R(z + h, z) = \{(x, y) \in R, z < x < z + h\}$. In view of the asymptotic conditions (2.4), we see that

$$F(z) = -\int_{R(z)} K(y)uu_x dydx, \tag{2.7}$$

where $R(z) = R(\infty, z)$, and

$$F'(z) = \int_{L(z)} K(y)uu_x dy. \tag{2.8}$$

A direct differentiation gives

$$F''(z) = \int_{L(z)} K(y)(u_x^2 + uu_{xx})dy = \int_{L(z)} (K(y)u_x^2 + u_y^2)dy. \tag{2.9}$$

The last equality is a consequence of Eq. (2.1) and the boundary conditions. If we multiply Eq. (2.1) by u_x and after an integration we see that the function

$$E(z) = \int_{L(z)} (K(y)u_x^2 - u_y^2)dy, \tag{2.10}$$

is a constant which must vanish because of the asymptotic condition (2.4). That is

$$\int_{L(z)} K(y)u_x^2 dy = \int_{L(z)} u_y^2 dy, \tag{2.11}$$

for every $z \geq 0$. It then follows that

$$F''(z) = 2\int_{L(z)} u_y^2 dy, \tag{2.12}$$

which after two quadratures implies that

$$F(z) = 2\int_{R(z)} (\xi - z)u_y^2 dyd\xi. \tag{2.13}$$

We then see that $F(z)$ defines a measure on the sub-class of the solutions that satisfy the asymptotic conditions.

3. Spatial decay

In this section we establish the spatial decay estimate. The argument is standard in the sense that it has been proposed in other easier situations, but we must adapt it to our particular case saving the difficulties proposed by the fact that $K(y)$ does not have a definite sign. We have

$$F(z) = \frac{1}{2}\int_{L(z)} K(y)u^2 dy \le \frac{(K^*)^2}{2}\int_{L(z)} u^2 dy \le \frac{(b_2 - b_1)^2 (K^*)^2}{2\pi^2}\int_{L(z)} u_y^2 dy, \tag{3.1}$$

where $(K^*)^2$ is the supremum of $|K(y)|$ and we have used the Poincaré inequality in the interval $[b_1, b_2]$. We then see that

$$F''(z) \ge a^2 F(z), \quad z \ge 0, \tag{3.2}$$

where $a = 4\pi/((b_2 - b_1)K^*)$. The last inequality implies that either the function $F(z)$ blows-up in an exponential way when z becomes unbounded or the decay estimate

$$F(z) \le F(0)\exp(-az), \quad z \ge 0, \tag{3.3}$$

is satisfied. The asymptotic conditions are only compatible with the last inequality. Thus, we have obtained the following theorem:

Theorem 3.1. *Let $u(x, y)$ be a solution of the problem determined by (2.1)–(2.4). Then, the following relation*

$$\int_{R(z)} (\xi - z)u_y^2 dyd\xi \le \frac{\int_{L(0)} K(y)f^2(y)dy}{4}\exp(-az), \quad z \ge 0, \tag{3.4}$$

is satisfied, where $a = 4\pi/((b_2 - b_1)K^*)$.

Poincaré's inequality implies the estimate

$$\int_{R(z)} (\xi - z)u^2 dyd\xi \le \frac{(b_2 - b_1)^2 \int_{L(0)} K(y)f^2(y)dy}{4\pi^2}\exp(-az), \quad z \ge 0. \tag{3.5}$$

Uniqueness of solutions in the class of functions considered is a direct consequence.

Remark. Let us consider the case that $K(y) = y, b_1 = -L, b_2 = L$ and $f(y)$ such that $f^2(-y) = f^2(y)$ for every y. The right hand side of the estimates vanishes. Thus, the problem determined by these functions, the boundary and asymptotic conditions previously proposed have only the null solution. Thus, the problem only has a solution in the case that $f(y) = 0$. This comment can be extended whenever $f(y)$ and $K(y)$ are such that $\int_{L(0)} K(y)f^2(y)dy \le 0$. That is, when $\int_{L(0)} K(y)f^2(y)dy \le 0$ and $f(y) \ne 0$, the problem determined by the boundary and the asymptotic conditions does not have a solution. This suggests that the class of functions $f(y)$ and $K(y)$ which have a solution satisfying the boundary and the asymptotic conditions is a relevant problem to be analyzed, but we don't consider it here.

We could also consider the nonlinear equation

$$K(y)u_{xx} + u_{yy} = g(u), \tag{3.6}$$

with conditions (2.2)–(2.4). The function $F(z)$ satisfies

$$F''(z) = \int_{L(z)} (K(y)u_x^2 + u_y^2 + g(u)u)dy, \tag{3.7}$$

and

$$E(z) = \int_{L(z)} (K(y)u_x^2 + 2G(u) - u_y^2)dy = E(0) \equiv 0, \tag{3.8}$$

where $G(0) = 0$ and $G'(s) = g(s)$. We obtain

$$F''(z) = \int_{L(z)} (2u_y^2 + g(u)u - 2G(u))dy. \tag{3.9}$$

In case that $g(u)u - 2G(u) \ge -2\epsilon u^2$ where $\epsilon < \pi^2/(b_2 - b_1)^2$ we can obtain a spatial decay estimate of the type of (3.4).

4. Further comments

It is also possible to obtain some instability results for the generalized Tricomi equation when we do not assume *a priori* asymptotic conditions. For Eq. (2.1) with the proposed boundary conditions the function $F(z)$ satisfies

$$F''(z) = 2\int_{L(z)} u_y^2 dy + E(0) \geq E(0). \tag{4.1}$$

We then obtain $F(z) \geq F(0) + F'(0)z + \frac{z^2}{2}E(0)$. In case that we assume that $E(0) > 0$ the solution becomes unbounded as z increases. In fact, we can see the exponential instability because of the inequality (3.2). A similar result could be obtained when $E(0) = 0$ and $F'(0) > 0$. The instability of solutions holds even in the case that the set of points y such that $K(y)$ is positive is *very small*, but with positive measure. For the nonlinear equation (3.6) the function $F(z)$ satisfies

$$F''(z) = \int_{L(z)} (2u_y^2 + g(u)u - 2G(u))dy + E(0). \tag{4.2}$$

When we assume that $g(u)u - 2G(u) \geq \alpha u^{2+\delta} - 2\epsilon u^2$ where ϵ satisfies the previous condition and α is strictly positive and $E(0) > 0$, we obtain that the function $F(z)$ satisfies

$$F''(z) \geq C_1 F^{1+\delta_1}, \tag{4.3}$$

where C_1 and δ_1 are two calculable positive constants. As $F(z)$ and $F'(z)$ are greater than zero (at least for z large enough when $E(0) > 0$), we see that

$$(F'(z))^2 \geq C_2 F^{2+\delta_1}(z) + (F'(z_0))^2 - C_2 F^{2+\delta_1}(z_0), \quad z \geq z_0. \tag{4.4}$$

One starts with this inequality and argues that u exists for all time. We separate variables and integrate to find

$$\begin{aligned} z - z_0 &\leq \int_{F(z_0)}^{F(z)} \frac{dF}{\sqrt{(F'(z_0))^2 - C_2 F^{2+\delta_1}(z_0) + C_2 F^{2+\delta_1}}} \\ &\leq \int_{F(z_0)}^{\infty} \frac{dF}{\sqrt{(F'(z_0))^2 - C_2 F^{2+\delta_1}(z_0) + C_2 F^{2+\delta_1}}} < \infty. \end{aligned} \tag{4.5}$$

This inequality leads to a contradiction and so the solution can not exist in a classical sense for all time. Thus, the nonexistence is proved whenever the initial conditions satisfy the suitable $E(0) > 0$.

One sees that an upper bound for the existence of solutions when $E(0) > 0$, $F(0) > 0$ and $F'(0) > 0$ is

$$Z_u = \int_0^{\infty} \frac{dF}{\sqrt{(F'(0))^2 - C_2 F^{2+\delta_1}(0) + C_2 F^{2+\delta_1}}}.$$

Whenever the subset of points where $K(y) > 0$ has a nonzero measure, we can always select boundary conditions such that $E(0) > 0$.

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