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The de Rham–Witt and $\mathbb{Z}_p$-cohomologies of an algebraic variety

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Abstract

We prove that, for a smooth complete variety X over a perfect field,

$$H^i(X, \mathbb{Z}_p(r)) \cong \mathrm{Hom}_{\mathrm{D}_c^b(R)}(\mathbb{1}, R\Gamma(W\Omega_X^\bullet)(r)[i]),$$

where $H^i(X, \mathbb{Z}_p(r)) = \varprojlim_n H^{i-r}(X_{\mathrm{et}}, \nu_n(r))$ (Amer. J. Math. 108 (2) (1986) 297–360), $W\Omega_X^\bullet$ is the de Rham–Witt complex on X (Ann. Scient. Ec. Num. Sup. 12 (1979b) 501–661), and $\mathrm{D}_c^b(R)$ is the triangulated category of coherent complexes over the Raynaud ring (Inst. Hautes. Etuder Sci. Publ. Math. 57 (1983) 73–212).

Keywords: Crystalline cohomology; de Rham–Witt complex; Triangulated category

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1. Introduction

According to the standard philosophy (cf. [2, 3.1]), a cohomology theory $X \mapsto H^i(X,r)$ on the algebraic varieties over a fixed field k should arise from a functor $R\Gamma$ taking values in a triangulated category D equipped with a t-structure and a Tate twist $D \mapsto D(r)$ (a self-equivalence). The heart $\mathsf{D}^\heartsuit$ of D should be stable under the Tate twist and have a tensor structure; in particular, there should be an essentially unique identity object $\mathbb{1}$ in $\mathsf{D}^\heartsuit$ such that $\mathbb{1} \otimes D \cong D \cong D \otimes \mathbb{1}$ for all objects in $\mathsf{D}^\heartsuit$. The cohomology theory should satisfy

$$H^i(X,r) \cong \mathrm{Hom}_{\mathsf{D}}(\mathbb{1}, R\Gamma(X)(r)[i]). \tag{1}$$

For example, motivic cohomology $H^i_{\mathrm{mot}}(X, \mathbb{Q}(r))$ should arise in this way from a functor to a category D whose heart is the category of mixed motives k. Absolute ℓ-adic étale cohomology $H^i_{\mathrm{et}}(X, \mathbb{Z}_\ell(r))$, $\ell \neq \mathrm{char}(k)$, arises in this way from a functor to a category D whose heart is the category of continuous representations of $\mathrm{Gal}(\bar{k}/k)$ on finitely generated $\mathbb{Z}_\ell$-modules [5]. When k is algebraically closed, $H^i_{\mathrm{et}}(X, \mathbb{Z}_\ell(r))$ becomes the familiar group $\varprojlim H^i_{\mathrm{et}}(X, \mu_{\ell^n}^{\otimes r})$ and lies in $\mathsf{D}^\heartsuit$; moreover, in this case, (1) simplifies to

$$H^i(X,r) \cong H^i(R\Gamma(X)(r)). \tag{2}$$

Now let k be a perfect field of characteristic $p \neq 0$, and let W be the ring of Witt vectors over k. For a smooth complete variety X over k, let $W\Omega^\bullet_X$ denote the de Rham–Witt complex of Bloch–Deligne–Illusie (see [10]). Regard $\Gamma = \Gamma(X,-)$ as a functor from sheaves of W-modules on X to W-modules. Then

$$H^i_{\mathrm{crys}}(X/W) \cong H^i(R\Gamma(W\Omega^\bullet_X))$$

[9, 3.4.3], where $H^i_{\mathrm{crys}}(X/W)$ is the crystalline cohomology of X [1]. In other words, $X \mapsto H^i_{\mathrm{crys}}(X/W)$ arises as in (2) from the functor $X \mapsto R\Gamma(W\Omega^\bullet_X)$ with values in $\mathsf{D}^+(W)$.

Let R be the Raynaud ring, let $\mathsf{D}(X,R)$ be the derived category of the category of sheaves of graded R-modules on X, and let $\mathsf{D}(R)$ be the derived category of the category of graded R-modules [11, 2.1]. Then Γ derives to a functor

$$R\Gamma \colon \mathsf{D}(X,R) \to \mathsf{D}(R).$$

When we regard $W\Omega^\bullet_X$ as a sheaf of graded R-modules on X, $R\Gamma(W\Omega^\bullet_X)$ lies in the full subcategory $\mathsf{D}^b_c(R)$ of $\mathsf{D}(R)$ consisting of coherent complexes [12, II 2.2], which Ekedahl has shown to be a triangulated subcategory with t-structure [11, 2.4.8]. In this note, we define a Tate twist (r) on $\mathsf{D}^b_c(R)$ and prove that

$$H^i(X, \mathbb{Z}_p(r)) \cong \mathrm{Hom}_{\mathsf{D}^b_c(R)}(\mathbb{1}, R\Gamma(W\Omega^\bullet_X)(r)[i]).$$

Here $H^i(X, \mathbb{Z}_p(r)) =_{\text{df}} \varprojlim_n H^{i-r}_{\text{et}}(X, \nu_n(r))$ with $\nu_n(r)$ the additive subsheaf of $W_n\Omega^r_X$ locally generated for the étale topology by the logarithmic differentials [14, §1], and $\mathbb{1}$ is the identity object for the tensor structure on graded R-modules defined by Ekedahl [11, 2.6.1]. In other words, $X \mapsto H^i(X, \mathbb{Z}_p(r))$ arises as in (1) from the functor $X \mapsto R\Gamma(W\Omega^\bullet_X)$ with values in $\mathsf{D}^b_c(R)$.

This result is used in the construction of the triangulated category of integral motives in [16].

It is a pleasure for us to be able to contribute to this volume: the $\mathbb{Z}_p$-cohomology was introduced (in primitive form) by the first author in an article whose main purpose was to prove a conjecture of Artin, and, for the second author, Artin's famous 18.701-2 course was his first introduction to real mathematics.

2. The Tate twist

According to the standard philosophy, the Tate twist on motives should be $N \mapsto N(r) = N \otimes \mathbb{T}^{\otimes r}$ with $\mathbb{T}$ dual to $\mathbb{L}$ and $\mathbb{L}$ defined by $Rh(\mathbb{P}^1) = \mathbb{1} \oplus \mathbb{L}[-2]$.

The Raynaud ring is the graded W-algebra $R = R^0 \oplus R^1$ generated by F and V in degree 0 and d in degree 1, subject to the relations $FV = p = VF$, $Fa = \sigma a \cdot F$, $aV = V \cdot \sigma a$, $ad = da$ $(a \in W)$, $d^2 = 0$, and $FdV = d$; in particular, R^0 is the Dieudonné ring $W_\sigma[F, V]$ [11, 2.1]. A graded R-module is nothing more than a complex

$$M^\bullet = (\cdots \to M^i \xrightarrow{d} M^{i+1} \to \cdots)$$

of W-modules whose components M^i are modules over R^0 and whose differentials d satisfy $FdV = d$. We define T to be the functor of graded R-modules such that $(TM)^i = M^{i+1}$ and $T(d) = -d$. It is exact and defines a self-equivalence $T : \mathsf{D}^b_c(R) \to \mathsf{D}^b_c(R)$.

The identity object for Ekedahl's tensor structure on the graded R-modules is the graded R-module

$$\mathbb{1} = (W, F = \sigma, V = p\sigma^{-1})$$

concentrated in degree zero [11, 2.6.1.3]. It is equal to the module $E_{0/1} =_{\text{df}} R^0/(F-1)$ of Ekedahl [3, p. 66].

There is a canonical homomorphism

$$\mathbb{1} \oplus T^{-1}(\mathbb{1})[-1] \to R\Gamma(W\Omega^\bullet_{\mathbb{P}^1})$$

(in $\mathsf{D}_c^b(R)$), which is an isomorphism because it is on $W_1\Omega^\bullet_{\mathbb{P}^1} = \Omega^\bullet_{\mathbb{P}^1}$ and we can apply Ekedahl's "Nakayama lemma" [11, 2.3.7]. See [8, I 4.1.11, p. 21], for a more general statement. This suggests our definition of the Tate twist r (for $r \geqslant 0$), namely, we set

$$M(r) = T^r(M)[-r]$$

for M in $\mathsf{D}_c^b(R)$.

Ekedahl has defined a nonstandard t-structure on $\mathsf{D}_c^b(R)$ the objects of whose heart Δ are called diagonal complexes [11, 6.4]. It will be important for our future work to note that $\mathbb{T} = T(\mathbb{1})[-1]$ is a diagonal complex: the sum of its module degree (-1) and complex degree $(+1)$ is zero. The Tate twist is an exact functor which defines a self-equivalence of $\mathsf{D}_c^b(R)$ preserving Δ.

3. Theorem and corollaries

Regard $W\Omega^\bullet_X$ as a sheaf of graded R-modules on X, and write $R\Gamma$ for the functor $\mathsf{D}(X, R) \to \mathsf{D}(R)$ defined by $\Gamma(X, -)$. As we noted above, $R\Gamma(W\Omega^\bullet_X)$ lies in $\mathsf{D}_c^b(R)$.

Theorem. *For any smooth complete variety X over a perfect field k of characteristic $p \neq 0$, there is a canonical isomorphism*

$$H^i(X, \mathbb{Z}_p(r)) \cong \mathrm{Hom}_{\mathsf{D}_c^b(R)}(\mathbb{1}, R\Gamma(W\Omega^\bullet_X)(r)[i]).$$

Proof. For a graded R-module $M^\bullet$,

$$\mathrm{Hom}(\mathbb{1}, M^\bullet) = \mathrm{Ker}(1 - F\colon M^0 \to M^0).$$

To obtain a similar expression in $\mathsf{D}^b(R)$ we argue as in Ekedahl [3, p. 90]. Let $\hat{R}$ denote the completion $\varprojlim R/(V^n R + dV^n R)$ of R [3, p. 60]. Then right multiplication by $1 - F$ is injective, and $\mathbb{1} \cong \hat{R}^0/\hat{R}^0(1 - F)$. As F is topologically nilpotent on $\hat{R}^1$, this shows that the sequence

$$0 \longrightarrow \hat{R} \xrightarrow{\cdot(1-F)} \hat{R} \longrightarrow \mathbb{1} \longrightarrow 0, \tag{3}$$

is exact. Thus, for a complex of graded R-modules M in $\mathsf{D}^b(R)$,

$$\mathrm{Hom}_{\mathsf{D}(R)}(\mathbb{1}, M) \overset{[7, 10.9]}{\cong} H^0(R\,\mathrm{Hom}(\mathbb{1}, M)) \overset{(3)}{\cong} H^0(R\,\mathrm{Hom}(\hat{R} \xrightarrow{\cdot(1-F)} \hat{R}, M)).$$

If M is complete in the sense of Illusie 1983, 2.4, then $R\,\mathrm{Hom}(\hat{R}, M) \cong R\,\mathrm{Hom}(R, M)$ [3, 5.9.3ii, p. 78], and so

$$\begin{aligned}\mathrm{Hom}_{\mathsf{D}(R)}(\mathbb{1}, M) &\cong H^0(\mathrm{Hom}(R \xrightarrow{\cdot(1-F)} R, M)) \\ &\cong H^0(\mathrm{Hom}(R, M) \xrightarrow{1-F} \mathrm{Hom}(R, M)). \end{aligned} \tag{4}$$

Following Illusie [11, 2.1], we shall view a complex of graded R-modules as a bicomplex $M^{\bullet\bullet}$ in which the first index corresponds to the R-grading: thus the j^{th} row $M^{\bullet j}$ of the bicomplex is the R-module $(\cdots \to M^{i,j} \to M^{i+1,j} \to \cdots)$, and the i^{th} column $M^{i\bullet}$ is a complex of (ungraded) R^0-modules. The jth-cohomology $H^j(M^{\bullet\bullet})$ of $M^{\bullet\bullet}$ is the graded R-module

$$(\cdots \to H^j(M^{i\bullet}) \to H^j(M^{i+1\bullet}) \to \cdots).$$

Now, $\mathrm{Hom}(R, M^{\bullet\bullet}) = M^{0\bullet}$, and so

$$H^0(\mathrm{Hom}(R, M^{\bullet\bullet}(r)[i])) = H^{i-r}(M^{r\bullet}). \tag{5}$$

The complex of graded R-modules $R\Gamma(W\Omega_X^\bullet)$ is complete [11, 2.4, Example (b), p. 33], and so (4) gives an isomorphism

$$\begin{aligned}&\mathrm{Hom}_{\mathsf{D}(R)}(\mathbb{1}, R\Gamma(W\Omega_X^\bullet)(r)[i]) \\ &\quad\cong H^0(\mathrm{Hom}(R, R\Gamma(W\Omega_X^\bullet)(r)[i]) \xrightarrow{1-F} \mathrm{Hom}(R, R\Gamma(W\Omega_X^\bullet)(r)[i])). \end{aligned} \tag{6}$$

The jth-cohomology of $R\Gamma(W\Omega_X^\bullet)$ is obviously

$$H^j(R\Gamma(W\Omega_X^\bullet)) = (\cdots \to H^j(X, W\Omega_X^i) \to H^j(X, W\Omega_X^{i+1}) \to \cdots)$$

[11, 2.2.1], and so (5) allows us to rewrite (6) as

$$\mathrm{Hom}_{\mathsf{D}(R)}(\mathbb{1}, R\Gamma(W\Omega_X^\bullet)(r)[i]) \cong H^{i-r}(R\Gamma(W\Omega_X^r) \xrightarrow{1-F} R\Gamma(W\Omega_X^r)).$$

This gives an exact sequence

$$\cdots \to \mathrm{Hom}(\mathbb{1}, R\Gamma(W\Omega_X^\bullet)(r)[i]) \to H^{i-r}(X, W\Omega_X^r) \xrightarrow{1-F} H^{i-r}(X, W\Omega_X^r) \to \cdots \tag{7}$$

On the other hand, there is an exact sequence [10, I 5.7.2]

$$0 \to \nu_\bullet(r) \to W_\bullet\Omega_X^r \xrightarrow{1-F} W_\bullet\Omega_X^r \to 0$$

of prosheaves on X_{et}, which gives rise to an exact sequence

$$\cdots \to H^i(X, \mathbb{Z}_p(r)) \to H^{i-r}(X, W_\bullet\Omega^r_X) \xrightarrow{1-F} H^{i-r}(X, W_\bullet\Omega^r_X) \to \cdots \tag{8}$$

[14, 1.10]. Here $\nu_\bullet(r)$ denotes the projective system $(\nu_n(r))_{n\geqslant 0}$, and $H^i(X, W_\bullet\Omega^r_X) = \varprojlim_n H^i(X, W_n\Omega^r_X)$ (étale or Zariski cohomology—they are the same).

Since $H^r(X, W\Omega^r_X) \cong H^r(X, W_\bullet\Omega^r_X)$ [9, 3.4.2, p. 101], the sequences (7) and (8) will imply the theorem once we check that there is a suitable map from one sequence to the other, but the right hand square in

$$\begin{array}{ccc} W\Omega^r_X & \xrightarrow{1-F} & W\Omega^r_X \\ \downarrow & & \downarrow \\ W_\bullet\Omega^r_X & \xrightarrow{1-F} & W_\bullet\Omega^r_X \end{array} \quad \xrightarrow{R\Gamma} \quad \begin{array}{ccc} R\Gamma W\Omega^r_X & \xrightarrow{1-F} & R\Gamma W\Omega^r_X \\ \downarrow & & \downarrow \\ R\Gamma W_\bullet\Omega^r_X & \xrightarrow{1-F} & R\Gamma W_\bullet\Omega^r_X \end{array}$$

gives rise to such a map. □

As in Milne [14, p. 309], we let $H^i(X, (\mathbb{Z}/p^n\mathbb{Z})(r)) = H^{i-r}_{\text{et}}(X, \nu_n(r))$.

Corollary 1. *There is a canonical isomorphism*

$$H^i(X, (\mathbb{Z}/p^n\mathbb{Z})(r)) \cong \text{Hom}_{D^b_c(R)}(\mathbb{1}, R\Gamma W_n\Omega^\bullet_X(r)[i]).$$

Proof. The canonical map $\nu_\bullet(r)/p^n\nu_\bullet(r) \to \nu_n(r)$ is an isomorphism [10, I 5.7.5, p. 598], and the canonical map $W\Omega^\bullet_X/p^n W\Omega^\bullet_X \to W_n\Omega^\bullet_X$ is a quasi-isomorphism [10, I 3.17.3, p. 577]. The corollary now follows from the theorem by an obvious five-lemma argument. □

Lichtenbaum [13] conjectures the existence of a complex $\mathbb{Z}(r)$ on X_{et} satisfying certain axioms and sets $H^i_{\text{mot}}(X, r) = H^i_{\text{et}}(X, \mathbb{Z}(r))$. Milne [15, p. 68] adds the "Kummer p-sequence" axiom that there be an exact triangle

$$\mathbb{Z}(r) \xrightarrow{p^n} \mathbb{Z}(r) \to \nu_n(r)[-r] \to \mathbb{Z}(r)[1].$$

Geisser and Levine [6, Theorem 8.5] show that the higher cycle complex of Bloch (on X_{et}) satisfies this last axiom, and so we have the following result.

Corollary 2. *Let $\mathbb{Z}(r)$ be the higher cycle complex of Bloch on X_{et}. Then there is a canonical isomorphism*

$$H^i_{\text{et}}(X, \mathbb{Z}(r) \xrightarrow{p^n} \mathbb{Z}(r)) \cong \text{Hom}_{D^b_c(R)}(\mathbb{1}, R\Gamma W_n\Omega^\bullet_X(r)[i]).$$

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