Resolutions of the Exterior and Symmetric Powers of a Module

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Let $R$ be a Noetherian ring, $M - R$-module. Assume, that

$$F: 0 \rightarrow F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0$$

is a finite free resolution of $M$. We construct the complexes $L_i F, S_i F$ which (under some assumptions) are finite free resolutions of modules $\wedge^i M, S^i M$. As an application we prove a conjecture of Buchsbaum and Eisenbud [1] on the structure theory for lower-order minors. This conjecture was proved by K. Lebelt for $Q$-algebras in [3].

1. Maps $A_i u, B_i u$

Let $u: F \rightarrow G$ be a linear map, $F, G$-free $R$-modules of finite type. We have a natural isomorphism of functors

$$\text{Hom}_R(F, G) \cong F^* \otimes G$$

on a category of free $R$-modules of finite type. We will write $\bar{u}$ for an image $e(u)$.

Now we can define the homomorphisms

$$A_i u: D_i F \otimes \wedge^i G \rightarrow D_{i+1} F \otimes \wedge^{i+1} G$$

as follows: $F^*$ acts on $D_i F$ by a module-action $S(F^*)$ on $DF$ (see [2]) and $G$ acts on $\wedge^i G$ by an exterior multiplication. Then $A_i u$ is a multiplication by $\bar{u}$ induced by above actions, and so is independent of a choice of a basis in $F$ and $G$. We have the homomorphisms $d^j : S_{i+1}(G^*) \rightarrow G^* \otimes S_i(G^*)$,

$$d^j(g^1_1 \cdots g^k_1) = \sum_{i=1}^{k} g^1_i \otimes g^1_{i+1} \cdots g^k_i.$$
where $g_i$ denotes a basis of $G^*$. $d_j^*$ then induces an action $G \otimes D_j G \to D_{j+1} G$.

Now we define the homomorphisms

$$B_{ij}: \bigwedge^i F \otimes D_j G \to \bigwedge^{i-1} F \otimes D_{j+1} G.$$

$B_{ij}$ is a multiplication by $\hat{a}$, where $F^*$ acts on $\bigwedge^i F$ by a module-action $(F)^*$ on $\bigwedge F$ (see [2]) and $G$ acts on $D_j G$ by $d_j$. Let $\{f_i\}$ be a basis in $F$, $\{g_i\}$ be a basis in $G$. Then,

$$A_{ij}u(f_1 \otimes \cdots \otimes f_r \otimes v) = \sum_{s=1}^r \sigma_{i+1} f_1 \otimes \cdots \otimes f_s \otimes u_f \otimes v,$$

$$B_{ij}(f_i \wedge \cdots \wedge f_s \otimes w) = (-1)^{i+s} f_1 \wedge \cdots \wedge f_i \wedge \cdots \wedge f_s \otimes u_{f_i} \cup w,$$

where $g_j \cup g_j^{(i)} \cdots g_j^{(s)} = g_j^{(i)} \cdots g_j^{(i+1)} \cdots g_j^{(s)}$.

### 2. Construction of the Complexes $L_i F$, $S_i F$

Let $a_0, \ldots, a_n$ be a sequence of natural numbers. We define functors

$$L(a_0, \ldots, a_n; F) = \bigwedge^{a_0} F_0 \otimes D_{a_1} F_1 \otimes \cdots \otimes \bigwedge^{a_n} F_n.$$

Now we put

$$(L_i F)_t = \bigoplus_{a_0, \ldots, a_n} L(a_0, \ldots, a_n; F).$$

The differential maps are defined as follows:

$$d: L(a_0, \ldots, a_n; F) \to L(b_0, \ldots, b_n; F)$$

is zero when $(b_0, \ldots, b_n) \neq (a_0, \ldots, a_{i-1} - 1, a_i, a_{i+1}, \ldots, a_n)$ for all $j$. When $(b_0, \ldots, b_n) = (a_0, \ldots, a_j + 1, a_{j+1}, \ldots, a_n)$, then

$$d = \pm 1 \otimes \cdots \otimes 1 \otimes A_{a_{j+1}, a_j} f_{j+1} \otimes 1 \cdots$$

if $j$ is even,

$$= \pm 1 \otimes \cdots \otimes 1 \otimes B_{a_{j+1}, a_j} f_{j+1} \otimes 1 \cdots$$

if $j$ is odd,

where the sign $\pm$ equals $(-1)^{\sigma}$, $\sigma = a_0 + 2a_1 + \cdots + (j + 1)a_j$.

We define the complexes $S_i F$ in a similar way. Putting

$$S(a_0, \ldots, a_n; F) = D_{a_1} F_0 \otimes \bigwedge^{a_1} F_1 \otimes D_{a_2} F_2 \cdots$$
we have
\[(S_i F)_i = \bigoplus_{a_i, \ldots, a_n} S(a_0, \ldots, a_n; F)\]

and \(d: S(a_0, \ldots, a_n; F) \to S(b_0, \ldots, b_n; F)\) is defined to be zero when \((b_0, \ldots, b_n) \neq (a_0, \ldots, a_j + 1, a_{j+1} - 1, \ldots, a_n)\) for all \(j\). When \((b_0, \ldots, b_n) = (a_0, \ldots, a_j + 1, a_{j+1} - 1, \ldots, a_n)\), then
\[
d = \begin{cases} 
\pm 1 \otimes \cdots \otimes 1 \otimes A_{a_{j+1}, a_j} f_{j+1} \otimes 1 \cdots & \text{if } j \text{ is odd} \\
\pm 1 \otimes \cdots \otimes 1 \otimes B_{a_{j+1}, a_j} f_{j+1} \otimes 1 \cdots & \text{if } j \text{ is even},
\end{cases}
\]
where \(\pm \) denotes \((-1)^\sigma; \ \sigma = a_0 + 2a_1 + \cdots + (j + 1)a_j\).

Now we are going to prove, that \(L_i F\) are indeed complexes (the proof for \(S_i F\) uses exactly the same arguments). We must verify three facts:

(a) The composition
\[
L(a_0, \ldots, a_n; F) \to L(a_0, \ldots, a_j + 1, a_{j+1} - 1, \ldots, a_n; F)
\]
\[
\to L(a_0, \ldots, a_j + 2, a_{j+1} - 2, \ldots, a_n; F);
\]
is zero,

(b) the following diagram is anti-commutative:

\[
\begin{array}{ccc}
L(a_0, \ldots, a_i; F) & \xrightarrow{d} & L(a_0, \ldots, a_j + 1, a_{j+1} - 1, \ldots, a_n; F) \\
\downarrow d & & \downarrow d \\
L(a_0, \ldots, a_i + 1, a_{i+1} - 1, \ldots, a_n; F) & \xrightarrow{d} & L(a_0, \ldots, a_i + 1, a_{i+1} - 1, \ldots, a_j + 1, \ldots, a_n; F)
\end{array}
\]
(we assume that two compositions exists);

(c) \(i < j\), but one of the compositions does not exist. This is the case \(j = i + 1, a_j = 0\).

Ad (a) It suffices to prove that the following compositions are zero:

\[
D_i F \otimes \bigwedge^j G \xrightarrow{A_{ij}a} D_{i-1} F \otimes \bigwedge^{j+1} G \xrightarrow{A_{i-1,j+1}a} D_{i-2} F \otimes \bigwedge^{j+2} G,
\]
\[
\bigwedge^i F \otimes D_j G \xrightarrow{B_{ij}a} \bigwedge^{i-1} F \otimes D_{j+1} G \xrightarrow{B_{i-1,j+1}a} \bigwedge^{i-2} F \otimes D_{j+2} G.
\]

1 The complexes \(L_i F, S_i F\) are independent of choice of basis because the following diagram is commutative

\[
S_{j+1} G^* \xrightarrow{d^{(1)}_j} G^* \otimes S_j G^* \\
S_{j+1} A \downarrow \quad \quad \quad \quad \quad \quad \quad \quad \downarrow A \otimes S_j A
\]
\[
S_{j+1} G^* \xrightarrow{d^{(2)}_j} G^* \otimes S_j G^*;
\]
where \(A\)—a change of a basis in \(G^*\), \(d^{(1,2)}_j\)—are two maps used in a definition of \(B_{ij}\) for two bases in \(G^*\).
We want to prove a criterion of exactness for $L_iF$ and $S_iF$. It is useful for this goal to compute the length of these complexes. We have:

\[
\text{length } L_iF = \begin{cases} 
  ni & \text{for } n \text{ odd} \\
  \min((n - 1)i + \dim F, ni) & \text{for } n \text{ even}, 
\end{cases}
\]

\[
\text{length } S_iF = \begin{cases} 
  ni & \text{for } n \text{ even} \\
  \min((n - 1)i + \dim F, ni) & \text{for } n \text{ odd}. 
\end{cases}
\]

3. Relations Between $L_iF, S_iF$ for Different Resolutions $F$ of $R$-Module $M$

In this Section $F'$ denotes a resolution of $M$ obtained from $F$ by adding a summand $R$ to $F_n$ and $F_{n-1}$ ($f'_n = f_n \oplus 1_R$). We are going to show, that the homology modules of $L_iF$ and $L_iF'$ are isomorphic. For this we need some lemmas.

**Lemma 1.** Let

\[
A: \cdots \to A_n \to A_{n-1} \to \cdots,
\]

\[
B: \cdots \to B_n \to B_{n-1} \to \cdots
\]

be two complexes. Let $e = (e_n: A_n \to B_{n-1})$ be a sequence of homomorphisms such that $de_{n+1} + e_n d = 0$ ($d$ denotes differentials of $A$ and $B$). We construct a sequence

\[
C(A, B, e): \cdots \to A_n \oplus B_n \to A_{n-1} \oplus B_{n-1} \to \cdots
\]

Then:

(a) $C(A, B, e)$ is a complex,

(b) $C(A, B, e)$ is exact if $e_n$ is an isomorphism for all $n$.

**Proof.** (a) $dd(x, y) = d(dx, dy + e_nx) = (ddx, d) dy + e_nx (+e_{n-1} dx) = (0, 0)$.

(b) Assume $(x, y) \mapsto 0$. Then $dx = 0$, $dy + e_nx = 0$.

Now for $x' = e_{n+1} y$ $dx' = x$, so $d(x', 0) = (x, y)$.

**Lemma 2.**

\[
D_i(F \oplus R) \cong \bigoplus_{0}^{i} D_j F,
\]

\[
\bigwedge^i (F \oplus R) \cong \bigwedge^i F \oplus \bigwedge^{i-1} F
\]

for all free $R$-modules $F$ of finite type, and these isomorphisms are natural with respect to $F$. 
We want to compare the complexes $L_i F'$ and $L_i F$ (the case of $S_i F'$ and $S_i F$ is analogous). From Lemma 2 we can easily deduce the following key decompositions:

(a) For $n$ even:

$$(L_i F')_t \cong \bigoplus_{j=0}^t \sum_{i} L_{t-j} F_{t-(n-1)j} \oplus \bigoplus_{j=0}^{i-1} L_{t-(j+1)} F_{t-(n-1)j-n}$$

where the skew map $e$ is an isomorphism of the $j$th piece onto the $j + 1$st ($e = \pm{id}$ on each summand $L(a_0, ..., a_n; F)$ and it maps the $j$th piece into the $j + 1$st) and $\oplus d$ denotes the sum of differentials of $L_i F$.

Now we can divide $L_i F'$ by $L_i F$. This kills the 0th piece in the first summand, and we see that after division the conditions of Lemma 1 are satisfied. We obtain

$$L_i F'/L_i F = C(K_-, K_-, e),$$

where $K_-= \bigoplus_{j=0}^t L_{t-j} F_{t-(n-1)j}$.

(b) For $n$ odd:

$$(L_i F')_t \cong \bigoplus_{j=0}^t L_{t-j} F_{t-(n-1)j} \oplus \bigoplus_{j=0}^{i-1} L_{t-(j+1)} F_{t-(n-1)j-n}$$

where the skew map $e$ is an isomorphism of the $j$th piece onto the $j + 1$st. After division by $L_i F$ we see that the conditions of Lemma 1 are satisfied and we conclude that $L_i F'/L_i F = C(M_, M_, e)$, where $M_= \bigoplus_{j=0}^{i-1} L_{t-j} F_{t-(n+1)}$, and the quotient is exact.

In this way we have proved the following proposition:

**Proposition 1.** Let

$$F: 0 \to F_n \xrightarrow{f_n} \cdots \xrightarrow{f_1} F_1 \xrightarrow{f_1} F_0,$$

$$F': 0 \to F_n \oplus R \xrightarrow{f_n \oplus 1} F_{n-1} \oplus R \cdots F_1 \xrightarrow{f_1} F_0$$

be two resolutions of $M$. Then $H.(L_i F) \cong H.(L_i F') H.(S_i F) \cong H.(S_i F')$. 

4. THE EXACTNESS CRITERION FOR $L_iF$ AND $S_iF$

We are going to prove the main result of this paper—the exactness criterion for the complexes $L_iF$, $S_iF$. First we restate the basic exactness theorem proved by Buchsbaum and Eisenbud.

**Theorem A.** Let $F: 0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0$ be a complex of free $R$-modules of finite type. Then $F$ is exact if and only if two following conditions are satisfied:

(a) rank $f_{k+1} + \text{rank} f_k = \text{rank} F_k$ for all $k$,

(b) depth $I(f_k) \geq k$ for all $k$.

**Theorem B** (Peskine–Szpiro). $F$ is exact if and only if $F \otimes R_p$ is exact for all $P$, $\text{depth } PR_p < n$.

Now we are ready to state our theorem.

**Theorem 1.** Let $F: 0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0$ be a finite free resolution, $M = \text{coker } f_1$. Then:

(a) $L_iF$ is exact if and only if:
- for all odd $j$ depth $I(f_j) \geq j$,
- for all even $j$ depth $I_{r_j-i+2}(f_j) \geq j$, depth $I_{r_j-i+1}(f_j) \geq j-1$, depth $I_{r_j}(f_j) \geq (j-1)i + 1$.

When $L_iF$ is exact then it is a finite free resolution of $\wedge^i M$.

(b) $S_iF$ is exact if and only if:
- for all even $j$ depth $I(f_j) \geq j$,
- for all odd $j$ depth $I_{r_j-i+1}(f_j) \geq j$, depth $I_{r_j-i+2}(f_j) \geq j-1$, depth $I_{r_j}(f_j) \geq (j-1)i + 1$.

When $S_iF$ is exact then it is a finite free resolution of $S_iM$.

**Proof.** We prove only the part (a) of the theorem. The proof of the part (b) is completely analogous.

Without loss of generality we can assume that $R$ is local. We proceed by induction on $n$:

For $n = 1$ we have the complexes $F: 0 \to F \to G$ and


e $L_iF: 0 \to D_iF \to D_{i-1}F \otimes G \to \cdots \to \wedge^i G$.

The length of $L_iF$ equals $i$, and $L_iF$ is minimal when $F$ is also. From the Peskine–Szpiro lemma (Theorem B) we conclude, that depth $I_k(f) \geq i$. Now assume that
depth \( R < i \). We can change the basis in \( F \) and \( G \) in such a way that \( f = f' \oplus 1: F' \oplus R \to G' \oplus R \) “Dividing by \( R \)” (Proposition 1) we continue by induction on \( \dim F \).

For \( n > 1 \) we proceed by induction on \( \dim F_n \).

(a) \( n \) is odd. We have \( \text{depth } I(\langle f_n \rangle) \geq n \) so, assuming \( \text{depth } R < n \) (lemma of Peskine–Szpiro) we see, that we can change the basis in \( F_n \) and \( F_{n-1} \) in such a way that \( f_n = f_n \oplus 1: F_n \oplus R \to F_{n-1} \oplus R \). Now we can divide by \( R \) and the proof is complete.

(b) \( n \) is even. We obtain \( \text{depth } I(\langle f_n \rangle) < n \cdot \) so we can divide by \( R \) in all localizations \( R_p \) \( \text{depth } PR_p < \text{length } L_i F \). We proceed by induction on \( n \):

\( n = 1 \). It suffices to verify exactness of \( L_i F \otimes R_p \) (\( \text{depth } PR_p < i \)). But in this case \( I(\langle f_1 \rangle) = R_p \) and after change of basis we can divide by \( R \). Now the induction on \( \dim F \) completes the proof.

\( n > 1 \). Once again we can assume depth \( R < \text{length } L_i F \), \( R \)-local is what guarantees \( I(\langle f_n \rangle) = R \). Dividing by \( R \) we finish the proof by induction on \( \dim F_n \).

**Corollary.** Assume that \( M \) is \( n(i - 1) + k \)-torsion-free (i.e., there exists an exact sequence

\[
0 \to M \to G_1 \to \cdots \to G_{n(i-1)+k}
\]

\( G_i \)-free, of finite type). Then the complexes \( L_i F, S_i F \) are exact and \( \wedge^i M, S_i M \) are \( k \)-torsion-free.

5. The Structure Theorem on Lower-Order Minors

As an application of our results we will prove the theorem on minors in finite free resolutions, which was conjectured by Buchsbaum and Eisenbud in [1] and proved for \( Q \)-algebras by Lebelt [3].

Let \( F: 0 \to F_n \to \cdots \to F_1 \to F_0 \) be a finite free resolution, \( r_k = \text{rank } f_k \). Buchsbaum and Eisenbud have proved the existence of maps \( a_k: R \to \wedge^{r_k} F_{k-1} \) such that the diagrams

\[
\begin{array}{ccc}
\wedge F_k & \xrightarrow{\wedge r_k f_k} & \wedge F_{k-1} \\
\downarrow a_k & & \downarrow a_{k-1} \\
R & & R
\end{array}
\]

are commutative for all \( k \) (we use here the canonical isomorphisms \( \wedge^{r_{k+1}} F_k \cong \wedge^{r_k} F_k \dim F_k - r_{k+1} = r_k \) by Theorem A).
We define the maps $a_k^j: \wedge^j F_{k-1} \rightarrow \wedge^{j+\tau_k} F_{k-1}$ as the compositions:

$$
\begin{align*}
\wedge F_{k-1} \xrightarrow{a_k^j} \wedge F_{k-1} \otimes \wedge F_{k-1} & \xrightarrow{i} \wedge F_{k-1}.
\end{align*}
$$

**Theorem 2.** For all $j$ such that $(j - 1)n < j(k - 1) - 2$ there exists a map $c_k^j: \wedge^j F_{k-1} \rightarrow \wedge^{\tau_{k-1} - j} F_k$ making the following diagram commutative.

$$
\begin{array}{ccc}
\wedge F_{k-1} & \xrightarrow{\tau_{k-1} \rightarrow \tau_k} & \wedge F_{k-2} \\
\downarrow & & \downarrow \\
\wedge F_{k-1} & \xrightarrow{c_k^j} & \wedge F_{k-2} \\
\end{array}
$$

**Proof.** We introduce the maps $d_f^k$:

$$
d_f^k: \wedge F_{k-1} \xrightarrow{a_k^j} \wedge F_{k-1} \otimes F_{k-1} \otimes F_k \xrightarrow{i} \wedge F_{k-1} \otimes F^*_k.
$$

The following lemma consists the properties of $d_f^k$ we need.

**Lemma 3.** (a) **The compositions**

$$
(\ast) \begin{align*}
\wedge F_{k-1} \xrightarrow{a_k^j} \wedge F_{k-1} \xrightarrow{d_f^k} \wedge F_{k-1} \otimes F^*_k,
\end{align*}
$$

$$
(\ast\ast) \begin{align*}
\wedge F_{k-2} \xrightarrow{\tau_{k-1} \rightarrow \tau_k} \wedge F_{k-1} \xrightarrow{d_f^k} \wedge F_{k-1} \otimes F^*_k.
\end{align*}
$$

are zero.

(b) **The sequence:**

$$
\begin{align*}
F_k \otimes \wedge F_{k-1} \xrightarrow{\tau_{k-1} \rightarrow \tau_k} \wedge F_{k-1} \otimes F^*_k \xrightarrow{d_f^k} \wedge F_{k-1} \otimes F^*_k
\end{align*}
$$

is a complex and, considered as a right part of a finite free resolution, it satisfies the conditions of Theorem A.

The existence of the maps $c_k^j$ is a consequence of the exactness of (by dualization and the universal property of kernel). The sequence is exact, because we can embed it into a long exact sequence by resolving the map $\wedge (f_k \otimes 1)$, which is the first map of $L_1 F^*$ where $F^*$ denotes the resolution

$$
0 \rightarrow F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_{k+1} \rightarrow F_k.
$$
$L_jF^*$ is exact and its cokernel is $2$-torsion-free when $\text{coker} f_k$ is $(n - k + 1)(j - 1) + 2$-torsion-free, i.e., $k - 1 \geq (n - k + 1)(j - 1) + 2$. This completes the proof of Theorem 2 modulo Lemma 3.

**Proof of Lemma 3.** (a) (*) It suffices to prove that $d'_j k a_k = 0$. We know that $I(a_{k+1})$ contains a nonzero divisor, so it suffices to show that $d'_j k \wedge^* f_k = 0$ but this is just [1, Lemma 3.2(a)].

(***) Lemma 3.2(c) in [1] states that $d'_j k = \pm (f_k \otimes 1)^* \wedge^*$. Now our statement is evident.

(b) The condition on depths is obvious, because $I(a_j) \supset I(f_k)$ and $I(d'_j k) \supset I(f_k)$ [1, Lemma 3.2(b), Theorem 3.1(b)]. Now we are going to show that the composition

$$F_k \otimes \bigwedge^{j-1} F_{k-1} \xrightarrow{\bigwedge^j (f_k \otimes 1)} \bigwedge^j F_{k-1} \xrightarrow{a_k^*} \bigwedge^{j+1} F_{k-1}$$

is zero. We can assume that $j = 1$. Let $x \in R$ be a nonzero divisor, $x = a_{k+1}(u)$. Then $xa_k(1) = \wedge^* f_k u$, so $xf_k u \wedge a_k(1) = 0$. The condition on ranks is verified by localizing at the set of all nonzero divisors. The proof of Lemma 3 is finished.

**REFERENCES**