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On the Variety of a Highest Weight Module

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1. INTRODUCTION

1.1. This work is a natural continuation of [19, 20, 22] which we refer to as I, II, III and whose notation we adopt. Throughout $\mathfrak{g}$ is a complex semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra for $\mathfrak{g}$, and $\lambda \in \mathfrak{h}^*$ is dominant and regular.

1.2. Let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. Detailed information on the primitive spectrum $\operatorname{Prim} U(\mathfrak{g})$ of $U(\mathfrak{g})$ was summarized by Conjecture 7.4 of [18]. Here a main question was to relate $\operatorname{Prim} U(\mathfrak{g})$ with the Springer correspondence. Parts (i), (ii) of this conjecture were established in I, II and a partial solution to (iii) was given in [21] sufficient, e.g., to treat $\mathfrak{g}$ of type A_n. Meanwhile Borho and Brylinski [5] have established (iii) for induced ideals, while Barbasch and Vogan [1, 2] have established a version of (iii) in which $\mathscr{V}(\operatorname{gr} I)$: $I \in \operatorname{Prim} U(\mathfrak{g})$ is replaced by a certain wavefront set (known to be contained in $\mathscr{V}(\operatorname{gr} I)$). This last work involves some case by case analysis and gives little indication as to why the Springer correspondence should arise in the study of $\operatorname{Prim} U(\mathfrak{g})$.

1.3. Let W be the Weyl group for the pair $(\mathfrak{g}, \mathfrak{h})$ and W_λ the subgroup of "integral reflections" relative to λ (notation I, 1.4). After Duflo, one has surjective maps $\mathfrak{h}^* \to \operatorname{Prim} U(\mathfrak{g}) \to \mathfrak{h}^*/W$ defined by $\mu \to \operatorname{Ann} L(\mu) \to Z(\mathfrak{g}) \cap \operatorname{Ann} L(\mu)$ (notation I, 1.3). In view of the Borho–Janzten translation principle it was natural to conjecture that the kernel of these maps should be given by some combinatorial property of W. In fact [16], for $\mathfrak{g}$ of type A_{n-1} and when $W = W_\lambda$ these kernels are precisely determined by the Robinson correspondence between the symmetric group S_n and the set of all pairs of standard tableaux associated to the partitions of n. Following this,

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Spaltenstein observed [30] that the Robinson correspondence was also appropriate to describe the Steinberg correspondence (see Section 9) introduced [35] to study the set of nilpotent elements in $\mathfrak{g}^*$. The results of Barbasch and Vogan [1, 2] extend this relationship between Prim $U(\mathfrak{g})$ and the Steinberg correspondence to arbitrary $\mathfrak{g}$. This becomes less precise outside type A_n and in addition the case-by-case nature of their analysis does not permit easy interpretation of this remarkable phenomenon.

1.4. Let $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ be the triangular decomposition of $\mathfrak{g}$ corresponding to the choice of $\mathfrak{h}$ and to the choice of a basis B for the set of non-zero roots R. Set $\mathfrak{b} = \mathfrak{n} \oplus \mathfrak{h}$. Identify $\mathfrak{g}$ with $\mathfrak{g}^*$ through the Killing form (and hence $\mathfrak{n}^*$ with $\mathfrak{n}^-$). Let $\mathbf{G}$ denote the algebraic adjoint group of $\mathfrak{g}$ and $\mathbf{B}$, $\mathbf{N}$, $\mathbf{N}^-$, $\mathbf{H}$ the subgroups of $\mathbf{G}$ corresponding to $\mathfrak{b}$, $\mathfrak{n}$, $\mathfrak{n}^-$, $\mathfrak{h}$.

1.5. Let M be a $U(\mathfrak{g})$ module generated by some finite-dimensional subspace M^0. After Bernstein [4, Sect. 1] the variety $\mathscr{V}(M)$ of zeros of $\operatorname{gr}(\operatorname{Ann} M^0)$ is independent of M^0. Again $U(\mathfrak{g})/\operatorname{Ann} M$ is cyclic as a left $U(\mathfrak{g})$ module and we set $\mathscr{V}\mathscr{A}(M) := \mathscr{V}(U(\mathfrak{g})/\operatorname{Ann} M)$. One observes that $\mathscr{V}\mathscr{A}(M)$ coincides with the variety of zeros of $\operatorname{gr}(\operatorname{Ann} M)$ (4.6) and is hence $\mathbf{G}$ stable. This gives the inclusion $\mathscr{V}\mathscr{A}(M) \supset \overline{\mathbf{G}\mathscr{V}(M)}$. For say simple highest weight modules one has equality of dimension [24, 6.3.14] in the above and hence

$$\mathscr{V}\mathscr{A}(M) = \overline{\mathbf{G}\mathscr{V}(M)} \qquad (*)$$

if say $\mathscr{V}\mathscr{A}(M)$ is irreducible (e.g., in type A_n [21]). More recently Borho and Bryslinki [5] have proved $(*)$ for $M \cong L(\mu)$ with $\mu \in \mathfrak{h}^*$ integral.

1.6. Our present aim is to compute $\mathscr{V}(M)$ for any simple highest weight module M. Here we shall present two methods. First we remark (see 10.1) that $\mathscr{V}(M)$ need not be irreducible. Yet a very general result of Gabber [11] asserts that for any finite-dimensional Lie algebra $\mathfrak{g}$ and any finitely generated $U(\mathfrak{g})$ module M which admits no non-zero submodule of strictly lower Gelfand–Kirillov dimension the variety $\mathscr{V}(M)$ is equidimensional. Consequently we need only to be able to compute the components of $\mathscr{V}(M)$ of maximum dimension. Again we can use Gabber's result (4.6) to show (for M simple) that $\mathscr{V}\mathscr{A}(M)$ is equidimensional. Consequently our methods can in principle be extended to establishing $(*)$ of 1.5.

1.7. Our first method originates from an observation of King [27] (used also by Barbasch and Vogan). Let $\mathscr{V}$ be a closed $\mathbf{H}$ stable subvariety of $\mathfrak{n}$ and $I(\mathscr{V})$ its ideal of definition in $S(\mathfrak{n}^-)$. To $\mathscr{V}$ we assign a polynomial $p_{\mathscr{V}}$ in $S(\mathfrak{h})$ which measures the growth rate of $\mathfrak{h}$ weight spaces of $S(\mathfrak{n}^-)/I(\mathscr{V})$.

Since $\mathscr{V}(L(\mu)) \supset \mathfrak{h} + \mathfrak{n}^-$ we may just consider $\mathscr{V}(L(\mu))$ to be a closed **B** stable subvariety of $\mathfrak{n}$. From King's observation we show (5.5) that $p_{\mathscr{S}(L(w\lambda))}$ is proportional to the Goldie rank polynomial $\tilde{p}_{w^{-1}}$ (II, 1.4) assigned to $U(\mathfrak{g})/\text{Ann}\, L(w^{-1}\lambda)$. Here $p_{\mathscr{S}(L(w\lambda))}$ differs from $p_{\mathscr{V}(L(w\lambda))}$ in that multiplicities are counted. Now let $\mathscr{N}$ denote the variety of nilpotent elements of $\mathfrak{g}$ which we recall decomposes into a finite union of **G** orbits. From the construction of Springer [33] one obtains an injective map $\text{Sp}: \mathscr{N}/\mathbf{G} \to \hat{W}$ with the property that for each nilpotent orbit $\mathscr{O}$ the number of irreducible components of $\mathscr{O} \cap \mathfrak{n}$ is just $\dim \text{Sp}(\mathscr{O})$. Denote these components by $\mathscr{V}_i$. Applying a technique used by Spaltenstein [31] (who showed that the $\mathscr{V}_i$ have all the same dimension) we show that $P_{\mathscr{O}} := \sum \mathbb{C} p_{\mathscr{V}_i}$ is a W submodule of $S(\mathfrak{h})$. Presumably $P_{\mathscr{O}}$ is irreducible of type $\text{Sp}(\mathscr{O}) \otimes \text{sg}$, where sg denotes the sign representation of W. Admitting this and 1.5(*) gives a natural and complete solution to part (iii) in Conjecture 7.4 of [18]. Moreover this isomorphism would imply the $p_{\mathscr{V}_i}$ to be linearly independent and so comparison of the $p_{\mathscr{V}_i}$ with $\tilde{p}_{w^{-1}}$ completely determines $\mathscr{V}(L(w\lambda))$. Of course this last step also requires a fairly explicit knowledge of the $p_{\mathscr{V}_i}$. We conjecture the precise form of these polynomials in 9.8. Our formula would in addition imply the required identification with the Springer correspondence, namely, that $P_{\mathscr{O}}$ is irreducible of type $\text{Sp}(\mathscr{O}) \otimes \text{sg}$.

1.8. For each $w \in W$, set $w(\mathfrak{n}) = \mathbb{C}\{X_{w\alpha} : \alpha \in R^+\}$. Then $\mathbf{G}(\mathfrak{n} \cap w(\mathfrak{n}))$ is an irreducible **G** stable subvariety of $\mathscr{N}$ and so admits a unique dense orbit which we denote by $\text{St}(w)$. After Steinberg [35, Sect. 4] the map $w \mapsto \text{St}(w)$ of W to $\mathscr{N}/\mathbf{G}$ is surjective. Furthermore, from [31] and [35] one easily shows (9.6) that the irreducible components of $\text{St}(w) \cap \mathfrak{n}$ are open in the closures of the $\mathscr{V}_1(w') := \mathbf{B}(\mathfrak{n} \cap w'(\mathfrak{n}))$: $w' \in St^{-1}(\text{St}(w))$. Using Enright's functor we develop an inductive procedure for relating $\mathscr{V}(L(-w\lambda))$ to $\overline{\mathscr{V}_1(w)}$. This shows for example when λ is integral that $\mathscr{V}(L(-w\lambda)) \supset \overline{\mathscr{V}_1(w)}$. Though equality fails in general, one expects for λ integral to have equality whenever one has equality of dimension. (One has equality of dimension in type A_n and a somewhat more precise result than the above (see 9.12, 9.14).) In any case it is clear that the above result gives a natural relationship between the kernel of the Duflo map and the Steinberg correspondence indicated in 1.4. To make this relationship precise we need to show that for each $w \in W$ there exists $\mu \in \mathfrak{h}^*$ (not necessarily regular) such that $\mathscr{V}(L(\mu)) = \overline{\mathscr{V}_1(w)}$. By a theorem of Gabber [10] this would imply that $\mathscr{V}_1(w)$ is involutive; but of course one easily sees (7.5) this to be the case directly and so our question can be viewed as a very special case of the obvious converse to Gabber's theorem.

1.9. One may ask, how does the variety of a highest weight module behave under the Enright functor? We show that (for integral weights) this is described by the action of the Weyl group on components as discussed briefly in 1.7. This indicates that the Enright functor can be implemented via the results of Beilinson and Bernstein [3] (or Brylinski and Kashiwara [6]) through the action defined by Springer [33] of W on the components of $\mathcal{O} \cap \mathfrak{n}$.

2. The Characteristic Polynomial

2.1. For each finite-dimensional $\mathbb{C}$ vector space V, we denote by V^* its dual and by $S(V)$ the symmetric algebra over V. For each ideal I of $S(V)$ we denote by $\mathscr{V}(I) \subset V^*$ its variety of zeros and for any subvariety $\mathscr{V} \subset V^*$ we denote by $I(\mathscr{V})$ the ideal of definition of the Zariski closure $\overline{\mathscr{V}}$ of $\mathscr{V}$.

2.2. Let δ be a semisimple endomorphism of V. We assume that the eigenvalues $\{k_1, k_2, ..., k_t\}$ of δ are strictly positive integers, which we shall eventually regard as variables. Extend δ to a derivation (also denoted by δ) of $S(V)$. Let M be a finitely generated $S(V)$ module and D a derivation of M satisfying $D(am) = \delta(a)m + a(Dm)$ for all $a \in S(V)$, $m \in M$. (We call D a δ-derivation of M.) We assume that D acts locally semisimply on M and that all its eigenvalues are positive integers. For each $n \in \mathbb{N}$ set $M_n = \{\xi \in M: D\xi = n\xi\}$ and define the Poincaré series of M through

$$R_M(x) := \sum_{n=0}^{\infty} (\dim M_n)\, x^n.$$

As is well known $R_M(x)$ takes the form

$$R_M(x) = \frac{f_M(x)}{\prod_{i=1}^{t} (1 - x^{k_i})}, \qquad (*)$$

where f_M is a polynomial. Now assume that M is a cyclic $S(V)$ module generated by an eigenvector of D which we can conveniently assume to have zero eigenvalue. Then we can write $R_M(x)$ in the form

$$R_M(x) = \sum_{l \in \mathbb{N}^t} c_l(M)\, x^{k \cdot l},$$

where $k \cdot l = \sum_{i=1}^{t} k_i l_i$ and the $c_l(M) \in \mathbb{C}$ depend on M but not on the k_i. Substitution in $(*)$ and multiplying out its denominator shows that there exists a finite subset $F_M \subset \mathbb{N}^t$ such that

$$f_M(x) = \sum_{l \in F_M} c'_l(M)\, x^{k \cdot l}, \qquad (**)$$

where the $c_l'(M)$ are linear combinations of the $c_l(M)$ and again independent of the k_i.

2.3. Retain the hypotheses and notation of 2.2. We consider the expression

$$\sum_{m=0}^{n} \dim M_m$$

as a function of n and the k_i: $i = 1, 2, \ldots, t$. By 2.2(*) it is a polynomial in n on residue classes mod($\prod k_i$). Fix a residue class. Then the corresponding polynomial has degree equal to the dimension d_M of the variety associated to M and we denote its leading coefficient by $r_M(k)$.

LEMMA. (i) *$r_M(k)$ is independent of the residue class chosen and takes the form*

$$r_M(k) = p_M(k) \Big/ \left(\prod_{i=1}^{t} k_i \right),$$

where p_M is a polynomial.

(ii) *If f_M is given by (**) then*

$$p_M(k) = \frac{(-1)^m}{m!} \sum_{l \in F_M} c_l'(M)(k \cdot l)^m,$$

where m is the smallest integer $\geqslant 0$ for which this expression is non-zero.

(iii) *p_M is homogeneous of degree $t - d_M$.*

Through the additivity of R_M on exact sequences and the fact that in (iii) the degree of p_M is determined by d_M it is enough to assume that M is generated by an eigenvector of D. Fix $n \in \mathbb{N}$ and take $x = e^{-i(\theta - i\varepsilon)}$ with θ, ε real and $\varepsilon > 0$. Set $n_0 = \deg f_M + 1$. We shall require a complex function h satisfying for m real

$$\int_{-\infty}^{\infty} e^{-i(\theta - i\varepsilon)m} h(\theta)\, d\theta = 1, \qquad n_0 \leqslant m \leqslant n,$$

$$= 0, \qquad \text{otherwise.}$$

By Fourier transform we obtain

$$h(\theta) = \frac{1}{2\pi} \int_{n_0}^{n} e^{i(\theta - i\varepsilon)m}\, dm = \frac{(e^{i(\theta - i\varepsilon)n} - e^{i(\theta - i\varepsilon)n_0})}{2\pi i(\theta - i\varepsilon)}.$$

By construction

$$\sum_{m=n_0}^{n} \dim M_m = \sum_{m=n_0}^{\infty} (\dim M_m) \int_{-\infty}^{\infty} e^{-i(\theta - i\varepsilon)m} h(\theta)\, d\theta$$
$$= \int_{-\infty}^{\infty} R_M(e^{-i(\theta - i\varepsilon)})\, h(\theta)\, d\theta$$
$$= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{in_0(\theta - i\varepsilon)}(e^{i(\theta - i\varepsilon)(n-n_0)} - 1) f_M(e^{-i(\theta - i\varepsilon)})\, d\theta}{(\theta - i\varepsilon) \prod_{l=1}^{t} (1 - e^{-i(\theta - i\varepsilon)k_l})}.$$

The integrand admits analytic continuation in the upper half-plane to a meromorphic function $e^{i\theta} g(\theta)$ with a pole at $\theta = i\varepsilon$. Take $n \geqslant n_0$. By the choice of n_0 and by the positivity of k_l it follows that $g(\theta) \to 0$ uniformly as $|\theta| \to \infty$ for $0 \leqslant \arg \theta \leqslant \pi$. Hence by Jordan's lemma we have

$$\sum_{m=n_0}^{n} \dim M_m = \operatorname{Res}_\varepsilon g(\theta)$$
$$= \frac{n^{t-s} p_M(k)}{(t-s)!\,(\prod k_l)} + O(n^{t-s-1}),$$

where s, $p_M(k)$ are determined by the Taylor expansion

$$f_M(e^{-i\theta}) = p_M(k)(i\theta)^s + O(\theta^{s+1}) \tag{$*$}$$

around $\theta = 0$. This gives (i).

Obviously $p_M(k) = [f_M(x)/(1-x)^s]_{x=1}$. If we write $1 - x = y$, then from 2.1($**$) we obtain

$$\frac{f_M(x)}{(1-x)^s} = \frac{1}{y^s} \sum_{l \in F_M} \left\{ c_l'(M) \left(\sum_{r=0}^{k \cdot l} (-1)^r \binom{k \cdot l}{r} y^r \right) \right\}.$$

Expand the binomial coefficient in powers of $k \cdot l$ and sum over l. The first non-vanishing sum of the form

$$\sum_{l \in F_M} c_l'(M)(k \cdot l)^m$$

gives a non-zero contribution to the coefficient of y^m and possibly to terms of higher degree in y. Hence (ii).

From ($*$) we obtain $s = \deg p_M(k)$. Yet $d_M = t - s$ and so $\deg p_M = t - d_M$. Hence (iii).

2.4. Take $V = \mathfrak{n}^-$ in 2.1 and identify $(\mathfrak{n}^-)^*$ with $\mathfrak{n}$ through the Killing form on $\mathfrak{g}$. Let $\mathscr{V}$ be an $\mathbf{H}$ stable subvariety of $\mathfrak{n}$. With $-\alpha \in R^+$,

$-\nu \in P(R)^{++}$, take δ to be the map $X_\alpha \to (\alpha, \nu) X_\alpha$ and $M = S(\mathfrak{n}^-)/I(\mathscr{V})$ in 2.2 (notation I, 1.3, 1.4, 1.6). Since $I(\mathscr{V})$ is $\mathfrak{h}$ stable hence δ stable there exists a unique δ derivation D of M satisfying $D(1 + I(\mathscr{V})) = 0$. Writing $r_{\mathscr{V}} = r_M$, $p_{\mathscr{V}} = p_M$ we obtain

COROLLARY. (i) $r_{\mathscr{V}} = p_{\mathscr{V}}/(\prod_{\alpha \in R^+} \alpha)$ *where* $p_{\mathscr{V}} \in S(\mathfrak{h})$.

(ii) $p_{\mathscr{V}}$ *is homogeneous of degree* (card $R^+ - \dim \mathscr{V}$).

2.5. For each $\alpha \in R^+$ let $\mathbf{N}_\alpha$ (resp. $\mathbf{N}_\alpha^-$) denote the subgroup of $\mathbf{N}$ (resp. $\mathbf{N}^-$) with Lie algebra $\mathbb{C}X_\alpha$ (resp. $\mathbb{C}X_{-\alpha}$). Let $\mathbf{H}_\alpha$ denote the subgroup of $\mathbf{H}$ with Lie algebra $\mathbb{C}H_\alpha$ ($H_\alpha \in \mathfrak{h}$ being the coroot to α) and $\mathbf{S}_\alpha$ the subgroup of $\mathbf{G}$ with Lie algebra $\mathfrak{s}_\alpha := \mathbb{C}X_\alpha \oplus \mathbb{C}H_\alpha \oplus \mathbb{C}X_{-\alpha}$. Let $s_\alpha \in W$ be the reflection defined by α.

LEMMA. *If* $\mathscr{V}$ *is* $\mathbf{S}_\alpha$ *stable, then* $s_\alpha p_{\mathscr{V}} = -p_{\mathscr{V}}$.

Take $-\nu \in P(R)^{++}$. The Poincaré series $R_M(x)$ for $M = S(\mathfrak{n}^-)/I(\mathscr{V})$ can be conveniently written

$$R_{\mathscr{V}}(x, \nu) = \sum_{\mu \in \mathbb{N}R^-} \dim(S(\mathfrak{n}^-)/I(\mathscr{V}))_\mu \, x^{(\mu, \nu)}. \qquad (*)$$

The hypothesis on $\mathscr{V}$ implies that $S(\mathfrak{n}^-)/I(\mathscr{V})$ is a direct sum of finite-dimensional $\mathfrak{s}_\alpha$ modules and so $\dim(S(\mathfrak{n}^-)/I(\mathscr{V}))_\mu = \dim(S(\mathfrak{n}^-)/I(\mathscr{V}))_{s_\alpha \mu}$ for all $\mu \in \mathbb{N}R^-$. In particular the non-vanishing of this weight space implies that $s_\alpha \mu \in \mathbb{N}R^-$. It follows that $R_{\mathscr{V}}(x, \nu)$ is also defined for all $-\nu \in s_\alpha(P(R)^{++})$ and that $R_{\mathscr{V}}(x, \nu) = R_{\mathscr{V}}(x, s_\alpha \nu)$. Hence $s_\alpha r_{\mathscr{V}} = r_{\mathscr{V}}$ and so $s_\alpha p_{\mathscr{V}} = -p_{\mathscr{V}}$.

2.6. Set $\mathbf{B}_\alpha = \mathbf{N}_\alpha \mathbf{H}_\alpha$, $\mathbf{B}_\alpha^- = \mathbf{N}_\alpha^- \mathbf{H}_\alpha$. Consider the $\mathbf{H}$ stable subvariety $\mathscr{V}$ of $\mathfrak{n}$ as a subvariety of $\mathfrak{g}$.

PROPOSITION. *Assume that* $\mathscr{V}$ *is irreducible and* $\mathbf{B}_\alpha^-$ *stable. If* $\mathbf{S}_\alpha \mathscr{V} \supsetneq \mathscr{V}$ *then there exists a positive integer* z *such that*

$$z p_{\mathbf{S}_\alpha \mathscr{V}} = (1/\alpha)(s_\alpha + 1) p_{\mathscr{V}}.$$

Set $I = I(\mathscr{V})$ and $J = I(\mathbf{S}_\alpha \mathscr{V})$. Since $(S(\mathfrak{n}^-)/J)^{X_{-\alpha}}$ is $\mathfrak{h}$ stable and finitely generated [13] as an algebra, it admits a Poincaré series $\tilde{R}_{\mathscr{V}}(x, \nu)$ satisfying the conclusions analogous to 2.4. We denote by $\tilde{r}_{\mathscr{V}}$,

$$\tilde{p}_{\mathscr{V}} := \tilde{r}_{\mathscr{V}} \left(\prod_{\alpha \in R^+} \alpha \right)$$

the functions which obtain from $\tilde{R}_{\mathscr{V}}(x, v)$. We shall compare $\tilde{p}_{\mathscr{V}}$ with $p_{\mathbf{S}_\alpha \mathscr{V}}$ and $p_{\mathscr{V}}$.

Since $\mathscr{V}$ is irreducible and $\mathbf{B}_\alpha^-$ stable, it follows that I is an $\mathfrak{h}$ stable prime ideal of $S(\mathfrak{n}^-)$ which is also $\operatorname{ad} X_\alpha$ stable. Again J is the largest ideal of $S(\mathfrak{n}^-)$ contained in I being $\mathfrak{s}_\alpha$ stable. It follows that J is prime and if $a \in I$ satisfies $(\operatorname{ad} X_{-\alpha})a = 0$, then $a \in J$. The latter implies that the restriction of the natural projection $\pi: S(\mathfrak{n}^-)/J \to S(\mathfrak{n}^-)/I$ to the subalgebra $(S(\mathfrak{n}^-)/J)^{X_{-\alpha}}$ is injective. We identify $(S(\mathfrak{n}^-)/J)^{X_{-\alpha}}$ as a prime subring of $S(\mathfrak{n}^-)/I$.

Let z denote the dimension of $\operatorname{Fract}(S(\mathfrak{n}^-)/I)$ over $\operatorname{Fract}(S(\mathfrak{n}^-)/J)^{X_{-\alpha}}$. We show that z is finite. Since $\operatorname{Fract}(S(\mathfrak{n}^-)/I)$ is already finitely generated over $\mathbb{C}$ it is enough to show that it is an algebraic extension of $\operatorname{Fract}(S(\mathfrak{n}^-)/J)^{X_{-\alpha}}$. Since $\operatorname{ad} X_{-\alpha}$ is a locally nilpotent derivation of $S(\mathfrak{n}^-)/J$ we obtain (notation I, 2.3) that $d((S(\mathfrak{n}^-)/J)^{X_{-\alpha}}) \geqslant d(S(\mathfrak{n}^-)/J) - 1 = d(S(\mathfrak{n}^-)/I)$, where the last equality follows from the hypothesis $\mathbf{S}_\alpha \mathscr{V} \supsetneq \mathscr{V}$ which since $\mathscr{V}$ is irreducible implies that $\dim \mathbf{S}_\alpha \mathscr{V} = \dim \mathscr{V} + 1$.

Since $\mathfrak{h}$ acts locally semisimply on $S(\mathfrak{n}^-)/I$ we can choose z weight vectors in $S(\mathfrak{n}^-)/I$ which are linearly independent over $\operatorname{Fract}(S(\mathfrak{n}^-)/J)^{X_{-\alpha}}$. By choice of z they generate a freee $(S(\mathfrak{n}^-)/J)^{X_{-\alpha}}$ submodule of rank z of $S(\mathfrak{n}^-)/I$ whose quotient has strictly lower Gelfand–Kirillov dimension. It easily follows that $z\tilde{p}_{\mathscr{V}} = p_{\mathscr{V}}$.

Let μ be a weight of $(S(\mathfrak{n}^-)/J)^{X_{-\alpha}}$. Then μ is a lowest weight of an $\mathfrak{s}_\alpha$ submodule of $S(\mathfrak{n}^-)/J$ whose weights have multiplicity one and form the α-string $\mu, \mu+\alpha, \mu+2\alpha, \dots, s_\alpha\mu$. Hence $(s_\alpha\mu, v) \geqslant 0$, for all $-v \in P(R)^{++}$ and so $\tilde{R}_{\mathscr{V}}(x, s_\alpha v)$ is defined. Furthermore, for all $-v \in P(R)^{++}$ one has

$$\begin{aligned} R_{\mathbf{S}_\alpha \mathscr{V}}(x, v) &= \sum_{\mu \in \mathbb{N}R^-} \dim(S(\mathfrak{n}^-)/J)_\mu \, x^{(\mu, v)} \\ &= \sum_{\mu \in \mathbb{N}R^-} (\dim(S(\mathfrak{n}^-)/J)_\mu^{X_{-\alpha}})(x^{(\mu,v)} + x^{(\mu+\alpha,v)} + \cdots + x^{(s_\alpha\mu,v)}) \\ &= \sum_{\mu \in \mathbb{N}R^-} \dim(S(\mathfrak{n}^-)/J)_\mu^{X_{-\alpha}} \left[\left(\frac{1 - x^{(s_\alpha\mu - \mu, v)}}{1 - x^{(\alpha,v)}} \right) x^{(\mu,v)} + x^{(s_\alpha\mu,v)} \right] \\ &= \frac{1}{(1 - x^{(\alpha,v)})} (\tilde{R}_{\mathscr{V}}(x, v) - \tilde{R}_{\mathscr{V}}(x, s_\alpha v)) + \tilde{R}_{\mathscr{V}}(x, s_\alpha v). \end{aligned}$$

The second term gives a contribution of lower degree in n, namely, of degree $\dim \mathscr{V} < \dim \mathbf{S}_\alpha \mathscr{V}$ and can therefore be ignored. The first term gives

$$r_{\mathbf{S}_\alpha \mathscr{V}}(v) = (1/\alpha)(\tilde{r}_{\mathscr{V}}(v) - \tilde{r}_{\mathscr{V}}(s_\alpha v)) = (1/\alpha)(1 - s_\alpha)\, \tilde{r}_{\mathscr{V}}(v).$$

Hence $p_{\mathbf{S}_\alpha \mathscr{V}} = (1/\alpha)(s_\alpha + 1)\, \tilde{p}_{\mathscr{V}}$.

2.7. We may of course reverse the roles of $\mathbf{B}_\alpha$, $\mathbf{B}_\alpha^-$ in 2.6. That is, we have the

PROPOSITION. *Let $\mathscr{V}$ be a* **B** *stable irreducible subvariety of* $\mathfrak{g}$. *If* $\mathfrak{n} \supset \mathbf{S}_\alpha \mathscr{V} \supsetneqq \mathscr{V}$ *then there exists a positive integer z such that* $z p_{\mathbf{S}_\alpha \mathscr{V}} = (1/\alpha)(s_\alpha + 1) p_{\mathscr{V}}$.

2.8. Fix $\alpha \in R^+$, $p \in S(\mathfrak{h})$. It is well known that $(1 - s_\alpha)p$ is divisible by α and so $p \mapsto (1/\alpha)(1 - s_\alpha)p$ is an element of End $S(\mathfrak{h})$. These maps occur in the study of the flag variety [7] and in particular have been used to describe the action of W on its cohomology space.

2.9. Take $\alpha \in R^+$ and let $\mathfrak{m}_\alpha$ denote the subvariety of $\mathfrak{n}$ defined by the equation $X_\alpha = 0$. If $\alpha \in B$ then $\mathfrak{m}_\alpha$ is also a subalgebra of $\mathfrak{n}$ and coincides with $\mathfrak{n} \cap s_\alpha(\mathfrak{n})$. Set $\mathfrak{m}_\alpha^- = {}^t\mathfrak{m}_\alpha$ (notation I, 2.1).

LEMMA. *If* $\mathscr{V} \not\subset \mathfrak{m}_\alpha$ *then*

$$p_{\mathscr{V} \cap \mathfrak{m}_\alpha} = -\alpha p_{\mathscr{V}}.$$

Set $I = I(\mathscr{V})$, $J = I(\mathscr{V} \cap \mathfrak{m}_\alpha)$. One has $J = I + S(\mathfrak{n}^-) X_{-\alpha}$. The hypothesis $\mathscr{V} \not\subset \mathfrak{m}_\alpha$ implies $X_{-\alpha} \notin I$. Since I is a prime ideal, $X_{-\alpha}$ is a non-zero divisor in $S(\mathfrak{n}^-)/I$. For each $\mu \in \mathbb{N}R^-$ choose a subspace V_μ of $(S(\mathfrak{n}^-)/I)_\mu$ whose image under the natural projection $S(\mathfrak{n}^-)/I \to S(\mathfrak{n}^-)/J$ is just $(S(\mathfrak{n}^-)/J)_\mu$. Set $V = \oplus V_\mu$ which identifies with $S(\mathfrak{m}_\alpha^-)/(I \cap S(\mathfrak{m}_\alpha^-))$.

Since $X_{-\alpha}$ is a non-zero divisor in $S(\mathfrak{n}^-)/I$ and generates $S(\mathfrak{n}^-)$ over $S(\mathfrak{m}_\alpha^-)$ it follows that the sum

$$\sum_{l \in \mathbb{N}} V X_{-\alpha}^l$$

is direct and equals $S(\mathfrak{n}^-)/I$. Hence

$$\begin{aligned} R_{\mathscr{V}}(x, v) &= \sum_{l \in \mathbb{N}} \sum_{\mu \in \mathfrak{h}^*} (\dim V_\mu)\, x^{(\mu - l\alpha, v)} \\ &= \frac{1}{1 - x^{-(\alpha, v)}} R_{\mathscr{V} \cap \mathfrak{m}_\alpha}(x, v), \end{aligned}$$

and so $p_{\mathscr{V} \cap \mathfrak{m}_\alpha} = -\alpha p_{\mathscr{V}}$ as required.

2.10. Let B' be a subset of B. Let $W_{B'}$ denote the subgroup of W generated by the $s_\alpha : \alpha \in B'$ and $w_{B'}$ the unique longest element of $W_{B'}$. Let $\mathfrak{p}_{B'} \supset \mathfrak{b}$ (notation I, 1.3) denote the unique parabolic subalgebra of $\mathfrak{g}$ with nilradical $\mathfrak{m}'_B := \mathfrak{n} \cap w_{B'}(\mathfrak{n})$. Let $\mathfrak{r}_{B'}$ denote the reductive part of $\mathfrak{p}_{B'}$ and set $\mathfrak{n}_{B'} = \mathfrak{r}_{B'} \cap \mathfrak{n}$. Let $\mathbf{G}_{B'}$ denote the subgroup of $\mathbf{G}$ with Lie algebra $\mathfrak{r}_{B'}$. Let $\mathscr{V}$ be a subvariety of $\mathfrak{n}_{B'}$. Following Lusztig and Spaltenstein [29] it is natural to define $\mathrm{Ind}(\mathscr{V}, \mathfrak{p}_{B'} \uparrow \mathfrak{g}_{B'}) := \mathscr{V} + \mathfrak{m}_{B'}$. If $\mathscr{V}$ is $\mathfrak{h}$ stable, then so is Ind $\mathscr{V}$ and $p_{\mathscr{V}}$ defined with respect to $\mathfrak{r}_{B'}$ coincides with $p_{\mathrm{Ind}\, \mathscr{V}}$. Furthermore if $\mathscr{V}$ is an irreducible component of $\mathscr{O} \cap \mathfrak{n}_{B'}$ for some nilpotent $\mathbf{G}_{B'}$ orbit $\mathscr{O}$, then $\operatorname{Ind} \mathscr{V}$ is an irreducible component of $(\operatorname{Ind} \mathscr{O}) \cap \mathfrak{n}$, where $\operatorname{Ind} \mathscr{O}$ is defined [29] to be the unique dense orbit in $G(\mathscr{O} + \mathfrak{m}_{B'})$. This is because $\mathscr{V} + \mathfrak{m}_{B'}$ is irreducible and $\dim(\mathscr{V} + \mathfrak{m}_{B'}) = \dim \mathscr{V} + \dim \mathfrak{m}_{B'} = \frac{1}{2}(\dim \mathscr{O} + 2 \dim \mathfrak{m}_{B'}) = \dim(\operatorname{Ind} \mathscr{O}) \cap \mathfrak{n}$. (See 4.7.)

3. Action of the Weyl Group

3.1. Let $\mathscr{O}$ be a $\mathbf{G}$ orbit of nilpotent elements of $\mathfrak{g}$ (briefly, a nilpotent orbit). Let $\{\mathscr{V}_i\}$ denote the set of irreducible components of $\mathscr{O} \cap \mathfrak{n}$. By [31] the $\mathscr{V}_i$ have all the same dimension and by [34, p. 134; 35, Sect. 4] we even have $\dim \mathscr{V}_i = \frac{1}{2} \dim \mathscr{O}$, $\forall i$, which we shall call the Spaltenstein–Steinberg equality. Since $\mathfrak{n}$ is $\mathbf{B}$ stable and $\mathbf{B}$ is irreducible, each $\mathscr{V}_i$ is $\mathbf{B}$ stable. Consequently $p_{\mathscr{V}_i}$ is defined for each $\mathscr{V}_i$ (notation 2.4). We set $P_{\mathscr{O}} := \sum \mathbb{C} p_{\mathscr{V}_i}$ which may be considered as a subspace of $S(\mathfrak{h})$.

Theorem. *$P_{\mathscr{O}}$ is a W submodule of $S(\mathfrak{h})$.*

Set $p_i = p_{\mathscr{V}_i}$. It is enough to show that $s_\alpha p_i \in P_{\mathscr{O}}$, $\forall \alpha \in B$, $\forall i$. Fix $\alpha \in B$ and set $\mathbf{P}_\alpha = \mathbf{B} \mathbf{N}_\alpha$ which is a parabolic subgroup of $\mathbf{G}$ with Lie algebra $\mathfrak{p}_\alpha := \mathfrak{b} \oplus \mathbb{C} X_{-\alpha}$ whose nilradical is $\mathfrak{m}_\alpha$ (notation 2.9). To compute $s_\alpha p_i$ we distinguish two cases.

Case 1. $\mathscr{V}_i \subset \mathfrak{m}_\alpha$. If this holds $\mathscr{V}_i$ is an irreducible component of $\mathscr{O} \cap \mathfrak{m}_\alpha$ and hence $\mathbf{P}_\alpha$ stable (since $\mathbf{P}_\alpha$ is irreducible). Hence by 2.5 we obtain $s_\alpha p_i = -p_i \in P_{\mathscr{O}}$ as required.

Case 2. $\mathscr{V}_i \not\subset \mathfrak{m}_\alpha$. Set $I_i = I(\mathscr{V}_i \cap \mathfrak{m}_\alpha)$ and let $I_i = \bigcap J_{ij}$ be a primary decomposition for I. For each j set $K_{ij} = \sqrt{J_{ij}}$, which is a prime ideal and satisfies $K_{ij}^n \subset J_{ij}$ for n suffiently large. Set $A = S(\mathfrak{n}^-)/K_{ij}$. Then $\operatorname{Fract} A \otimes_A (S(\mathfrak{n}^-)/J_{ij})$ is finite dimensional over $\operatorname{Fract} A$ of dimension say y_{ij}. Just as in 2.6 it follows that $p_{S(\mathfrak{n}^-)/J_{ij}} = y_{ij} p_{S(\mathfrak{n}^-)/K_{ij}}$. On the other hand, K_{ij} is the ideal of definition of an irreducible component of $\mathscr{V}_i \cap \mathfrak{m}_\alpha$. By Krull's theorem these components have all the same dimension, so from the above and 2.9 we obtain

$$\sum_j y_{ij} p_{\mathscr{V}(K_{ij})} = p_{\mathscr{V}_i \cap \mathfrak{m}_\alpha} = -\alpha p_{\mathscr{V}_i}.$$

To proceed further it is convenient to further distinguish two cases:

Case 2a. *$\mathscr{V}(K_{ij})$ is $\mathbf{P}_\alpha$ stable.* When this holds we obtain $(s_\alpha + 1) p_{\mathscr{V}(K_{ij})} = 0$ by 2.5.

Case 2b. $\mathscr{V}_{ij} := \mathbf{P}_\alpha \mathscr{V}(K_{ij}) \supsetneq \mathscr{V}(K_{ij})$. Since $\mathscr{V}(K_{ij})$ is $\mathbf{B}$ stable it follows that $\dim \mathscr{V}_{ij} = 1 + \dim \mathscr{V}(K_{ij}) = \dim \mathscr{V}_i$. Since $\mathfrak{m}_\alpha$ is $\mathbf{P}_\alpha$ stable we have $\mathscr{V}_{ij} \subset \mathbf{P}_\alpha(\mathscr{V}_i \cap \mathfrak{m}_\alpha) = \mathbf{P}_\alpha \mathscr{V}_i \cap \mathfrak{m}_\alpha \subset \mathscr{O} \cap \mathfrak{n}$. Since the $\mathscr{V}_i$ have all the same dimension we conclude that $\mathscr{V}_{ij}$ is dense in a component of $\mathscr{O} \cap \mathfrak{n}$. In this case $p_{\mathscr{V}_{ij}} \in P_\mathscr{O}$. Furthermore by 2.7 there exists a positive integer z_{ij} such that $(1/\alpha)(s_\alpha + 1)\, p_{\mathscr{V}(K_{ij})} = z_{ij}\, p_{\mathscr{V}_{ij}}$.

Combining the above results we obtain

$$
\begin{aligned}
(s_\alpha - 1)\, p_{\mathscr{V}_i} &= (1/\alpha)(s_\alpha + 1)(-\alpha p_{\mathscr{V}_i}) \\
&= (1/\alpha)(s_\alpha + 1) \sum_j y_{ij}\, p_{\mathscr{V}(K_{ij})} \\
&= \sum_j y_{ij} z_{ij}\, p_{\mathscr{V}_{ij}} \in P_\mathscr{O},
\end{aligned}
$$

as required.

3.2. In 3.1 we observe that the $\{\mathscr{V}_{ij}\}_j$ are just the components of $\mathscr{O} \cap \mathfrak{n}$ which are contained in $\mathfrak{m}_\alpha$ and whose intersection with $\mathscr{V}_i$ has codimension one. Furthermore for each such j the product $y_{ij} z_{ij}$ is a positive integer which can be defined without reference to the polynomials $p_{\mathscr{V}_i}$. Yet it is not obvious that these coefficients define a representation of W on the free abelian group $\bigoplus \mathbb{Z} \mathscr{V}_i$ since we do not know the $p_{\mathscr{V}_i}$ to be linearly independent. It is not even obvious that the $p_{\mathscr{V}_i}$ are cyclic vectors for $P_\mathscr{O}$. However from Spaltenstein's analysis [31] it easily follows that a subset of the $p_{\mathscr{V}_i}$ cannot span a strict W submodule of $P_\mathscr{O}$.

3.3. We should like to show that $P_\mathscr{O}$ is irreducible of type $\mathrm{Sp}(\mathscr{O}) \otimes \mathrm{sg}$ (notation 1.7). Here we remark that the appearance of sg should be thought of arising from omission of the product of the positive roots from denominators. We shall say that an orbit $\mathscr{O}$ is of Springer type if this does indeed hold. Since $\dim(\mathrm{Sp}(\mathscr{O}) \otimes \mathrm{sg}) = \dim \mathrm{Sp}(\mathscr{O})$ and the latter is just the number of irreducible components of $\mathscr{O} \cap \mathfrak{n}$ for this to hold it is enough that $P_\mathscr{O}$ admits a subrepresentation of type $\mathrm{Sp}(\mathscr{O}) \otimes \mathrm{sg}$ and furthermore this will imply the $p_{\mathscr{V}_i}$ to be linearly independent. Actually we should like to show (in the language of (II, 1.1)) that $\mathrm{Sp}(\mathscr{O}) \otimes \mathrm{sg}$ is a univalent W module and so determines a unique submodule $\mathrm{Sp}_\mathscr{O}$ of $S(\mathfrak{h})$. (This condition on $\mathscr{O}$ is termed property (B) by Lusztig and Spaltenstein [9].) Given that this holds we should in fact like to show that $\mathrm{Sp}_\mathscr{O} = P_\mathscr{O}$.

Now in the notation of 2.10 assume that $\mathscr{O}$ is a nilpotent $\mathbf{G}_{B'}$ orbit in $\mathfrak{r}_{B'}$.

LEMMA. *Suppose* $\mathscr{O}$ *has property* (B). *Then*

(i) $\mathrm{Ind}\, \mathscr{O}$ *has property* (B) *and* $\mathrm{Sp}_{\mathrm{Ind}\, \mathscr{O}} = \mathbb{C} W \mathrm{Sp}_\mathscr{O}$.

(ii) *If* $P_{\mathscr{O}} = Sp_{\mathscr{O}}$, *then* $P_{\operatorname{Ind}\mathscr{O}} = \operatorname{Sp}_{\operatorname{Ind}\mathscr{O}}$.

(i) is just [29, 3.5] and then (ii) follows from the remarks above and in 2.10.

4. Associated Varieties

4.1. Let $\mathfrak{a}$ be a finite-dimensional Lie algebra and let $\mathscr{F}(\mathfrak{a})$ denote the category of finitely generated $U(\mathfrak{a})$ modules. Take $M \in \operatorname{Ob}\mathscr{F}(\mathfrak{a})$ and pick a finite-dimensional subspace M^0 of M such that $M = U(\mathfrak{a})M^0$. We define a filtration in M compatible with the canonical filtration $\{U^j(\mathfrak{a})\}_{j\in\mathbb{N}}$ of $U(\mathfrak{a})$ by setting $M^j = U^j(\mathfrak{a})M^0$, $\forall j \in \mathbb{N}$. Then $\operatorname{gr} M$ is a finitely generated module over the graded algebra $\operatorname{gr}(U(\mathfrak{a}))$ and the latter we identify with the symmetric algebra $S(\mathfrak{a})$. After Bernstein [4] the radical $\sqrt{\operatorname{gr}\operatorname{Ann} M}$ is independent of the choice of the generating subspace M^0 and we set $\mathscr{V}(M) = \mathscr{V}(\operatorname{gr}\operatorname{Ann} M)$. Given $a \in U(\mathfrak{a})$ we let $\deg a$ be the smallest integer $m \geqslant 0$ such that $a \in U^m(\mathfrak{a})$.

Lemma. *Choose* $a \in U(\mathfrak{a})$ *and set* $m = \deg a$.

(i) $a \in \operatorname{Ann} M^0 \Rightarrow aM^j \subset M^{j+m-1}$, $\forall j \in \mathbb{N}$ *and so* $\operatorname{gr}\operatorname{Ann} M^0 \subset \operatorname{Ann}\operatorname{gr} M$.

(ii) $\operatorname{Ann}\operatorname{gr} M \subset \sqrt{\operatorname{gr}\operatorname{Ann} M^0}$.

(i) is clear and (ii) follows from [4, Prop. 1.4].

4.2. After Bernstein [4] or using the fact that $\oplus U^j(\mathfrak{a})$ is noetherian (cf. [24, 7.1.6]) we obtain

Lemma. *Let* $0 \to M_1 \to M_2 \to M_3 \to 0$ *be an exact sequence of objects in* $\mathscr{F}(\mathfrak{a})$. *One has*

$$\mathscr{V}(M_2) = \mathscr{V}(M_1) \cup \mathscr{V}(M_3).$$

4.3. Given $M \in \operatorname{Ob}\mathscr{F}(\mathfrak{a})$ choose M^0 as in 4.1. Then for any finite-dimensional $U(\mathfrak{g})$ module E the tensor product $M \otimes E$ (with the diagonal action of $\mathfrak{g}$) is generated over $M^0 \otimes E$ and so $M \otimes E \in \operatorname{Ob}\mathscr{F}(\mathfrak{a})$.

Lemma. $\mathscr{V}(M) = \mathscr{V}(M \otimes E)$.

Filter $M \otimes E$ through $(M \otimes E)^j := U^j(\mathfrak{a})(M^0 \otimes E) = M^j \otimes E$. Then $\operatorname{gr}(M \otimes E)$ is isomorphic to $(\operatorname{gr} M) \otimes E$ as an $S(\mathfrak{g})$ module, where E is given the structure of a trivial $S(\mathfrak{g})$ module. Hence $\operatorname{Ann}\operatorname{gr}(M \otimes E) = \operatorname{Ann}\operatorname{gr} M$ and so the assertion of the lemma follows from 4.1.

4.4. Given $M \in \mathrm{Ob}\,\mathscr{F}(\mathfrak{a})$, let $d(M)$ denote its Gelfand–Kirillov dimension. One has $d(M) = \dim \mathscr{V}(M)$. Call M smooth if $d(M) = d(N)$ for every submodule $N \neq 0$ of M. Gabber informs me that he has established the following [11].

THEOREM. *If M is smooth, then $\mathscr{V}(M)$ is equidimensional.*

4.5. Let M be a $U(\mathfrak{a})$ module and consider $U(\mathfrak{a})/\mathrm{Ann}\,M$ as a left $U(\mathfrak{a})$ module. Obviously $U(\mathfrak{a})/\mathrm{Ann}\,M$ is generated by the image 1 of the identity of $U(\mathfrak{a})$. Furthermore $\mathrm{Ann}\,1 = \mathrm{Ann}\,M$. Thus if we define $\mathscr{V}\mathscr{A}(M) := \mathscr{V}(U(\mathfrak{a})/\mathrm{Ann}\,M)$ it follows trivially that $\mathscr{V}\mathscr{A}(M) = \mathscr{V}(\mathrm{gr}\,\mathrm{Ann}\,M)$.

4.6. Now assume that M is a simple $U(\mathfrak{a})$ module. Then the prime ring $U(\mathfrak{a})/\mathrm{Ann}\,M$ is smooth as a left $U(\mathfrak{a})$ module [25, 2.3]. Hence by 4.4.

COROLLARY. *Let M be a simple $U(\mathfrak{a})$ module. Then $\mathscr{V}\mathscr{A}(M)$ is equidimensional.*

4.7. Let $\mathscr{R}$ denote the Bernstein–Gelfand–Gelfand category of "regular" $U(\mathfrak{g})$ modules (notation I, 2.2). Each $M \in \mathrm{Ob}\,\mathscr{R}$ has finite length and admits a locally finite action of $\mathfrak{b}$. Thus $\mathscr{V}(M)$ is a closed **B** stable subvariety of $\mathfrak{n}$ which identifies with $\mathscr{V}(\mathrm{gr}\,\mathrm{Ann}_{U(\mathfrak{n}^-)}\,M^0)$ for any finite-dimensional subspace M^0 of M which generates M as a $U(\mathfrak{n}^-)$ module.

We call an irreducible subvariety of $\mathfrak{n}$, *orbital* if it is dense in the closure of an irreducible component of $\mathscr{O} \cap \mathfrak{n}$ for some nilpotent orbit $\mathscr{O}$. Every closed orbital subvariety of $\mathfrak{n}$ is **B** stable; but (except for direct sums of copies of $sl(2)$) the variety $\mathbb{C}X_\beta$: β a highest root, is **B** stable but not orbital. By the Spaltenstein–Steinberg equality an irreducible subvariety $\mathscr{V}$ of $\mathfrak{n}$ is orbital if and only if $\dim \mathscr{V} \geqslant \frac{1}{2} \dim \mathbf{G}\mathscr{V}$ and then equality holds.

PROPOSITION. *Take $M \in \mathrm{Ob}\,\mathscr{R}$. Then every irreducible component of $\mathscr{V}(M)$ is closed and orbital.*

By 4.2 we can assume M to be simple. Now $\mathbf{G}\mathscr{V}(M)$ is a finite union of nilpotent orbits contained in $\mathscr{V}\mathscr{A}(M)$. Since $\mathscr{V}(M)$ is equidimensional (4.4) it is enough to show that $\dim \mathscr{V}(M) \geqslant \frac{1}{2} \dim \mathscr{V}\mathscr{A}(M)$, or equivalently that $d(M) \geqslant \frac{1}{2} d(U(\mathfrak{g})/\mathrm{Ann}\,M)$. The latter holds for any $U(\mathfrak{g})$ module M finitely generated over $U(\mathfrak{n}^-)$ [15, 2.6].

Remark. In 7.4 we give an alternative proof of 4.7 not requiring 4.4.

5. Comparison with Goldie Rank Polynomials

5.1. Take $w \in W_\lambda$, $\mu \in \Lambda^+$ regular, $-\nu \in P(R)^{++}$ (notation I, 1.4, 1.6; III, 1.1) and set

$$R_w(x, \mu, \nu) = \sum_{\xi \in \mathbb{N}R^-} \dim L(w\mu)_\xi \, x^{(\xi, \nu)},$$

(notation I, 1.3; II, 3.2).

Define the Jantzen matrix (II, 1.3) through

$$\operatorname{ch} L(w\mu) = \sum_{w' \in W_\lambda} a(w, w') \operatorname{ch} M(w'\mu).$$

From [8, 7.5.6, 7.5.7] this gives

$$R_w(x, \mu, \nu) = \sum_{w' \in W_\lambda} \frac{a(w, w')\, x^{(w'\mu - \rho, \nu)}}{\prod_{\alpha \in R^+} (1 - x^{-(\alpha, \nu)})}.$$

Just as in Section 2 we can associate a function $p_w(\mu, \nu)$ polynomial in ν to the above Poincaré series. From 2.3(ii) we obtain (up to a scalar) that

$$p_w(\mu, \nu) = \sum_{w' \in W_\lambda} a(w, w')(w'\mu, \nu)^{m_w},$$

where m_w is the smallest integer $\geqslant 0$ for which the above expression is nonzero. When we take $\nu = \delta$ we see that $p_w(\mu, \delta)$ is just the polynomial $\tilde{p}_w(\mu)$ defined in (II, 1.4) and shown there to determine the Goldie ranks of the primitive quotients $U(\mathfrak{g})/J(w\mu){:}\ \mu \in \Lambda^+$ (notation I, 1.3). Slightly generalizing the analysis of II, King [27] showed that $\tilde{p}_w(\mu)$ divides $p_w(\mu, \nu)$ for any $\nu \in -P(R)^{++}$. This implies that $p_w(\mu, \nu)$ factorizes as polynomials in μ, ν which by the symmetry of the given expansion yields

$$p_w(\mu, \nu) = \tilde{p}_w(\mu)\, \tilde{p}_{w^{-1}}(\nu). \qquad (*)$$

We remark that as a consequence of the work of various authors [3, 6, 9, 28] the $a(w, w')$ can be considered to be known and are in fact determined by a purely combinatorial recipe involving only the specification of (W_λ, B_λ) as a Coxeter group. However, this is not so useful in computing the $\tilde{p}_w$ and even the above factorization cannot yet be established purely combinatorially.

5.2. For each $\xi \in \mathfrak{h}^*$ let e_ξ denote the canonical generator of $M(\xi)$ and $\bar{e}_\xi$ its image in $L(\xi)$. Set $I(\xi) = \operatorname{Ann}_{U(\mathfrak{n}^-)} \bar{e}_\xi$. Now take $w \in W_\lambda$, $\mu \in \Lambda^+$ and set $J_\Lambda(w) = \sqrt{I(w\mu)}$ (or simply, $J(w)$). By [14, 3.4] $J_\Lambda(w)$ is independent of

$\mu \in \Lambda^+$ for which $p_w(\mu) \neq 0$ (see III, 1.1). We set $\mathscr{W}_\Lambda(w) = \mathscr{V}(J_\Lambda(w))$ (or simply, $\mathscr{W}(w)$). Of course $\mathscr{V}(L(w\mu)) = \mathscr{W}_\Lambda(w)$, $\forall \mu \in \Lambda^+ : p_w(\mu) \neq 0$. Let $\{\mathscr{V}_i\}$ denote the set of orbital varieties of dimension $d(L(w\mu)) = \operatorname{card} R^+ - m_w$.

LEMMA. *Fix $w \in W_\lambda$. There exist integers $l_w^i \geqslant 0$ (independent of μ) such that*

$$\sum l_w^i p_{\mathscr{V}_i} = \tilde{p}_{w^{-1}}.$$

Let $I(w\mu) = \bigcap I_i(\mu)$ be a primary decomposition of $I(w\mu)$ and set $J_i = \sqrt{I_i(\mu)}$. (One has $J(w) = \bigcap J_i$ and so the J_i are independent of μ.) Set $A_i = S(\mathfrak{n}^-)/J_i$, $\mathscr{V}_i = \mathscr{V}(J_i)$. Let $y_w^i(\mu)$ denote the dimension (which is finite) of $\operatorname{Fract} A_i \otimes_{A_i} (S(\mathfrak{n}^-)/I_i(\mu))$ over $\operatorname{Fract} A_i$. Then

$$\sum y_w^i(\mu) p_{\mathscr{V}_i}(\nu) = p_w(\mu, \nu) \qquad (*)$$

where the sum is over all components of maximal dimension. Substituting from 5.1(*) and evaluating at say $\mu = \rho$ gives the conclusion of the lemma.

5.3. This result can be used (10.1) to show that $\mathscr{W}(w)$ is not always irreducible. Yet by Gabber's result (4.4) it is always equidimensional. Let us show how 5.2 can be used to determine its components.

Let us recall that $\mathbb{C} W \tilde{p}_w = \mathbb{C} W \tilde{p}_{w^{-1}}$ (because every left cell contains an involution III, 4.1, 4.6) and the latter is a simple univalent W module (II, 5.4). Denote this module by P. By 5.2 there exists a component $\mathscr{V}_i$ of $\mathscr{W}(w)$ such that P is a submodule of $P_{\mathscr{O}}$ where $\mathscr{O}$ is the unique nilpotent orbit dense in $\mathbf{G}\mathscr{V}_i$. Now if $P_{\mathscr{O}}$ is simple (e.g., if $\mathscr{O}$ is of Springer type, 3.3) it follows that $P_{\mathscr{O}} = P$ and so $p_{\mathscr{W}_i(w)} \in P$. Moreover, $\mathscr{O}$ satisfies property (B) of 3.3, since P is univalent. Now for distinct orbits $\mathscr{O}$, $\mathscr{O}'$ of Springer type we must have $P_{\mathscr{O}} \neq P_{\mathscr{O}'}$ and so we obtain the

THEOREM. *Take $\mu \in \Lambda^+$, $w \in W_\lambda$ with $p_w(\mu) \neq 0$. Suppose that all the nilpotents orbits of dimension $2d(L(w\mu))$ in $\mathfrak{g}^*$ are of Springer type. Then $\mathbf{G}\mathscr{W}(w)$ admits a unique dense nilpotent orbit $\mathscr{O}$ and this satisfies the property* (B) *of Lusztig and Spaltenstein. Furthermore* $\operatorname{Sp}_{\mathscr{O}} = \mathbb{C} W \tilde{p}_w$.

5.4. Since non-induced orbits are rather few and very rarely have the same dimension it is plausible that one could combine the analysis of 3.3 and 5.3 to show (inductively) that all the nilpotent orbits are of Springer type. This would of course involve some case by case analysis and is not really what we have in mind. Nevertheless, admitting the equality $\overline{\mathbf{G}\mathscr{W}(w)} = \mathscr{V}\mathscr{A}(L(w\lambda))$ (cf. [6]) this would establish part (iii) of conjecture [18, 7.4].

Note that the hypothesis that all the orbits are of Springer type implies that $p_{\mathscr{V}_i}$ are linearly independent. Since the $\tilde{p}_{w^{-1}}$ are given by 5.1, a knowledge of the $p_{\mathscr{V}_i}$ allows one to determine $\mathscr{W}(w)$ via 5.2.

5.5. (*Notation* 5.2). Given that the $p_{\mathscr{V}_i}$ are linearly independent (e.g., if all nilpotent orbits are of Springer type) then in 5.2 we must have that $y^i_w(\mu) = l^i_w \tilde{p}_w(\mu)$, in particular the y^i_w are all proportional to the Goldie rank polynomial $\tilde{p}_w$. Actually we can prove this last assertion without knowledge of linear independence. We define for each $M \in \mathrm{Ob}\,\mathscr{R}$ the associated "scheme" $\mathscr{S}(M) \in \bigoplus \mathbb{N}\mathscr{V}_i$ of M as follows.

Choose a finite-dimensional generating subspace M^0 of M considered as a $U(\mathfrak{n}^-)$ module and set $I = \mathrm{gr}(\mathrm{Ann}_{U(\mathfrak{n}^-)} M^0)$. Take a primary decomposition $I = \bigcap I_i$ set $J_i = \sqrt{I_i}$, $\mathscr{V}_i = \mathscr{V}(J_i)$ and let y^i denote the multiplicity of I_i with respect to J_i as defined in 5.2. By 4.7 each $\mathscr{V}_i$ is an orbital variety and we set

$$\mathscr{S}(M) = \sum y^i \mathscr{V}_i, \qquad p_{\mathscr{S}(M)} = \sum y^i p_{\mathscr{V}_i}.$$

A refinement of Bernstein's theorem (see [24, 6.3.2]) shows that $\mathscr{S}(M)$ is independent of the choice of M^0. Furthermore, if E is a finite-dimensional module, taking $M^0 \otimes E$ as a generating subspace for $M \otimes E$ shows as in 4.3 that $\mathscr{S}(M \otimes E) = (\dim E)\,\mathscr{S}(M)$. Again in the situation of 4.2 we have $\mathscr{S}(M_2) = \mathscr{S}(M_1) + \mathscr{S}(M_3)$. This "additivity principle" implies exactly as in [36, Sect. 4] that in 5.2 the y^i_w are W harmonic polynomials on $\mathfrak{h}^*$. Comparison with the corresponding additivity principle for Goldie rank (I, 5.11) shows that the transformation matrices for y^i_w under the action of W_λ are exactly the same as those for the $\tilde{q}_w := z_w \tilde{p}_w$ (notation I, 5.11; II, 5.5, Remark 1). It follows that $\mathbb{C}Wy^i_w$ isomorphic as a W_λ module to a homomorphic image of (the $\mathbb{C}$ linear span of the elements of) the left cell of W_λ containing w (see III, 4.6, 4.11). Evaluating 5.2(∗) at say $v = \rho$ we obtain for each $w \in W$ integers $k_i > 0$ such that

$$\sum y^i_w(\mu)\, k_i = \tilde{p}_w(\mu).$$

Since $y^i_w(\mu) \geqslant 0$, $\forall \mu \in \Lambda^+$ and Λ^+ is Zariski dense it follows that $\deg y^i_w \leqslant \deg \tilde{p}_w$. Now from the analysis of (II, 5.4) one has that $\mathbb{C}W_\lambda \tilde{p}_w$ is a simple W_λ module and furthermore is the only simple factor of the left cell of W_λ containing w which can be realized by polynomials on $\mathfrak{h}^*$ of degree $\leqslant \deg \tilde{p}_w$. Thus $\mathbb{C}W_\lambda y^i_w = \mathbb{C}W_\lambda \tilde{p}_w$ for each i and more precisely since the transformation matrices of y^i_w, $\tilde{q}_w$ coincide, Schur's lemma applied to the irreducibility of $\mathbb{C}W_\lambda \tilde{p}_w$ gives rational numbers r_i such that $y^i_w = r_i \tilde{q}_w$ for all w in a given left cell of W_λ. Up to the overall factor $\tilde{q}_w(\mu)$ it follows that $\mathscr{S}(L(w\mu))$ is independent of the left cell containing w. Indeed $\mathbb{Q}p_{\mathscr{S}(L(w\mu))} = \mathbb{Q}\tilde{p}_{w^{-1}}$ and so taking account of (II, 5.5) we have established the

LEMMA. *For all $w, w' \in W_\lambda$ one has*

$$\mathbb{Q}\mathscr{S}(L(w\lambda)) = \mathbb{Q}\mathscr{S}(L(w'\lambda)) \Leftrightarrow J(w^{-1}\lambda) = J(w'^{-1}\lambda).$$

Remarks. We may identify $\mathbb{Q}\mathscr{S}(M)$ with $\mathscr{V}(M)$ if $\mathscr{V}(M)$ is irreducible. Although the above result can also be obtained if it is known that all orbits are of Springer type, here we have actually obtained a more precise result which gives $l_w^i = r_i z_w$.

6. COMPARISON WITH ENRIGHT COMPLETION

6.1. Let $M \in \text{Ob}\,\mathscr{R}$ and choose $\alpha \in B$. Call M *α-finite* if $X_{-\alpha}$ acts locally finitely on M and *α-free* if M admits no non-zero $U(\mathfrak{g})$ submodule which is α-finite. Since $\text{ad}\, X_{-\alpha}$ is a locally nilpotent derivation of $U(\mathfrak{g})$ the multiplicatively closed subset T_α of $U(\mathfrak{g})$ generated by $X_{-\alpha}$ is Ore in $U(\mathfrak{g})$. We set $U(\mathfrak{g})_{X_{-\alpha}} = T_\alpha^{-1} U(\mathfrak{g})$ and $M_{X_{-\alpha}} = U(\mathfrak{g})_{X_{-\alpha}} \otimes_{U(\mathfrak{g})} M$. Then M is α finite (resp. α-free) if and only if $M_{X_{-\alpha}} = 0$ (resp. the canonical map $M \to M_{X_{-\alpha}}$ is injective). This shows that M is α-free if and only if it admits no non-zero finite-dimensional $U(\mathfrak{s}_\alpha)$ submodule. Again

LEMMA *Take $M \in \text{Ob}\,\mathscr{R}$. Then the following four conditions are equivalent.*

(i) *M is α-finite.*

(ii) *M is a direct sum of finite-dimensional $\mathfrak{s}_\alpha$ modules.*

(iii) *$s_\alpha p_{\mathscr{V}_i} = -p_{\mathscr{V}_i}$, for each component $\mathscr{V}_i$ of $\mathscr{V}(M)$.*

(iv) *$\mathscr{V}(M) \subset \mathfrak{m}_\alpha$.*

(i) $\Rightarrow$ (ii) is clear. (ii) $\Rightarrow$ (iii) by 2.5. (iii) $\Rightarrow$ (iv) by 4.7 and Case 2 in 3.1. (iv) $\Rightarrow$ (i) is clear.

6.2. Take $M \in \text{Ob}\,\mathscr{R}$, $\alpha \in B$. We define the Enright completion $C_\alpha M$ of M to be the largest $U(\mathfrak{s}_\alpha)$ submodule of $M_{X_{-\alpha}}$ on which $X_{-\alpha}$ acts locally finitely. Since $\text{ad}\, X_\alpha$ acts locally finitely on $U(\mathfrak{g})$ this is just the largest $U(\mathfrak{g})$ submodule of M belonging to $\text{Ob}\,\mathscr{R}$. The functor $M \to C_\alpha M$ on $\text{Ob}\,\mathscr{R}$ is left exact, takes α-free modules to α-free modules and commutes with the functor $M \mapsto E \otimes M$ where E is a finite-dimensional $U(\mathfrak{g})$ module. (For all this see [23] noting [23, 2.12] in particular.)

Assume that M is α-free. From the canonical embedding $M \to M_{X_{-\alpha}}$ we may identify M with an essential submodule of $C_\alpha M$. We say that M is α-complete if $M = C_\alpha M$.

Choose $\mu \in \mathfrak{h}^*$, set $\hat{\mu} = W\mu$, and let $\mathscr{R}_{\hat{\mu}}$ denote the full subcategory of $\mathscr{R}$

of all modules annihilated by some power of the maximal ideal of $Z(\mathfrak{g})$ (notation I, 1.2) corresponding to $\hat{\mu} \in \mathfrak{h}^*/W \cong \operatorname{Max} Z(\mathfrak{g})$. Then C_α restricts to a functor on $\mathscr{R}_{\hat{\mu}}$ which is the identity unless $(\alpha^\vee, w\mu) \in \mathbb{N}^+$ for some $w \in W$.

6.3. Assume $\mathfrak{g} = sl(2)$. The restriction of C_α to $\mathscr{R}_{\hat{\mu}}$ is the identity unless $(\alpha^\vee, \mu) \in \mathbb{Z} - \{0\}$. Assume that $-(\alpha^\vee, \mu) \in \mathbb{N}^+$. An easy exercise (cf. [24, 4.3.5]) shows that $\mathscr{R}_{\hat{\mu}}$ admits just five indecomposable non-isomorphic objects. Of these only three are α-free, namely, $V(\mu) := M(\mu)$, $P(\mu) := M(s_\alpha \mu)$, and the non-trivial extension $T(\mu)$ of $P(\mu)$ by $V(\mu)$. Of these only $P(\mu)$, $T(\mu)$ are α-complete. The $\mathfrak{s}_\alpha$ module $V(\mu)$ is simple, $C_\alpha V(\mu) = P(\mu)$ and the quotient $E(\mu)$ is a simple $\mathfrak{s}_\alpha$ module of dimension $-(\mu, \alpha^\vee)$.

Lemma. *Assume* $-(\mu, \alpha^\vee) \in \mathbb{N}^+$.

(i) $\operatorname{Hom}(P(\mu), V(\mu)) = 0$.

(ii) *Assume* $\dim E \leqslant -(\mu, \alpha^\vee)$. *Any α-free quotient of* $P(\mu) \otimes E$ *is a direct summand and α-complete.*

(i) is clear. For (ii) observe that the hypothesis implies that $P(\mu) \otimes E$ is a direct sum of Verma modules. (Use the action of $Z(\mathfrak{g})$ and [8, 7.6.14]). Then any quotient which is not a direct summand would admit a finite-dimensional summand and hence not be α-free. Finally, since C_α is left exact any direct summand of the α-complete module $P(\mu) \otimes E$ is α-complete.

6.4. Take $M \in \operatorname{Ob} \mathscr{R}$, $\alpha \in B$, and assume that M is α-free. Choose a finite-dimensional subspace $M^0 \subset M$ (resp. $(C_\alpha M)^0 \subset C_\alpha M$) which generates M (resp. $C_\alpha M$) over $U(\mathfrak{n}^-)$. We can assume that $M^0 \subset (C_\alpha M)^0 \subset \mathbb{C}[X_{-\alpha}^{-1}] M^0$ and that M^0, $(C_\alpha M)^0$ are $\mathfrak{b}$ stable. Set $N = \mathbb{C}[X_{-\alpha}](C_\alpha M)^0$ which is a regular α-free $\mathfrak{s}_\alpha$ submodule of $C_\alpha M$. Since N is contained in the α-completion of $\mathbb{C}[X_{-\alpha}] M^0$ we have $\dim(N/\mathbb{C}[X_{-\alpha}] M^0) < \infty$ so we can assume that

$$N = (C_\alpha M)^0 + \mathbb{C}[X_{-\alpha}] M^0. \qquad (*)$$

Set $M^j = U^j(\mathfrak{n}^-) M^0$, $(C_\alpha M)^j = U^j(\mathfrak{n}^-)(C_\alpha M)^0$, $\forall j \in \mathbb{N}$.

Lemma. *For all $j \in \mathbb{N}$ one has*

$$C_\alpha M \cap \mathbb{C}[X_{-\alpha}^{-1}](C_\alpha M)^j \subset (C_\alpha M)^j + \mathbb{C}[X_{-\alpha}] M^j.$$

Set $K^j = U^j(\mathfrak{m}_\alpha^-)N$, $K = U(\mathfrak{m}_\alpha^-)N$. Then K coincides with $C_\alpha M$ and so is

α-complete and α-free. Its $\mathfrak{s}_\alpha$ submodule K^j is hence also α-free. We show that it is α-complete. We have embeddings

$$K^j \longrightarrow C_\alpha K^j \xrightarrow{\varphi} K.$$

Let π denote the projection of $U(\mathfrak{m}_\alpha^-)$ onto $U^j(\mathfrak{m}_\alpha^-)$ defined by taking an $\mathfrak{s}_\alpha$ module complement to $U^j(\mathfrak{m}_\alpha^-)$. It defines a surjection $\pi': K \to K^j$ of $\mathfrak{s}_\alpha$ modules whose restriction to K^j is the identity map. Then $\pi'\varphi: C_\alpha K^j \to K^j$ is surjective and is the identity when restricted to K^j. Thus if K^j were not α-complete it would admit a direct summand isomorphic to a simple $\mathfrak{s}_\alpha$ Verma module $V(\mu)$ (see 6.3) and we should have maps $V(\mu) \hookrightarrow C_\alpha V(\mu) \twoheadrightarrow V(\mu)$ whose composition is the identity. This contradicts 6.3(i). We obtain for all $j \in \mathbb{N}$ that

$$C_\alpha M \cap \mathbb{C}[X_{-\alpha}^{-1}](C_\alpha M)^j \subset C_\alpha M \cap \mathbb{C}[X_{-\alpha}^{-1}]\, K^j = C_\alpha K^j = K^j.$$

Yet by (*)

$$K^j = U^j(\mathfrak{m}_\alpha^-)N \subset U^j(\mathfrak{m}_\alpha^-)((C_\alpha M)^0 + \mathbb{C}[X_{-\alpha}]\, M^0) \subset (C_\alpha M)^j + \mathbb{C}[X_{-\alpha}]\, M^j,$$

as required.

6.5. Proposition. *Take $M \in \operatorname{Ob} \mathscr{R}$, $\alpha \in B$. Then*

$$\mathscr{V}(C_\alpha M) \subset \overline{\mathbf{P}_\alpha \mathscr{V}(M)}.$$

Set $N = \ker(M \to M_{X_{-\alpha}})$. Since localization is exact we have $C_\alpha(M/N) \xrightarrow{\sim} C_\alpha M$, so by 4.2 it is enough to assume that M is α-free. Again by 4.2 it is enough to show that $\mathscr{V}(C_\alpha M/M) \subset \overline{\mathbf{P}_\alpha \mathscr{V}(M)}$.

Define M^j, $(C_\alpha M)^j$ as in 6.4. Since M^0 (resp. $(C_\alpha M)^0$) is $\mathfrak{b}$ stable we have $M^j = U^j(\mathfrak{g})\, M^0$ (resp. $(C_\alpha M)^j = U^j(\mathfrak{g})(C_\alpha M)^0$). Set $I = \operatorname{Ann} M^0$ and let J be the largest $\mathfrak{p}_\alpha$ stable ideal of $S(\mathfrak{g})$ contained in $\operatorname{gr} I$. Then J is homogeneous and is a direct sum of simple finite-dimensional $\mathfrak{s}_\alpha$ modules. Let V be a finite-dimensional $\mathfrak{s}_\alpha$ submodule of J of homogeneous elements of degree m. Choose $v \in V$. Since $V \subset \operatorname{gr} I$, there exists $u \in I$ such that $\operatorname{gr} u = v$. Again for each $l \in \mathbb{N}$ we have $\operatorname{gr}_m((\operatorname{ad} X_{-\alpha})^l\, u) = (\operatorname{ad} X_{-\alpha})^l \operatorname{gr}_m(u) = (\operatorname{ad} X_{-\alpha})^l\, v \in V$ and so there exists $u_l \in I$ such that $u_l - (\operatorname{ad} X_{-\alpha})^l\, u \in U^{m-1}(\mathfrak{g})$. Then by 4.1(i) we obtain $((\operatorname{ad} X_{-\alpha})^l\, u)\, M^j \subset M^{j+m-1}$, $\forall j, l \in \mathbb{N}$. Then for all $j \in \mathbb{N}$,

$$\begin{aligned} u(C_\alpha M)^j &\subset (C_\alpha M \cap \mathbb{C}[X_{-\alpha}^{-1}]\, U^j(\mathfrak{n}^-)\, M^0) \\ &\subset C_\alpha M \cap \mathbb{C}[X_{-\alpha}^{-1}] \left(\sum_{l \in \mathbb{N}} (\operatorname{ad} X_{-\alpha})^l\, u \right) M^j \\ &\subset C_\alpha M \cap \mathbb{C}[X_{-\alpha}^{-1}]\, M^{j+m-1}, \qquad \text{by the above,} \\ &\subset (C_\alpha M)^{j+m-1} + \mathbb{C}[X_{-\alpha}]\, M^j, \qquad \text{by 6.4.} \end{aligned}$$

Setting $((C_\alpha M)/M)^j = ((C_\alpha M)^j + M)/M$, this means that $u(C_\alpha M/M)^j \subset (C_\alpha M/M)^{j+m-1}$ and so $v = \mathrm{gr}\, u \in \mathrm{Ann}_{S(\mathfrak{g})}(\mathrm{gr}(C_\alpha M/M)) \subset \sqrt{\mathrm{gr}\, Ann(C_\alpha M/M)^0}$, by 4.1(ii). Since v, V are arbitrary this gives $\mathscr{V}(C_\alpha M/M) \subset \mathscr{V}(J) = \overline{\mathbf{P}_\alpha \mathscr{V}(I)} = \overline{\mathbf{P}_\alpha \mathscr{V}(M)}$, as required.

Remark. $\mathbf{P}_\alpha \mathscr{V}(M)$ is closed since $\mathscr{V}(M)$ is closed and $\mathbf{B}$ stable, while $\mathbf{P}_\alpha/\mathbf{B}$ is complete [34, p. 68].

6.6. Take $w \in W_\lambda$ and let $C(w)$ (resp. $C'(w)$) denote the left (resp. right) cell of W_λ to which w belongs (III, 4.6). One has (III, 4.6) that $w' \in C(w) \Leftrightarrow J(w'\lambda) = J(w\lambda) \Leftrightarrow \mathbb{C}\tilde{p}_w = \mathbb{C}\tilde{p}_{w'}$ (II, 5.5). Again by (II, 5.2(i)) $w' \in C'(w) \Leftrightarrow w'^{-1} \in C(w^{-1})$.

LEMMA. (*notation* 5.2).

(i) *$\mathscr{W}(w)$ is independent of the right cell of W_λ to which w belongs.*

(ii) *Suppose $B_\lambda \subset B$. Then $\overline{\mathbf{G}\mathscr{W}(w)}$ is independent of the left cell of W_λ to which w belongs.*

The hypothesis of (i) is equivalent to $J(w^{-1}\lambda) = J(w'^{-1}\lambda)$. Then by [12, 3.8] there exists a finite-dimensional $U(\mathfrak{g})$ module E such that $L(w\lambda)$ is a submodule of $L(w'\lambda) \otimes E$. By 4.3 we obtain $\mathscr{W}(w) := \mathscr{V}(L(w\lambda)) \subset \mathscr{V}(L(w'\lambda) \otimes E) = \mathscr{V}(L(w'\lambda)) =: \mathscr{W}(w')$. Interchanging w, w' reverses the inclusion.

(ii) Take $\alpha \in B_\lambda$ and let U_α denote the functor on $\mathscr{R}$ defined in 4.13. (It is a subfunctor of the coherent continuation functor θ_α defined by suitable tensoring with finite-dimensional modules.) By [23, 3.2] one has that $L(w'\lambda)$ is a subquotient of $(C_\alpha L(w\lambda)/L(w\lambda)) \Leftrightarrow L(w'^{-1}\lambda)$ is a subquotient of $U_\alpha L(w^{-1}\lambda)$. Thus the completion functors C_α: $\alpha \in B_\lambda$ generate the left cells just as the coherent continuation functors θ_α: $\alpha \in B_\lambda$ generate the right cells. Now by 4.2, 6.5 we have $\mathscr{W}(w') \subset \overline{\mathbf{P}_\alpha \mathscr{W}(w)}$ if $L(w'\lambda)$ is a subquotient of $C_\alpha L(w\lambda)$ and so eventually $\mathscr{W}(w') \subset \overline{\mathbf{G}\mathscr{W}(w)}$ if $w' \in C(w)$. Hence (ii).

6.7. We remark that if all nilpotent orbits are of Springer type then 6.6(i) obtains from 5.2 and 6.6(ii) from 5.3 without need of the technical restriction $B_\lambda \subset B$. Also 6.6(i) obtains from 5.5 but the proof here is simpler. Thus the main interest of 6.5 is not so much computational; but to indicate the possible relation between the Enright functor and the Springer action of W on the cohomology of fixed point sets of the flag variety. This is made more precise as follows.

LEMMA. *Take $\mu \in h^*$ and $\alpha \in B$ such that $-(\alpha^\vee, \mu)$ is a positive integer. If all nilpotent orbits having dimension $2d(L(\mu))$ are of Springer type, then*

$$\mathscr{V}(C_\alpha(L(\mu)) = \mathscr{V}(L(\mu)) \cup \overline{(\mathbf{P}_\alpha \mathscr{V}(L(\mu)) \cap \mathfrak{m}_\alpha}).$$

Set $\mathscr{W} = \mathscr{V}(L(\mu))$, $\mathscr{W}' = \mathscr{V}(C_\alpha(L(\mu)))$, $\mathscr{W}'' = \overline{\mathbf{P}_\alpha \mathscr{W} \cap \mathfrak{m}_\alpha}$.

We show that $\mathscr{W}' = \mathscr{W} \cup \mathscr{W}''$ by comparing the characteristic polynomials p_i of irreducible components $\mathscr{V}_i$ of both sides. By 4.7 these components are all orbital and our hypothesis that all the nilpotent orbits of dimension $2d(L(\mu))$ are of Springer type implies that their characteristic polynomials are linearly independent.

If $p = \sum k_i p_i$ where $k_i \in \mathbb{N}$ we denote by $[p]$ the set $\{p_i : k_i \neq 0\}$. If $\mathscr{V}$ is a union of some of the $\mathscr{V}_i$ then $[p_{\mathscr{V}}]$ determines this union, so what we have to show is that $[p_{\mathscr{W}'}] = [p_{\mathscr{W}}] \cup [p_{\mathscr{W}''}]$.

Let $\mathscr{V}^{(1)}$ denote the union of the components of $\mathscr{W}$ contained in $\mathfrak{m}_\alpha$ and $\mathscr{V}^{(2)}$ the union of those components not contained in $\mathfrak{m}_\alpha$. By definition

$$[p_{\mathscr{W}}] = [p^{(1)}] \cup [p^{(2)}]: p^{(i)} = p_{\mathscr{V}^{(i)}},$$

whereas by the analysis of 3.1

$$[p_{\mathscr{W}''}] = [p^{(1)}] \cup [(s_\alpha - 1) p^{(2)}].$$

We can assume μ regular and write $\mu = w\lambda$. If we identify $M(w'\lambda)$ with w' in the Grothendieck group, then $L(w\lambda)$ identifies with

$$\mathbf{a}(w) := \sum_{w' \in W_\lambda} a(w, w')\, w' \in \mathbb{C} W_\lambda$$

and by [33, 3.2] (as noted in 6.5), $C_\alpha(L(w\lambda))$ corresponds to $s_\alpha \mathbf{a}(w)$. Carrying through the analysis of 5.1, 5.2 on $C_\alpha(L(w\lambda))$ it follows that there exist non-negative integers l_i, l_i' such that

$$\sum l_i p_i = \tilde{p}_{w^{-1}}, \qquad \sum l_i' p_i = s_\alpha \tilde{p}_{w^{-1}}, \tag{$*$}$$

where $l_i \neq 0 \Leftrightarrow \mathscr{V}_i \subset \mathscr{W}$ (resp. $l_i' \neq 0 \Leftrightarrow \mathscr{V}_i \subset \mathscr{W}'$).

Now by 3.1, $s_\alpha p_i = -p_i$ if $\mathscr{V}_i \subset \mathfrak{m}_\alpha$, whereas $s_\alpha p_i$ is a non-negative integer combination of the p_j if $\mathscr{V}_i \not\subset \mathfrak{m}_\alpha$. Hence by $(*)$

$$[p_{\mathscr{W}'}] = [p^{(1)}] \cup [s_\alpha p^{(2)}]$$

as required.

Remarks. One expects the set $\mathscr{V}^{(1)}$ defined above to be empty. If this holds then $\mathscr{V}(C_\alpha(L(\mu))/L(\mu)) = \overline{\mathbf{P}_\alpha \mathscr{V}(L(\mu)) \cap \mathfrak{m}_\alpha}$. One may also ask if this equality is valid with $\mathscr{V}$ replaced by $\mathscr{S}$ (notation 5.5) in which the right-hand side is computed through taking primary decomposition (as in 3.1). The main object of this would be to obtain a geometric interpretation of the z_w (see III, introduction) and hence a process for computing these important coefficients.

7. INVOLUTIVE VARIETIES

7.1. Let $\mathfrak{a}$ be a finite-dimensional Lie algebra. The Poisson bracket structure on $S(\mathfrak{a})$ is the unique bilinear antisymmetric pairing $(a, b) \to \{a, b\}$ of $S(\mathfrak{a}) \times S(\mathfrak{a}) \to S(\mathfrak{a})$ satisfying $\{ab, c\} = a\{b, c\} + b\{a, c\}$, $\forall a, b, c \in S(\mathfrak{a})$ and whose restriction to $\mathfrak{a}$ is just the Lie bracket on $\mathfrak{a}$. One calls a subvariety $\mathscr{V}$ of $\mathfrak{a}^*$ *involutive* if $I(\mathscr{V})$ is closed under the Poisson bracket. If $\mathscr{V}$ is involutive it is easy to check that all its irreducible components are involutive. On the other hand, it can very easily happen that an ideal I of $S(\mathfrak{a})$ is closed under Poisson bracket and yet $\mathscr{V}(I)$ is not involutive.

7.2. Let $\mathfrak{a}$ be an algebraic Lie algebra and **A** its adjoint group (acting in $\mathfrak{a}^*$). Let $\mathscr{O}$ be an **A** orbit and set $J = J(\mathscr{O})$. Since $\{S(\mathfrak{a}), J\} \subset J$ the Poisson bracket structure on $S(\mathfrak{a})$ defines in an obvious fashion a Poisson bracket structure on the affine algebraic variety $(\mathscr{O}, S(\mathfrak{g})/J)$.

LEMMA. *Let $\mathscr{V}$ be an involutive subvariety of $\mathscr{O}$. Then* $\dim \mathscr{V} \geqslant \frac{1}{2} \dim \mathscr{O}$.

This is fairly easy and a fairly well-known result. Take $x \in \mathscr{V}$ in general position. For each $f \in S(\mathfrak{a})/J$, let ξ_f denote the Hamiltonian vector field paired to the differential df through the Kirillov–Kostant two-form ω. One has $\xi_f g = \{f, g\}$ and so the hypothesis that $\mathscr{V}$ is involutive implies that $\xi_f(x): f \in I(\mathscr{V})$ belongs to the tangent space T_x of $\mathscr{V}$ at x. From the relation $\langle \omega, (\xi_f(x), \xi_g(x)) \rangle = \{f, g\}(x)$, $\forall f, g \in S(\mathfrak{a})/J$ it then follows that T_x is coisotropic with respect to ω and so $\dim \mathscr{V} = \dim T_x \geqslant \dim \mathscr{O}$, as required.

7.3. From now on we take $\mathfrak{a} = \mathfrak{g}$.

LEMMA. *Let $\mathscr{O}$ be a nilpotent orbit and $\mathscr{V}$ an involutive subvariety of $\mathscr{O} \cap \mathfrak{n}$. Then* $\dim \mathscr{V} = \dim \mathscr{O} \cap \mathfrak{n}$.

By the Spaltenstein–Steinberg equality we have $\dim \mathscr{V} \leqslant \dim(\mathscr{O} \cap \mathfrak{n}) = \frac{1}{2} \dim \mathscr{O}$. The opposite inequality follows from 7.2.

7.4. It follows from 7.3 that any irreducible involutive subvariety of $\mathscr{O} \cap \mathfrak{n}$ is dense in the closure of a component of $\mathscr{O} \cap \mathfrak{n}$, i.e., it is orbital

(4.7). From this remark we obtain an alternative proof of 4.7 using the fact that $\mathscr{V}(M)$ is always involutive [10, Thm. 1].

7.5. Take $w \in W$ and recall (1.8) that $\mathbf{G}(\mathfrak{n} \cap w(\mathfrak{n}))$ admits a unique dense orbit $\mathrm{St}(w)$. Set $\mathscr{V}_1(w) = \mathbf{B}(\mathfrak{n} \cap w(\mathfrak{n}))$, $\mathscr{V}_2(w) = \mathscr{V}_1(w) \cap \mathrm{St}(w)$, $\mathscr{V}(w) = \overline{\mathscr{V}_2(w)} \cap \mathrm{St}(w)$. Obviously $\mathscr{V}_2(w)$ is dense in $\mathscr{V}_1(w)$.

LEMMA. *$\mathscr{V}(w)$ is an involutive subvariety of* $\mathrm{St}(w)$.

Set $S(w) = \{\beta \in R^+ : w\beta \in R^-\}$ and $\mathfrak{n}_w^- = \mathbb{C}\{X_{-\beta} : \beta \in S(w^{-1})\}$. One easily checks that $\mathfrak{n}_w^-$ is the orthogonal of $\mathfrak{n} \cap w(\mathfrak{n})$ in $\mathfrak{n}^-$. It follows that the ideal I of definition of $\mathscr{V}(w)$ in $S(\mathfrak{g})$ takes the form $I = S(\mathfrak{g})\mathfrak{b} + J$, where J is the largest $\mathbf{B}$ stable ideal of $S(\mathfrak{n}^-)$ contained in $S(\mathfrak{n}^-)\,\mathfrak{n}_w^-$. Now $\mathfrak{n}_w^-$ is a subalgebra of $\mathfrak{n}^-$ and so $\{J, J\} \subset S(\mathfrak{n}^-)\,\mathfrak{n}_w^-$. Yet $\{J, J\}$ is $\mathbf{B}$ stable, hence $\{J, J\} \subset J$ which gives $\{I, I\} \subset I$, as required.

7.6. COROLLARY. *$\mathscr{V}(w)$ is an irreducible component of* $\mathrm{St}(w) \cap \mathfrak{n}$.

7.7. Take $w \in W$. Define $\operatorname{supp} w$ to be the smallest subset B' of B such that $w \in W_{B'}$ (notation 2.10). Now let $\mathbf{B}'$ denote the Borel subgroup relative to the reductive subalgebra $\mathfrak{r}_{B'}$ (notation 2.10) having Lie algebra $\mathfrak{h} \oplus \mathfrak{n}_{B'}$. Given $B' = \operatorname{supp} w$, we obtain $\mathbf{B}(\mathfrak{n} \cap w(\mathfrak{n})) = \mathbf{B}'(\mathfrak{n}_{B'} \cap w(\mathfrak{n}_{B'})) + \mathfrak{m}_{B'}$. Thus $\mathscr{V}_1(w)$ takes the form $\mathrm{Ind}(\mathscr{V}, \mathfrak{p}_{B'} \uparrow \mathfrak{g}_{B'})$ (notation 2.10). A similar remark applies to $\mathrm{St}(w)$.

7.8. To see the relevance of induced varieties we first need the general

LEMMA. *Let $\mathfrak{a}$ be a subalgebra of $\mathfrak{q}$ and I a left ideal of $U(\mathfrak{a})$. With respect to canonical filtrations*

(i) $\mathrm{gr}(U(\mathfrak{q})I) = S(\mathfrak{q})\,\mathrm{gr}\, I$.

(ii) *If $\mathfrak{m}$ is an ideal of $\mathfrak{q}$ complementing $\mathfrak{a}$ in $\mathfrak{q}$ then* $\mathrm{gr}(U(\mathfrak{q})(I + \mathfrak{m})) = S(\mathfrak{q})(\mathrm{gr}\, I + \mathfrak{m})$.

(i) obtains from the fact that $U(\mathfrak{q})$ is free as a right $U(\mathfrak{a})$ module. For (ii) use the direct sum decomposition $U(\mathfrak{q})I = I \oplus U(\mathfrak{m})\,\mathfrak{m}I$ and the fact that $\mathfrak{m}I \subset U(\mathfrak{q})\mathfrak{m}$.

7.9. Let $\mathscr{R}(\mathfrak{r}_{B'})$ denote the category of regular $\mathfrak{r}_{B'}$ modules. Take $M \in \mathrm{Ob}\,\mathscr{R}(\mathfrak{r}_{B'})$ and consider M as a $U(\mathfrak{p}_{B'})$ module through the trivial action of $\mathfrak{m}_{B'}$. Then $\mathrm{Ind}(M, \mathfrak{p}_{B'} \uparrow \mathfrak{g}) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_{B'})} M$ (simply, $\mathrm{Ind}\, M$) is a regular $U(\mathfrak{g})$ module. Furthermore

LEMMA. $\mathscr{V}(\mathrm{Ind}\, M) = \mathrm{Ind}\, \mathscr{V}(M)$.

By the fact that $U(\mathfrak{g})$ is free as a right $U(\mathfrak{p}_{B'})$ module and 4.2 we can assume that M is cyclic, say $M \cong U(\mathfrak{r}_{B'})/I$ and then $\operatorname{Ind} M \cong U(\mathfrak{g})/U(\mathfrak{g})(I + \mathfrak{m}_{B'})$. Use of 7.8 gives $\mathscr{V}(\operatorname{Ind} M) = \mathscr{V}(\operatorname{gr}(U(\mathfrak{g})(I + \mathfrak{m}_{B'}))) = \mathscr{V}(S(\mathfrak{g})(\operatorname{gr} I + \mathfrak{m}_{B'})) = \mathscr{V}(\operatorname{gr} I) + \mathfrak{m}_{B'} = \operatorname{Ind} \mathscr{V}(M)$, where we have used that $\mathfrak{r}_{B'} \oplus \mathfrak{m}_{B'} \oplus \mathfrak{m}_{B'}^- = \mathfrak{g}$, and $\mathfrak{m}_{B'}$, $\mathfrak{m}_{B'}^-$ are paired by the Killing form.

7.10. Take $B' \subset B$ and let $L_{B'}(\mu) \in \operatorname{Ob} \mathscr{O}(\mathfrak{r}_{B'})$ denote the simplest highest weight module with highest weight $\mu - \rho$. The following result is due to Jantzen and follows from [14, 4.12].

PROPOSITION. *Take $-\mu \in \mathfrak{h}^*$ dominant. Then for all $w \in W_{B'}$ one has*

$$\operatorname{Ind}(L_{B'}(w\mu), \mathfrak{p}_{B'} \uparrow \mathfrak{g}) \cong L(w\mu).$$

8. SIMPLE HIGHEST WEIGHT MODULES

8.1. Take $w \in W$ and recall that $\mathscr{V}(w)$ (notation 7.4) is involutive. The fundamental question to which we are arriving is the following. Does there exist $\mu \in \mathfrak{h}^*$ antidominant such that $\mathscr{V}(L(w\mu)) = \overline{\mathscr{V}(w)}$? By 7.7–7.10 this question is reduced to the case when $\operatorname{supp} w = B$. In particular it has a positive answer when $w = w_{B'}$ for then $L_{B'}(w_{B'}\mu)$ (with say $\mu = -\rho$) is finite dimensional. For the moment there seems to be no general construction which works for arbitrary $w \in W$. The main result of this section is nevertheless a partial step in this direction. In this we shall fix $-\mu \in \mathfrak{h}^*$ dominant, regular and $w \in W$, $\alpha \in B$ such that $k := -(\alpha^\vee, w\mu)$ is a positive integer. Now for each $\nu \in \mathfrak{h}^*$ let e_ν denote the canonical generator of $M(\nu)$ (having weight $\nu - \rho$) and $\bar{e}_\nu$ its image in $L(\nu)$. In the following sections we shall establish the following result (notation 7.5).

PROPOSITION. $\operatorname{Ann}_{U(\mathfrak{n}^-)} \bar{e}_{w\mu} \subset U(\mathfrak{n}^-)\,\mathfrak{n}_w^-$ *implies* $\operatorname{Ann}_{U(\mathfrak{n}^-)} \bar{e}_{s_\alpha w\mu} \subset U(\mathfrak{n}^-)\,\mathfrak{n}_{s_\alpha w}^-$.

8.2. Retain the above hypothesis.

LEMMA.

(i) $\mathfrak{n}_w^- \subset \mathfrak{m}_\alpha^-$.

(ii) $\mathfrak{n}_{s_\alpha w}^- = s_\alpha(\mathfrak{n}_w^-) \oplus \mathbb{C}X_{-\alpha}$.

(iii) $[X_\alpha, \mathfrak{n}_w^-] \subset \mathfrak{n}_w^-$.

Since $-\mu$ is dominant and $(w^{-1}\alpha^\vee, -\mu) > 0$ we obtain $w^{-1}\alpha \in R^+$ and so $\alpha \notin S(w^{-1})$. Hence (i). Again $\alpha \in B$ and an elementary calculation shows that $S((s_\alpha w)^{-1}) = s_\alpha(S(w^{-1})) \cup \{\alpha\}$. Hence (ii). Finally if $\beta \in R^+$, $w^{-1}\beta \in R^-$ and $\beta - \alpha \in R$; then $\beta - \alpha \in R^+$ and $w^{-1}(\beta - \alpha) = w^{-1}\beta - w^{-1}\alpha \in R^+$, which implies (iii).

8.3. The proof of 8.1 obviously requires a good description of the maximal submodule $\overline{M(\nu)}$ of $M(\nu)$. This is provided by the truth of the Kazhdan–Lusztig conjecture in the general case [9, 28] which through [23, 5.1] can be used to describe $\overline{M(s_\alpha w\mu)}$ in terms of $\overline{M(w\mu)}$ using the Enright completion functor C_α. This result may be stated as follows. Recall that C_α is left exact and so $C_\alpha \overline{M(w\mu)}$ identifies with a submodule of $C_\alpha M(w\mu) \cong M(s_\alpha w\mu)$ [23, 2.5]. Again by [8, 7.6.23] we may also consider $M(w\mu)$ as a submodule of $M(s_\alpha w\mu)$. Furthermore

THEOREM. $\overline{M(s_\alpha w\mu)} = M(w\mu) + C_\alpha(\overline{M(w\mu)})$.

8.4. From now on set $X = X_\alpha$, $Y = X_{-\alpha}$. Recall that $k := -(\alpha^\vee, w\mu)$.

COROLLARY.

$$\operatorname{Ann}_{U(\mathfrak{n}^-)} \bar{e}_{s_\alpha w\mu} = U(\mathfrak{n}^-)\, Y^k + (\mathbb{C}[Y^{-1}](\operatorname{Ann}_{U(\mathfrak{n}^-)} \bar{e}_{w\mu})\, Y^k) \cap U(\mathfrak{n}^-).$$

Set $e = e_{s_\alpha w\mu}$, $f = e_{w\mu}$. Recall [8, 7.1.15] that $Y^k e$ can be identified with f. Take $a \in \operatorname{Ann}_{U(\mathfrak{n}^-)} \bar{e}$. Then by 8.3 we have $ae = bY^k e + ce$, for some $b, c \in U(\mathfrak{n}^-)$ such that $ce \in C_\alpha(\overline{M(w\mu)})$. Since $\operatorname{Ann}_{U(\mathfrak{n}^-)} e = 0$ it follows that $a = bY^k + c$. Now consider $m \in C_\alpha(\overline{M(w\mu)})$. By definition of C_α there exists $l \in \mathbb{N}$ such that $Y^l m \in \overline{M(w\mu)}$ so we must have $Y^l m = df = dX^k_{-\alpha} e$, for some $d \in \operatorname{Ann}_{U(\mathfrak{n}^-)} \bar{f}$. Taking $m = ce$ it follows that $c \in Y^{-l}(\operatorname{Ann}_{U(\mathfrak{n}^-)} \bar{f})\, Y^k \cap U(\mathfrak{n}^-)$. Conversely, if this holds then $ce \in (\overline{M(w\mu)})_Y$ and since $Xe = 0$ and $(\operatorname{ad} X)^t c = 0$ for t sufficiently large we have that $ce \in C_\alpha \overline{M(w\mu)}$ by the definition of C_α. From the above observations the corollary follows.

8.5. The conclusion of 8.1 follows from 8.4 if we can show that

$$(\mathbb{C}[Y, Y^{-1}](U(\mathfrak{m}_\alpha^-)\, \mathfrak{n}_w^-)\, Y^k) \cap U(\mathfrak{n}^-) \subset U(\mathfrak{n}^-)\, s_\alpha(\mathfrak{n}_w^-) \bmod U(\mathfrak{n}^-)Y. \quad (*)$$

In the next few sections we shall develop some machinery for doing this.

8.6. Set $H = H_\alpha$, $\mathfrak{s} = \mathbb{C}X \oplus \mathbb{C}H \oplus \mathbb{C}Y$, $\mathfrak{b} = \mathbb{C}X \oplus \mathbb{C}H$, $\mathfrak{b}^- = \mathbb{C}H \oplus \mathbb{C}Y$. Fix a finite-dimensional $\mathfrak{s}$ module E and let F be a $\mathfrak{b}$ submodule of E. We consider $\mathbb{C}[Y]$ to be a $\mathfrak{b}^-$ module for adjoint action and take $E\mathbb{C}[Y]$ to be the $\mathfrak{b}^-$ module $E \otimes_{\mathbb{C}} \mathbb{C}[Y]$ given a right $\mathbb{C}[Y]$ module structure by right multiplication in the second factor. Then $E\mathbb{C}[Y]$ admits a left $\mathbb{C}[Y]$ module structure through $Yez = [Y, e]z + eYz$: $e \in E$, $z \in \mathbb{C}[Y]$. For example, take

E to be an $\mathfrak{s}$ submodule of $U(\mathfrak{m}_\alpha^-)$. Then $E\mathbb{C}[Y]$ identifies with the subspace of $U(\mathfrak{n}^-)$ generated by E over $\mathbb{C}[Y]$. Set $|Y\rangle := E\mathbb{C}[Y]Y$ (which we note is distinct from $\langle Y| := YE\mathbb{C}[Y]$) and let π denote the canonical projection of $E\mathbb{C}[Y]$ onto $E\mathbb{C}[Y]/|Y\rangle$. We identify $\operatorname{Im}\pi$ with E itself. For each $k \in \mathbb{N}^+$ set $\Phi_k(F) := \pi(\mathbb{C}[Y, Y^{-1}]FY^k \cap E\mathbb{C}[Y])$ considered as a subspace of E. Obviously $\Phi_k(F)$ is ad H stable. Furthermore

LEMMA. $[Y, \Phi_k(F)] \subset \Phi_k(F)$. *(Recall $k > 0$.)*

Take $0 \neq \bar{f} \in \Phi_k(F)$. We can write $f \in \pi^{-1}(\bar{f}) \subset E\mathbb{C}[Y]$ as $f = Y^{-j}aY^k$ for some $a \in \mathbb{C}[Y]F$. Now $j > 0$ otherwise $\bar{f} = 0$ and so $Yf = Y^{-(j-1)}aY^k \in \mathbb{C}[Y, Y^{-1}]FY^k$. Yet $[Y, \bar{f}] = [Y, \pi(f)] = \pi[Y, f] = \pi(Yf) \in \Phi_k(F)$, as required.

8.7. PROPOSITION. *$\Phi_k(F) \subset \Phi_{k+1}(F)$ with equality for $k > \dim E$.*

It is enough to show that any ad H weight vector $\bar{f} \in \Phi_k(F)$ satisfies $\bar{f} \in \Phi_{k+1}(F)$. For such a choice we can write $f = Y^{-j}aY^k$, $a \in \mathbb{C}[Y]F$ a weight vector. Furthermore we can choose weight vectors $f_i \in F$ such that each product $Y^i f_i$ has weight independent of i and such that

$$a = \sum_{i=0}^{m} d_i Y^k f_i \colon d_i \in \mathbb{C}.$$

Moreover, this expression lies in $\mathbb{C}[Y]F$ for arbitrary choices of the d_i. For $j > 0$ we can write

$$Y^{-j}aY^k = \sum_{i=0}^{m} \sum_{t=0}^{\infty} d_i c_{j-i,t}((\operatorname{ad} Y)^t f_i)\, Y^{k+i-j-t}.$$

We see that if j and the d_i are chosen so that

$$\sum_{i=0}^{m} d_i c_{j-i,t+i}(\operatorname{ad} Y)^{t+i} f_i = 0 \qquad (*)$$

for all $t > k - j$, then $Y^{-j}aY^k \in E\mathbb{C}[Y]$ and $\bar{f} := \pi(Y^{-j}aY^k)$ is just the above expression evaluated at $t = k - j$. If k is replaced by $k + 1$ then we can replace j by $j + 1$ so that the conditions on t are unchanged. To show $\bar{f} \in \Phi_{k+1}(F)$ it is enough to observe that the dependence of $c_{j-i,t+i}$ can be compensated by an alteration in the d_i. This follows from the formula

$$c_{j-i,t+i} = \frac{(-1)^{t+i}}{(t+i)!} \frac{(j+t-1)!}{(j-i)!}$$

from which we see that replacement of d_i by $d_i/(j-i)!$ and cancellation of the $(j+t-1)!$ in each equation $(*)$ removes this dependence on j. Similarly if k is replaced by $k-1$ then we can replace j by $j-1$ (to obtain $\bar{f} \in \bar{\Phi}_{k-1}(F)$) as long as $j > 1$. Now if $k > \dim E$ then $Y^{-1}aY^{k+1} \in |Y\rangle$ for all $a \in \mathbb{C}[Y]F$ which together with the first observation proves the last part.

8.8. Let π' denote the canonical projection of $E\mathbb{C}[Y]$ onto $E\mathbb{C}[Y]/\langle Y|$ and for each $k \in \mathbb{N}^+$, set $\psi_k(F) := \pi'(\mathbb{C}[Y, Y^{-1}]\, FY^k \cap E\mathbb{C}[Y])$, considered as a subspace of E. Obviously $\psi_k(F)$ is ad H stable. In contrast to the $\Phi_k(F)$ these spaces need not be increasing in k (unless say E is a simple module). For any $\mathbb{C}[H]$ module V on which H acts locally semisimply, let $\Omega(V)$ denote the set of weights for which the corresponding H weight subspace of V is non-zero.

A comparison of the rules for computing ψ_k and Φ_k gives the

LEMMA. *For all $k \gg 0$ one has $\Omega(\psi_k(F)) = \Omega(\Phi_k(F))$.*

8.9. So far we have not used that F is stable for the action of X. This plays the role described in the following. Set $\mathfrak{h} = \mathbb{C}H$ and choose $\nu \in \mathfrak{h}^*$ such that $(\nu, \alpha^\vee) = k$. Then the sl(2) Verma module $M(\nu)$ with canonical generator e_ν admits the submodule $M(s_\alpha \nu)$ with canonical generator $e_{s_\alpha} = Y^k e_\nu$. In particular $M(\nu)$ is α-complete and $E\mathbb{C}[Y]\, e_\nu$ identifies in the obvious fashion with the α-free and α-complete $\mathfrak{s}$ module $E \otimes M(\nu)$. Set $I(\nu) := \mathbb{C}[Y]\, FY^k e_\nu$, which because F is $\mathfrak{b}$ stable identifies with a submodule of $E \otimes M(s_\alpha \nu)$. Finally set $J(\nu) := (\mathbb{C}[Y, Y^{-1}]\, FY^k \cap E\mathbb{C}[Y])\, e_\nu$ which identifies with a submodule of $E \otimes M(\nu)$. Just as in the proof of 8.4 one shows that there are embeddings $I(\nu) \hookrightarrow J(\nu) \hookrightarrow C_\alpha(I(\nu))$.

LEMMA. *If $k > \dim E$, then $J(\nu) = C_\alpha(I(\nu))$.*

As in 8.7 we take $a \in \mathbb{C}[Y]F$ to be a weight vector of the form $a = \sum Y^i f_i$: $f_i \in F$. We must show that if $Y^{-j}a \notin E\mathbb{C}[Y]$ then $X^t Y^{-j} a e_\nu \neq 0$, $\forall t \in \mathbb{N}$. In this we can assume $f_0 \neq 0$, and $j > 0$ without loss of generality. Then

$$XY^{-j}ae_\nu = [X, Y^{-j}a]\, e_\nu$$
$$= \sum_{i=0}^{m} ([X, Y^{i-j}] f_i e_\nu + Y^{i-j}[X, f_i]\, e_\nu).$$

Since $\mathrm{Ann}_{E\mathbb{C}[Y]} e_\nu = 0$, the above expression can only vanish if $[X, Y^{-j}] f_0 e_\nu = 0$. Let us write $[H, f_0] = l f_0$. Through the hypothesis $k > \dim E$, we have $k + l > 0$ and so

$$[X, Y^{-j}] f_0 e_\nu = -jY^{-j-1}(H + j + 1) f_0 e_\nu$$
$$= -jY^{-j-1}(k + l + j) f_0 e_\nu \neq 0.$$

Finally F is $\mathfrak{b}$ stable so $[X, Y^{-j}a]\, e_\nu = Y^{-j-1}be_\nu$ for some (uniquely determined) $b \in F$ and from the above calculation $Y^{-j-1}b \notin E\mathbb{C}[Y]$. The assertion of the lemma results.

8.10. By 8.7 it makes sense to define

$$\Phi(F) = \lim_{k \to \infty} \Phi_k(F).$$

LEMMA. $\Omega(\Phi(F)) = s_\alpha(\Omega(F))$.

Take $k > \dim E$. By a standard argument (cf. [8, 7.6.14]), $I(\nu)$ has composition factors isomorphic to the Verma modules $M(s_\alpha \nu + \mu_i)$: $\mu_i \in \Omega(F)$. By the hypothesis on k those modules are simple and have different central characters. Then by 8.9 (cf. 6.3) it follows that $J(\nu)$ is a direct sum of Verma modules $M(\nu + s_\alpha \mu_i)$: $\mu_i \in \Omega(F)$. By definition of $\psi_k(F)$ we may write

$$\begin{aligned} J(\nu) &= \psi_k(F)\, e_\nu \operatorname{mod}(YE\mathbb{C}[Y]\, e_\nu) \\ &= \psi_k(F)\, e_\nu \operatorname{mod} Y(E \otimes M(\nu)). \end{aligned}$$

Yet $E \otimes M(\nu)$ is also a direct sum (6.3(ii)) of Verma modules (in which $J(\nu)$ embeds) and so we conclude that $\psi_k(F)\, e_\nu$ is mod $Y(E \otimes M(\nu))$ the span of the highest weight spaces of $J(\nu)$. Hence $\Omega(\psi_k(F)) = s_\alpha(\Omega(F))$, which combined with 8.8 gives the assertion of the lemma.

8.11. Let E' be a finite-dimensional $\mathfrak{s}$ module and let $E'E$ denote an $\mathfrak{s}$ quotient of $E' \otimes E$. For example, let E, E' be $\mathfrak{s}_\alpha$ submodules of $U(\mathfrak{m}_\alpha^-)$ and let $E'E$ be defined by the multiplication in $U(\mathfrak{m}_\alpha^-)$. The following result would be trivial for the tensor product $E' \otimes E$; but is a rather subtle question in general.

PROPOSITION. $\Phi(E'F) = E'\Phi(F)$.

Choose k, ν as before. We identify $E'E\mathbb{C}[Y]\, e_\nu$ with $E'E \otimes M(\nu)$, where the latter is defined to be the image $(E' \otimes E) \otimes M(\nu)$ under the map $E' \otimes E \twoheadrightarrow E'E$. Since $\operatorname{Ann}_{E'E\mathbb{C}[Y]}\, e_\nu = 0$ it suffices to show that

$$(\mathbb{C}[Y, Y^{-1}]\, E'FY^k \cap E'E\mathbb{C}[Y])\, e_\nu = E'(\mathbb{C}[Y, Y^{-1}]\, FY^k \cap E\mathbb{C}[Y])\, e_\nu,$$

for k sufficiently large.

Now the right-hand side is just $E'J(\nu)$ whereas an easy calculation shows that the left-hand side contains $E'J(\nu)$ and is contained in $C_\alpha(E'J(\nu))$. Taking $k > \dim E$ it follows from 8.9 that $J(\nu) = C_\alpha J(\nu)$ and moreover (cf. 8.10) is a direct sum of α-complete Verma modules $M(\nu + s_\alpha \mu_i)$: $\mu_i \in \Omega(F)$. Now

$E'J(\nu)$ is α-free and a quotient of $E' \otimes J(\nu) \cong \oplus (E' \otimes M(\nu + s_\alpha \mu_i))$: $\mu_i \in \Omega(F)$, so taking $k > \dim(E \otimes E')$ it follows from 6.3(ii) that $E'J(\nu)$ is α-complete.

8.12. Suppose that E can be written as a direct sum of simple $\mathfrak{s}$ modules E_i such that $F = \oplus (F \cap E_i)$. For example, take $E = \mathfrak{m}_\alpha^-$, $F = \mathfrak{n}_{w'}^-$.

COROLLARY. $\Phi(E'F) = E's_\alpha(F)$.

By 8.11 it is enough to show that $\Phi(F) = s_\alpha(F)$ and through the hypothesis on E, F we can assume E simple without loss of generality. When this holds the assertion follows from 8.6 and 8.10.

8.13. Writing $U(\mathfrak{m}_\alpha^-)$ as a direct sum of finite-dimensional $\mathfrak{s}_\alpha$ modules $(*)$ of 8.5 easily follows from 8.12. (We remark that more careful estimates of k show that equality holds in $(*)$ of 8.5.) This proves 8.1.

8.14. Proposition 8.1 is applied through the following

LEMMA. *Take $w, w' \in W$. If $\mathrm{Ann}_{U(\mathfrak{n}^-)} \bar{e}_{w\mu} \subset U(\mathfrak{n}^-)\, \mathfrak{n}_{w'}^-$, then $\mathscr{V}(L(w\mu)) \supset \overline{\mathscr{V}(w')}$.*

By 7.8 the hypothesis implies that $\mathrm{gr}\, \mathrm{Ann}_{U(\mathfrak{n}^-)} \bar{e}_{w\mu} \subset S(\mathfrak{n}^-)\, \mathfrak{n}_{w'}^-$ and so $\mathscr{V}(L(w\mu)) \subset \mathfrak{n} \cap w'(\mathfrak{n})$. Finally use that $\mathscr{V}(L(w\mu))$ is closed and $\mathbf{B}$ stable.

8.15. In the so-called integral case we obtain the following fairly satisfactory result.

THEOREM. *Take $-\mu \in \mathfrak{h}^*$ dominant, regular and suppose that $B_\mu \subset B$. Then for all $w \in W_\mu$ one has $\mathscr{V}(L(w\mu)) \supset \overline{\mathscr{V}(w)}$.*

Since $\mathrm{Ann}_{U(\mathfrak{n}^-)} \bar{e}_\mu = 0$, we obtain $\mathrm{Ann}_{U(\mathfrak{n}^-)} \bar{e}_{w\mu} \subset U(\mathfrak{n}^-)\, \mathfrak{n}_w^-$ by induction on the length of w. Then the conclusion follows from 8.14.

Remarks. It is likely that $\mathscr{V}(L(w\mu))$ is irreducible in the above situation. When $\mathfrak{g}$ has only type A_n factors one has equality of dimension (see 9.14). If B_μ does not have only type A_n factors then equality of dimension fails in 8.15. Yet it can still happen that equality holds sufficiently often for one to be then able to determine $\mathscr{V}(L(w\mu))$ through 6.6(i).

8.16. In the non-integral case the appropriate generalization of 8.15 should *not* be considered to be 8.15 with the technical restriction $B_\mu \subset B$ omitted. Although this might be true it is not the best result. This is illustrated by 10.1.

8.17. It is obvious that we can generalize 8.1 in the following possibly useful fashion. Replace $\mathfrak{n}_w^-$ by any $\mathbb{C}H_\alpha \oplus \mathbb{C}X_\alpha$ stable subspace $\mathfrak{m}$ of $\mathfrak{m}_\alpha^-$ and $\mathfrak{n}_{s_\alpha w}^-$ by $s_\alpha(\mathfrak{m}) \oplus \mathbb{C}X_{-\alpha}$. On the other hand, it is unfortunate that the converse of 8.14 fails. This is because $\mathrm{gr}(\mathrm{Ann}_{U(\mathfrak{n}^-)} \bar{e}_{w\mu}) \subset S(\mathfrak{n}^-)\, \mathfrak{n}_{w'}^-$ does not imply that $\mathrm{Ann}_{U(\mathfrak{n}^-)} \bar{e}_{w\mu} \subset U(\mathfrak{n}^-)\, \mathfrak{n}_{w'}^-$. For example, take $\mathfrak{g}$ of type $sl(3)$ with $\mu = -\rho$, $w = s_\alpha s_\beta$, $w' = s_\alpha$, where $B = \{\alpha, \beta\}$. This calculation illustrates nicely the theory developed in 8.6–8.12.

8.18. We can now answer positively a question raised implicitly in (III, 5.4). Take $-\mu \in \mathfrak{h}^*$ dominant, regular and given $w \in W_\mu$ let $l_\mu(w)$ denote its reduced length defined with respect to B_μ. If μ is integral we set $l_\mu(w) = l(w)$.

LEMMA. *For all* $w \in W_\mu$ *one has* $d(L(w\mu)) \geqslant \mathrm{card}\, R^+ - l_\mu(w)$ *with equality if and only if* $w = w_{B'}$ *for some* $B' \subset B_\mu$.

By the truth of the Kazhdan–Lusztig conjecture in the general case [9, 28] and (II, 5.1) the value of $\mathrm{card}\, R^+ - d(L(w\mu))$ depends only on the specification of (W_μ, B_μ) as a Coxeter group. Consequently we can assume that $B = B_\mu$ without loss of generality. Then by 8.15, $d(L(w\mu)) = \dim \mathscr{V}(L(w\mu)) \geqslant \dim \mathscr{V}(w) \geqslant \dim(\mathfrak{n} \cap w(\mathfrak{n})) = \mathrm{card}\, R^+ - l(w)$. This proves the first part. In the second part "if" is already given by [15, 2.8, 3.5]. For "only if" we note that the second inequality above is an equality only if $\mathfrak{n} \cap w(\mathfrak{n})$ is $\mathbf{B}$ stable. An easy exercise shows that this implies w to be of the prescribed form.

9. STEINBERG'S CONSTRUCTION

9.1. In order to formulate our conjecture concerning the characteristic polynomial $p_{\mathscr{V}}$ associated to an orbital variety $\mathscr{V}$ we need first to make explicit some straightforward consequences of Steinberg's construction [35] and Spaltenstein's equidimensionality theorem [31]. In this it is customary to consider unipotent rather than nilpotent elements. Let $\mathscr{U}$ denote the set of unipotent elements of $\mathbf{G}$. One has $\mathbf{N} = \mathscr{U} \cap \mathbf{B}$. For each $g \in \mathbf{G}$, $S \subset \mathbf{G}$, set $g(S) = gSg^{-1} := \{gsg^{-1} : s \in S\}$. For each $w \in W$ fix a representative in $\mathbf{G}$ (also denoted by w) set $\mathscr{U}_1(w) = \mathbf{B}(\mathbf{N} \cap w(\mathbf{N}))$, denote the unique conjugacy class dense in $\mathbf{G}(\mathbf{N} \cap w(\mathbf{N}))$ by $\mathrm{St}(w)$ and put $\mathscr{U}_2(w) = \mathscr{U}_1(w) \cap \mathrm{St}(w)$, $\mathscr{U}(w) = \overline{\mathscr{U}_2(w)} \cap \mathrm{St}(w)$. Given $u \in \mathscr{U}$, let $\mathbf{Z}_\mathbf{G}(u)$ denote its centralizer in $\mathbf{G}$ and set $\mathbf{A}(u) = \mathbf{Z}_\mathbf{G}(u)/\mathbf{Z}_\mathbf{G}^0(u)$, where $\mathbf{Z}_\mathbf{G}^0(u)$ denotes the connected component of the identity in $\mathbf{Z}_\mathbf{G}(u)$.

9.2. Set $X_w = \mathbf{B}w\mathbf{B}$. One has the classical result

$$\mathbf{G} = \coprod_{w \in W} X_w, \qquad \bar{X}_w = \coprod_{y \leqslant w} X_y, \tag{$*$}$$

where $\leqslant$ denotes the Bruhat order. Let θ: $\mathbf{G} \times \mathbf{G} \to \mathbf{G}$ be defined by $\theta(x, y) = x^{-1}y$ and set $Y_w = \mathscr{O}^{-1}(X_w)$. Taking $\mathbf{K} := \{(g, g): g \in \mathbf{G}\}$ one finds that $Y_w = \mathbf{K}(1, w)(\mathbf{B} \times \mathbf{B})$ and so $\bar{Y}_w$ is a disjoint union of certain Y_y: $y \in W$. Hence $\theta((\mathbf{G} \times \mathbf{G}) \setminus \bar{Y}_w) = \mathbf{G} \setminus \theta(\bar{Y}_w)$ and since θ is open, it follows that $\theta(\bar{Y}_w)$ is closed and so contains $\bar{X}_w$. Yet $\bar{Y}_w \subset \theta^{-1}(\bar{X}_w)$ as $\mathscr{O}$ is continuous which combined with our previous observation gives $\theta(\bar{Y}_w) = \bar{X}_w$. From $(*)$ we then obtain

$$\bar{Y}_w = \coprod_{y \leqslant w} Y_y. \tag{$**$}$$

Now let Z_w denote the image of Y_w in $\mathbf{G}/\mathbf{B} \times \mathbf{G}/\mathbf{B}$. Similar reasoning to the above gives

$$\bar{Z}_w = \coprod_{y \leqslant w} Z_y.$$

9.3. Let $\mathscr{B}$ denote the variety of all Borel subgroups of $\mathbf{G}$ which we identify (as usual) with $\mathbf{G}/\mathbf{B}$ through the isomorphism of $g\mathbf{B} \to g\mathbf{B}g^{-1}$. Similarly $\mathscr{B} \times \mathscr{B}$ is identified with $\mathbf{G}/\mathbf{B} \times \mathbf{G}/\mathbf{B}$ and then Z_w is just the $\mathbf{K}$ orbit generated by $(\mathbf{B}, w(\mathbf{B})) \in \mathscr{B} \times \mathscr{B}$. For each conjugacy class $\mathscr{C}$ of unipotent elements we set, following Steinberg [34, p. 134],

$$S(\mathscr{C}) = \{(u, \mathbf{B}_1, \mathbf{B}_2) \in (\mathscr{C} \times \mathscr{B} \times \mathscr{B}): u \in \mathbf{B}_1 \cap \mathbf{B}_2\},$$

$$S_w(\mathscr{C}) = \{(u, \mathbf{B}_1, \mathbf{B}_2) \in S(\mathscr{C}): (\mathbf{B}_1, \mathbf{B}_2) \in Z_w\}.$$

Let π: $S(\mathscr{C}) \to \mathscr{C}$ denote the projection onto the first factor.

LEMMA. *Take $u \in \mathscr{C}$. Then*

(i) $\{\overline{S_w(\mathscr{C})}: w \in \mathrm{St}^{-1}(\mathscr{C})\}$ *is the set of (distinct) irreducible components of $S(\mathscr{C})$.*

(ii) *The map $\overline{S_w(\mathscr{C})} \to \overline{S_w(\mathscr{C}) \cap \pi^{-1}(u)}$: $w \in \mathrm{St}^{-1}(\mathscr{C})$ is a one-to-one correspondence of the irreducible components of $S(\mathscr{C})$ onto the $\mathbf{A}(u)$ orbits of irreducible components of $\mathscr{B}_u \times \mathscr{B}_u$.*

Let X be an irreducible component of $\mathscr{B}_u \times \mathscr{B}_u$. By [35, 3.1], $Y := \{(g(u), g(X)): g \in \mathbf{G}\}$ is an irreducible component of $S(\mathscr{C})$ and every component so obtains. Obviously $\pi^{-1}(u) \cap Y = \mathbf{Z}_{\mathbf{G}}(u)(X) = \mathbf{A}(u)(X)$ and so $\dim Y = \dim X + \dim \mathscr{C} = 2 \dim \mathscr{B}_u + \dim \mathscr{C} = \operatorname{card} R$, by [31]. Then by [35, 3.3] $Y = \overline{S_w(\mathscr{C})}$ for some unique $w \in W$ such that $\mathscr{C} \cap \mathbf{N} \cap w(\mathbf{N})$ is dense in $\mathbf{N} \cap w(\mathbf{N})$, equivalently for some unique $w \in \mathrm{St}^{-1}(\mathscr{C})$. Hence (i) and it remains to show that the inclusion $\overline{S_w(\mathscr{C}) \cap \pi^{-1}(u)} \subset \overline{S_w(\mathscr{C})} \cap \pi^{-1}(u)$: $w \in \mathrm{St}^{-1}(\mathscr{C})$ is an equality. Now $\dim(S_w(\mathscr{C}) \cap \pi^{-1}(u)) = \dim S_w(\mathscr{C}) - \dim \mathscr{C} = \dim \overline{S_w(\mathscr{C})} - \dim \mathscr{C}$ [35, 3.3a], so we have equality of dimension. On the other hand, the left-hand side is $\mathbf{Z}_{\mathbf{G}}(u)$ stable, whereas the right-hand side is a $\mathbf{Z}_{\mathbf{G}}(u)$ orbit of an irreducible component of $\mathscr{B}_u \times \mathscr{B}_u$. Hence (ii).

9.4. Fix $u \in \mathbf{N} \cap \mathscr{C}$. Following Spaltenstein [31] we define a map φ from $\mathscr{B}_u \times \mathscr{B}_u$ to the $\mathbf{B} \times \mathbf{B}$ orbits in $\mathbf{N} \cap \mathscr{C} \times \mathbf{N} \cap \mathscr{C}$ through $\varphi(g_1(\mathbf{B}), g_2(\mathbf{B})) = (\mathbf{B}(g_1^{-1}(u)), \mathbf{B}(g_2^{-1}(u)))$. This sets up a bijection $\bar{\varphi}$ between the set of $\mathbf{Z}_{\mathbf{G}}(u) \times \mathbf{Z}_{\mathbf{G}}(u)$ orbits in $\mathscr{B}_u \times \mathscr{B}_u$ and the $\mathbf{B} \times \mathbf{B}$ orbits in $\mathbf{N} \cap \mathscr{C} \times \mathbf{N} \cap \mathscr{C}$, and which also maps bijectively the set of $\mathbf{A}(u) \times \mathbf{A}(u)$ orbits of components of $\mathscr{B}_u \times \mathscr{B}_u$ to the set of components of $\mathbf{N} \cap \mathscr{C} \times \mathbf{N} \cap \mathscr{C}$.

LEMMA. *Take $w \in W$. Then $\varphi(S_w(\mathscr{C}) \cap \pi^{-1}(u))$ is contained in $\mathscr{U}_2(w) \times \mathscr{U}_2(w^{-1})$. If $w \in \mathrm{St}^{-1}(\mathscr{C})$ then its closure is $\mathscr{U}(w) \times \mathscr{U}(w^{-1})$.*

One has

$$\begin{aligned}\varphi(S_w(\mathscr{C}) \cap \pi^{-1}(u)) &= \varphi\{(g(\mathbf{B}), gw(\mathbf{B})) : g(\mathbf{B} \cap w(\mathbf{B})) \ni u\} \\ &= \{\mathbf{B}(g^{-1}(u)), \mathbf{B}(w^{-1}(g^{-1}(u))) : g(\mathbf{N} \cap w(\mathbf{N})) \cap \mathscr{C} \ni u\} \\ &\subset \mathscr{U}_2(w) \times \mathscr{U}_2(w^{-1}),\end{aligned}$$

which proves the first part. Recalling [31] that $\dim \mathbf{Z}_{\mathbf{G}}(u) - \dim \mathbf{B} = \dim \mathscr{B}_u - \dim \mathscr{C} \cap \mathbf{N}$, it follows from 9.3(ii) that $\dim \varphi(S_w(\mathscr{C}) \cap \pi^{-1}(u)) = 2\dim(\mathscr{C} \cap N)$ which then gives the second part.

9.5. COROLLARY. (i) *Every irreducible component of $\mathbf{N} \cap \mathscr{C} \times \mathbf{N} \cap \mathscr{C}$ takes the form $\mathscr{U}(w) \times \mathscr{U}(w^{-1})$ for some $w \in \mathrm{St}^{-1}(\mathscr{C})$.*

(ii) $\mathscr{U}(w) \times \mathscr{U}(w^{-1}) \subset \bigcup_{y \leqslant w} (\mathscr{U}_2(y) \times \mathscr{U}_2(y^{-1}))$.

(i) is an immediate consequence of 9.3 and 9.4. For (ii) observe that $S_w(\mathscr{C})$ is just the inverse image of Z_w under the projection $\pi: S(\mathscr{C}) \to \mathscr{B} \times \mathscr{B}$. Since π is continuous $\pi^{-1}(\overline{Z}_w) \supset \overline{\pi^{-1}(Z_w)} = \overline{S_w(\mathscr{C})}$. Then (ii) follows from 9.2(∗∗∗) and 9.4.

Remark. (i) in a slightly weaker form is due to Spaltenstein [30, Lemma 1].

9.6. Let us return to nilpotent elements. From 9.5 we obtain the

LEMMA. *Let $\mathscr{O}$ be a nilpotent orbit.*

(i) *Every irreducible component* $\mathcal{O} \cap \mathfrak{n}$ *takes the form* $\mathscr{V}(w)$ *for some* $w \in \mathrm{St}^{-1}(\mathcal{O})$. *In particular every such component is involutive and* $\dim \mathscr{V}(w) = \dim \mathscr{V}(w^{-1})$.

(ii) $\mathscr{V}(w) \subset \bigcup_{y \leqslant w} \mathscr{V}_2(y)$.

9.7. We can use 9.6 to give a much more explicit description of the behaviour of components under the action of the Weyl group (as defined in Section 3).

PROPOSITION. *Take* $w \in W$, $\alpha \in B$. *Then* $\mathscr{V}(w) \subset \mathfrak{m}_\alpha \Leftrightarrow s_\alpha w < w$ *and if either hold* $\mathscr{V}(w)$ *is* $\mathbf{P}_\alpha$ *stable. Otherwise*

$$\mathfrak{m}_\alpha \cap (\mathbf{P}_\alpha \mathscr{V}(w)) \subset \left(\bigcup_{z < w \mid s_\alpha z < z} \mathscr{V}(z) \right) \cup \mathscr{V}(w) \cup \mathscr{V}(s_\alpha w).$$

Recall that $\mathscr{V}_1(w)$ is the $\mathbf{B}$ stable variety generated by the subspace $\mathscr{C}\{X_\beta: \beta \in R^+ \setminus S(w^{-1})\}$. Since $\alpha \in S(w^{-1}) \Leftrightarrow s_\alpha w < w$, this gives the first part.

Now suppose that $s_\alpha w > w$. Then one has $\mathbf{P}_\alpha w\mathbf{B} = \mathbf{B}w\mathbf{B} \cup \mathbf{B}s_\alpha w\mathbf{B}$. Since $\mathfrak{m}_\alpha$ is $\mathbf{P}_\alpha$ stable we obtain

$$\begin{aligned} \mathfrak{m}_\alpha \cap \mathbf{P}_\alpha(\mathscr{V}_1(w)) &= \mathbf{P}_\alpha(\mathfrak{m}_\alpha \cap \mathscr{V}_1(w)), \\ &= \mathbf{P}_\alpha(\mathfrak{m}_\alpha \cap w(\mathfrak{n})), \\ &= \mathfrak{m}_\alpha \cap (\mathbf{P}_\alpha w\mathbf{B})(\mathfrak{n}), \\ &= \mathscr{V}_1(s_\alpha w) \cup (\mathfrak{m}_\alpha \cap \mathscr{V}_1(w)), \end{aligned}$$

and hence $\mathfrak{m}_\alpha \cap \mathbf{P}_\alpha(\mathscr{V}_2(w)) = \mathscr{V}_2(s_\alpha w) \cup (\mathfrak{m}_\alpha \cap \mathscr{V}_2(w))$. Now consider $\mathscr{V}(w) \setminus \mathscr{V}_2(w)$ which has codimension $\geqslant 1$ in $\mathscr{V}(w)$ and is $\mathbf{B}$ stable. Hence $\dim \mathbf{P}_\alpha(\mathscr{V}(w) \setminus \mathscr{V}_2(w)) \leqslant \dim \mathscr{V}(w)$. Since all components hava the same dimension it follows that $\mathfrak{m}_\alpha \cap \mathbf{P}_\alpha(\mathscr{V}(w) \setminus \mathscr{V}_2(w))$ is precisely those components of $\mathrm{St}(w) \cap \mathfrak{n}$ which lie in $\mathfrak{m}_\alpha$ and whose union contains $\mathscr{V}(w) \setminus \mathscr{V}_2(w)$. Thus the second part of the lemma follows from 9.6.

9.8. From 9.7 we can give an explicit description of $\mathfrak{m}_\alpha \cap \mathbf{P}_\alpha(\mathscr{V}(w))$ as a variety without reference to the polynomials $p_{\mathscr{V}(y)}$. The result essentially coincides with the Kazhdan–Lusztig description [26, Sect. 7] taking account of 9.3–9.5). Yet it is not completely obvious that our action of W on the $p_{\mathscr{V}(y)}$ coincides with Springer's action of W on top cohomology. This is because we still have to show that multiplicities coincide. Following Kazhdan–Lusztig [26, Sect. 5] we set

$$S = \{(u, \mathbf{B}_1, \mathbf{B}_2) \in (\mathscr{U} \times \mathscr{B} \times \mathscr{B}): u \in B_1 \cap B_2\}$$

$$S_w = \{(u, \mathbf{B}_1, \mathbf{B}_2) \in S: (\mathbf{B}_1, \mathbf{B}_2) \in Z_w\}.$$

Observe that $S(\mathscr{C})$ (resp. $S_w(\mathscr{C})$) is the inverse image of $\mathscr{C}$ in S (resp. S_w).

The S_w: $w \in W$ are irreducible of dimension $2r := \operatorname{card} R$ and form a disjoint union of S. Hence the $\bar{S}_w$: $w \in W$ are the irreducible components of S and their classes $[\bar{S}_w]$ are a basis for the top homology $H_{4r}(S)$ of S (with rational coefficients). By [26, Sect. 5] $H_{4r}(S)$ is a W–W bimodule isomorphic to the two-sided regular representation of W. There is hence a matrix C with coefficients $C(y, w)$: $y, w \in W$ defined by

$$y[\bar{S}_1] = \sum_{w \in W} C(y, w)[\bar{S}_w].$$

Furthermore $C(y, y) = 1$ and $C(y, w) = 0$ unless $y \leqslant w$ [26, Sect. 7]. Thus C is invertible and we let A denote its inverse whose entries we denote by $A(y, w)$. Then A determines the left W module structures of $H_{4r}(S)$ (and as a right W module is determined through the relation $y[\bar{S}_1] = [\bar{S}_1]\, y^{-1}$; [26, Sect. 5].

Conjecture. *For each $w \in W$, one has (up to a non-zero scalar)*

$$p_{\mathscr{V}(w)} = \sum_{y \in W} A(w, y)\, y \rho^m,$$

where m is the least integer $\geqslant 0$ such that the right hand is non-zero. (In particular, $m = \deg p_{\mathscr{V}(w)} = \operatorname{card} R^+ - \dim \mathscr{V}(w)$.)

It would follow from this conjecture that the natural action of W on the $p_{\mathscr{V}(w)}$ coincides with the left action of W on the W gradation of $H_{4r}(S)$ associated to the W filtration [26, 6.1]

$$(H_{4r}(S))^m := \oplus \{\mathbb{C}[\bar{S}_w] : \dim \operatorname{St}^{-1}(w) \leqslant 2m\}.$$

In particular the $p_{\mathscr{V}(w)}$ would generate simple modules isomorphic to the Springer module defined on $\operatorname{St}^{-1}(w)$. In the language of 3.3 all orbits would then be of Springer type and (what is important for us) the $\mathbb{C}p_{\mathscr{V}(w)}$ for distinct $\mathscr{V}(w)$ would form a direct sum in $S(\mathfrak{h})$. Actually we would obtain the following more precise result. Observe [26, Sects. 5, 6] that the left (resp. right) action of W on $H_{4r}(S)$ corresponds to an action of W on the left (resp. right) $\mathscr{B}$ factor. Through the description (9.3–9.5) of irreducible components of each $S(\mathscr{C})$ it follows that the right action of W fixes the irreducible component $\mathscr{V}(w)$ defined by the left factor. Consequently the conjecture must give that up to scalars

$$p_{\mathscr{V}(w)} = \sum_{y \in W} A(w, y)\, y w' \rho^m, \qquad m = \deg p_{\mathscr{V}(w)},$$

for all $w' \in W$. Through the analysis of (II, Sect. 2) it then follows that the $p_{\mathscr{V}(w)}$ generate univalent (II, Sect. 2) W modules. We conclude that $\operatorname{St}(w)$

and hence every nilpotent orbit satisfies condition (B) of Lusztig and Spaltenstein.

9.9. There is an obvious analogy with conjecture 9.8 and the formula (II, 5.1) for the Goldie rank polynomials. In type A_n, Kazhdan and Lusztig conjecture [26, Sect. 7] that C (notation 9.8) coincides with the Jantzen matrix (for integral weights). Through 5.2 and the conjecture this would imply that we have equality in 8.15. In general C should be some degenerate version of the Jantzen matrix, which satisfies similar positivity requirements (cf. [26, Sect. 7]). In type B_2 for example it is known that these positivity requirements do not determine the Jantzen matrix and in fact one can even see from the early calculations of Jantzen [14, 3.16] how to choose C in type B_2 to verify conjecture 9.8. Carrying our analogy further we may conjecture that this matrix determines the order relations between the $\mathscr{V}(w)$ (defined by taking closures) through the notion of a left cell of W introduced in [17] and here defined relative to the basis

$$\mathbf{A}(w) := \sum_{y \in W} A(w, y)\, y, \qquad w \in W,$$

of $\mathbb{C}W$.

9.9. Let $\mathscr{V}$ be an orbital variety and define $\tau_-(\mathscr{V}) = \{\alpha \in B : \mathscr{V} \subset \mathfrak{m}_\alpha\}$. From 9.7 it is immediate that $\tau_-(\mathscr{V}(w)) = \tau_-(w) := \tau(w^{-1}) := S(w^{-1}) \cap B$.

Lemma. *Take $-\mu \in \mathfrak{h}^*$ dominant and regular. Then every component $\mathscr{V}$ of $\mathscr{V}(L(w\mu))$ satisfies $\tau_-(\mathscr{V}) \supset \tau_-(w)$.*

Since $\alpha \in \tau_-(w) \Leftrightarrow s_\alpha w < w$ it follows that $L(w\mu)$ is a quotient of $M(w\mu)/M(s_\alpha w\mu)$ and so $X^k_{-\alpha} \bar{e}_{w\mu} = 0$ for $k \geqslant (\alpha^\vee, w\mu)$. Consequently $\mathscr{V}(L(w\mu)) \subset \mathfrak{m}_\alpha$ which implies the required assertion.

9.10. Since the characteristic polynomials $p_{\mathscr{V}}$ behave via 3.1 rather like the formal character of a highest weight module under coherent continuation (as we already remarked in 6.7) we may apply the Vogan calculus [37, Sect. 3] to their analysis. For simplicity we shall do this just in type A_n and here we can essentially reduce to the integral case (but see 10.2 for the type of corrections required).

9.11. Take λ integral and assume that B is simple of type A_n. Take $w \in W$ and choose adjacent simple roots α, β such that $\alpha \notin \tau_-(w)$, $\beta \in \tau_-(w)$. (This is always possible except in the "trivial" cases $w \in \{\mathrm{Id}, w_B\}$.) Define

$$\begin{aligned} T_{\alpha\beta}(w) &= s_\alpha w, \qquad \beta \notin \tau_-(s_\alpha w) \\ &= s_\beta w, \qquad \text{otherwise.} \end{aligned}$$

LEMMA. $\overline{\mathfrak{m}_\alpha \cap (\mathbf{P}_\alpha \mathscr{V}(w))}$ *admits precisely one component* $\mathscr{V}$ *satisfing* $\alpha \in \tau_-(\mathscr{V})$, $\beta \notin \tau_-(\mathscr{V})$. *Furthermore* $\mathscr{V} = \mathscr{V}(T_{\alpha\beta}(w))$.

Since in type A_n every orbit is of Springer type it follows that the $p_{\mathscr{V}}$ for distinct $\mathscr{V}$ are linearly independent. Set $p = p_{\mathscr{V}(w)}$. By 3.1 and the hypothesis we may write $s_\alpha p = p + p^\alpha + p^{\alpha\beta}$ where p^α (resp. $p^{\alpha\beta}$) denotes the sum $\sum y_i z_i p_{\mathscr{V}_i}$, where $\mathscr{V}_i$ runs over the components of $\overline{\mathfrak{m}_\alpha \cap \mathbf{P}_\alpha \mathscr{V}(w)}$ for which $\alpha \in \tau_-(\mathscr{V}_i)$, $\beta \notin \tau_-(\mathscr{V}_i)$ (resp. $\alpha, \beta \in \tau_-(\mathscr{V}_i)$). Similarly we may write $s_\beta p^\alpha = p^\alpha + q^\beta + q^{\alpha\beta}$. Then $s_\beta s_\alpha p = -p + p^\alpha + q^\beta + q^{\alpha\beta} - p^{\alpha\beta}$. On the other hand, $s_\alpha(s_\beta s_\alpha p) = s_\beta s_\alpha s_\beta p = -s_\beta s_\alpha p$ and consequently the terms in $s_\beta s_\alpha p$ correspond to components $\mathscr{V}_i$ for which $\alpha \notin \tau_-(\mathscr{V}_i)$ must cancel and so $q^\beta = p$. Thus $p^\alpha \neq 0$ which proves that $\overline{\mathfrak{m}_\alpha \cap (\mathbf{P}_\alpha \mathscr{V}(w))}$ has at least one component of the required type. If we write p_i^α to denote the characteristic polynomials of these components, then we have $\sum y_i z_i p_i^\alpha = p^\alpha$ and as in the above we may write $s_\beta p_i^\alpha = p_i^\alpha + q_i^\beta + q_i^{\alpha\beta}$. Then $\sum y_i z_i q_i^\beta = q^\beta = p$ and so $q_i^\beta \in \mathbb{C}p$ for each i. (Recall that $y_i z_i > 0$, and the fact that p is the characteristic polynomial of an irreducible variety.) Now we may write $s_\alpha q_i^\beta = q_i^\beta + r_i^\alpha + r_i^{\alpha\beta}$ and the argument in the first part shows that $r_i^\alpha = p_i^\alpha$. We conclude that the p_i^α are all proportional, so there is exactly one component of the required type. Now suppose $\beta \notin \tau_-(s_\alpha w)$, then we show that $\mathscr{V}(s_\alpha w)$ is the required component. Through the analysis of 9.7 it is enough to show that $\dim \mathscr{V}(s_\alpha w) = \dim \mathscr{V}(w)$ and for this it is enough to observe $\mathscr{V}_1(s_\alpha w) \not\subset \mathfrak{m}_\alpha \cap \mathscr{V}_1(w)$. This follows because otherwise $\mathscr{V}_1(s_\alpha w) \subset \mathscr{V}_1(w) \subset \mathfrak{m}_\beta$ contradictory to the hypothesis $\beta \notin \tau_-(s_\alpha w)$. Finally if $\beta \in \tau_-(s_\alpha w)$, then $\alpha \in \tau_-(s_\beta w)$, $\beta \notin \tau_-(w)$ and so $T_{\alpha\beta}(s_\beta w) = w$. Hence $\mathscr{V}(w)$ is the unique component of $\overline{\mathfrak{m}_\beta \cap \mathbf{P}_\beta \mathscr{V}(s_\beta w)}$ satisfying the first part of the lemma. From the identity $q^\beta = p$ it follows that $\mathscr{V}(s_\beta w)$ is the desired component of $\overline{\mathfrak{m}_\alpha \cap \mathbf{P}_\alpha \mathscr{V}(w)}$.

9.12. Recall 8.15. Assume $\mathfrak{g}$ simple of type A_n.

COROLLARY. *Take* $-\mu \in \mathfrak{h}^*$ *dominant, integral, and regular. Then for each* $w \in W$, $\mathscr{V}(w)$ *is the unique component of* $\mathscr{V}(L(w\mu))$ *such that* $\tau_-(\mathscr{V}(w)) = \tau_-(w)$.

Choose $\alpha, \beta \in B$ as in 9.11 and set $w' = T_{\alpha\beta}(w)$. Let n_w be the number of components of $\mathscr{V}(L(w\mu))$ with the required property. We show that $n_{w'} \geqslant n_w$. Then as $\mathscr{V}(L(w\mu))$ does not depend on the left cell of W to which w belongs we can assume that the right cell containing w also contains $w_{B'}$ for some $B' \subset B$ (by the classification of double cells, cf. [16]). Yet $\mathscr{V}(L(w_{B'}\mu)) = \mathscr{V}(w_{B'})$ and so $n_{w_{B'}} = 1$. Yet by Knuth's theorem (cf. [37, 6.4]) the $T_{\alpha\beta}$ operators generate the right cells. Thus the above inequality gives $n_w \leqslant 1$, $\forall w$ and hence the conclusion of the corollary.

Consider $C_\alpha L(w\mu)$. By Vogan's calculus [37, Sect. 3; 23, 3.2] it follows that this module admits exactly one subquotient $L(w''\mu)$ such that $\alpha \in \tau_-(w'')$, $\beta \notin \tau_-(w'')$ and furthermore one finds that $w'' = w'$. Let $\mathscr{V}(w_i)$: $i = 1, 2, \ldots, n_w$, denote the components of $\mathscr{V}(L(w\mu))$ satisfying the hypothesis of the corollary. Since every other simple quotient $L(y\mu)$ of $C_\alpha L(w\mu)$ satisfies $\alpha, \beta \in \tau_-(y)$ it follows from 6.7 and 9.7 that the $\mathscr{V}(T_{\alpha\beta}(w_i))$ are components of $L(w'\mu)$. Since $\mathscr{V}(T_{\alpha\beta}(w_i)) = \mathscr{V}(T_{\alpha\beta}(w_j)) \Leftrightarrow \mathscr{V}(w_i) = \mathscr{V}(w_j)$ (as in the proof of 9.11, cf. [37, 3.6]) it follows that $n_{w'} \geqslant n_w$, as required.

9.13. We may also use 9.11 to determine which of the $\mathscr{V}(w)$ coincide (in type A_n). Choose w, α, β as in 9.11. Set $w' = T_{\alpha\beta}(w)$.

LEMMA. $\mathscr{V}(w^{-1}) = \mathscr{V}(w'^{-1})$.

Since $T_{\beta\alpha}(T_{\alpha\beta}(w)) = w$, we only have to show this when $T_{\alpha\beta}(w) = s_\alpha w$. Since $S(w) \subset S(s_\alpha w)$ then (cf. 7.5) we have $\mathscr{V}(w^{-1}) \supset \mathscr{V}((s_\alpha w)^{-1})$ trivially. Yet $\dim \mathscr{V}(w) = \dim \mathscr{V}(w^{-1})$ by 9.5(i) (or by 7.6 using $\mathrm{St}(w) = \mathrm{St}(w^{-1})$), whereas $\dim \mathscr{V}(w) = \dim \mathscr{V}(s_\alpha w)$ by 9.11. Consequently, $\dim \mathscr{V}(w^{-1}) = \dim \mathscr{V}((s_\alpha w)^{-1})$ which proves the lemma.

9.14. Let $\Phi: w \mapsto (A(w), B(w))$ denote the Robinson bijection [16, Sect. 2] of

$$W \cong S_{n+1} \rightrightarrows \bigcup_{\xi \in P(n+1)} (\mathbf{St}(\xi) \times \mathbf{St}(\xi)),$$

where $P(n+1)$ denotes the set of partitions of n and for each $\xi \in P(n+1)$, $\mathbf{St}(\xi)$ denote the set of standard tableaux corresponding to ξ.

COROLLARY. *For all* $w, w' \in W$, $B' \subset B$

(i) $\mathscr{V}(w) = \mathscr{V}(w') \Leftrightarrow B(w) = B(w')$.

(ii) $\{\mathscr{V}(w): A(w) = A(w_{B'})\}$ *is the set of irreducible components of* $\mathrm{St}(w_{B'})$.

(i) $\Leftarrow$ obtains from 9.13 using Knuth's theorem (as in 9.12). Then $\Rightarrow$ obtains from 9.6(i), and 3.3 which implies that the total number of distinct orbital varieties is just (in type A_n)

$$\sum_{\sigma \in \hat{W}} \dim \sigma = \sum_{\xi \in P(n+1)} \operatorname{card}(\mathbf{St}(\xi)).$$

(One may alternatively apply the analysis of [37, 6.5] to 9.11.)

Since $\mathrm{St}(w) = \mathrm{St}(w^{-1})$ and $A(w) = B(w^{-1})$ it follows from (i) that the $\{\mathscr{V}(w): w \in \Phi^{-1}(\mathbf{St}(\xi) \times \mathbf{St}(\xi))\}$ generate the same nilpotent orbit which takes the form $\mathrm{St}(w_{B'})$ for some $B' \subset B$, where $w_{B'} \in \Phi^{-1}(\mathbf{St}(\xi) \times \mathbf{St}(\xi))$. Hence (ii).

Remarks. This result in a slightly weaker form is due to Spaltenstein [30, Proposition 9.8]. From it we easily see that $w \mapsto \mathscr{U}(w) \times \mathscr{U}(w^{-1})$: $w \in W$ (notation 9.5) is (in type A_n) just the Robinson bijection. The identification of orbital varieties with standard tableaux which results was first obtained by Spaltenstein [32]. From say [16, Thm. 1; 15, 2.6] and the Spaltenstein–Steinberg equality (3.1) it follows that we have $\dim \mathscr{V}(w) = \dim \mathscr{V}(L(w\mu))$ in 9.12.

10. An Example

10.1. Take $\mathfrak{g}$ simple of type G_2. Set $B = \{\alpha_1, \alpha_2\}$ with α_1 the short root. The nilpotent orbit generated by the short root eigenvector X_{α_1} is eight dimensional and cannot be induced from any proper parabolic subalgebra. It is known that these are exactly two completely prime primitive ideals whose associated variety is the closure of this orbit. These ideals are maximal ideals in the primitive fibres $\mathbf{X}_{\hat{\lambda}}$ (notation I, 1.5) when W_λ is of type $A_1 \times A_1$. Now $\text{card}(W/W_\lambda) = 3$ and so the W_λ-dominant chamber is a union of three W-chambers. Specifically we may take $\lambda = \frac{1}{2}(\omega_1 + \omega_2)$ (where ω_i is the fundamental weight corresponding to α_i) and then $\lambda_1 := s_{\alpha_1}\lambda = -\frac{1}{2}(\omega_1 - 2\omega_2)$ and $\lambda_2 := s_{\alpha_2}\lambda = 2\omega_1 - \frac{1}{2}\omega_2$ are also dominant. On the other hand, the Springer representation associated to the eight-dimensional orbit is two-dimensional and so its intersection with $\mathfrak{n}$ has two irreducible components $\mathscr{V}_1, \mathscr{V}_2$, where we choose $\mathscr{V}_1 = \mathscr{V}(w_B s_{\alpha_1})$.

Lemma. $\mathscr{V}(L(\lambda_i)) = \overline{\mathscr{V}_i}$: $i = 1, 2$, $\mathscr{V}(L(\lambda)) = \overline{\mathscr{V}_1 \cup \mathscr{V}_2}$.

The Goldie polynomials associated to $L(\lambda)$, $L(\lambda_1)$, $L(\lambda_2)$ are, respectively, $p := (\alpha_1 + \alpha_2)(3\alpha_1 + \alpha_2)$, $p_1 := s_{\alpha_1} p = (2\alpha_1 + \alpha_2)\alpha_2$, $p_2 := s_{\alpha_2} p = \alpha_1(3\alpha_1 + 2\alpha_2)$ (cf. II, 6.3). Observe that $p = p_1 + p_2$. Then by 5.2 it is enough to show that $\mathbb{C}p_{\mathscr{V}_i} = \mathbb{C}p_i$: $i = 1, 2$. Now $\mathscr{V}_1 = \overline{\mathbf{B}X_{\alpha_1}}$ and so $\mathscr{V}_1 \subset \mathfrak{m}_{\alpha_2}$. Through 3.1 this gives $s_{\alpha_2} p_{\mathscr{V}_1} = -p_{\mathscr{V}_1}$. Again $\mathscr{V}_2 \not\subset \mathfrak{m}_{\alpha_1}$ and so by 3.1 again $(s_{\alpha_1} - 1) p_{\mathscr{V}_1} = z p_{\mathscr{V}_2}$ for some integer $z \geqslant 0$ which by 3.2 must (in this case) be strictly positive. In particular $s_{\alpha_1} p_{\mathscr{V}_2} = -p_{\mathscr{V}_2}$. Now by 5.2 applied to $L(\lambda_2)$ there exist $l_1, l_2 \in \mathbb{N}$ such that

$$l_1 p_{\mathscr{V}_1} + l_2 p_{\mathscr{V}_2} = p_2.$$

If $l_1 \neq 0$, then applying s_{α_1} to both sides we obtain $s_{\alpha_1} p_{\mathscr{V}_1} = -p_{\mathscr{V}_1}$ which gives the contradiction $z = -2$. Hence $\mathbb{C}p_{\mathscr{V}_2} = \mathbb{C}p_2$. Applying 5.2 to $L(\lambda_1)$ we obtain

$$l_1' p_{\mathscr{V}_1} + l_2' p_{\mathscr{V}_2} = p_1$$

which if $l_2' \neq 0$ gives the contradiction $s_{\alpha_2} p_2 = -p_2$. Hence $\mathbb{C}p_{\mathscr{V}_1} = \mathbb{C}p_1$, as required.

10.2. The above result illustrates two phenomena. First, unlike the situation for the primitive ideals, it can happen that $\mathscr{V}(L(\mu)) \neq \mathscr{V}(L(s_\alpha \mu))$: $\mu \in \mathfrak{h}^*$, $\alpha \in B$ even when $(\alpha^\vee, \mu)$ is not an integer. Secondly that $\mathscr{V}(L(\mu))$: $\mu \in \mathfrak{h}^*$ need not be an irreducible variety. Actually these are both quite common phenomena for non-integral λ, and particularly easy to detect in the case when card $B_\lambda <$ card B through arguments similar to the above. Yet one can hope to show as in the above case that for the appropriate nilpotent orbit $\mathscr{O}$ one can always find a set of $w \in W$ such that the $V(L(w\lambda))$ run through the closures of the irreducible components of $\mathfrak{n} \cap \mathscr{O}$.

APPENDIX: INDEX OF NOTATION

Symbols appearing frequently are given below in order of appearance. (See also [19, 20, 22].)

1.1. $\mathfrak{g}$, $\mathfrak{h}$, λ
1.2. $U(\mathfrak{g})$
1.3. W, W_λ
1.4. $\mathfrak{n}$, $\mathfrak{n}^-$, B, R, $\mathfrak{b}$, $\mathbf{G}$, $\mathbf{B}$, $\mathbf{N}$, $\mathbf{N}^-$, $\mathbf{H}$
1.5. $\mathscr{V}(M)$, $\mathscr{V}\mathscr{A}(M)$
1.7. $I(\mathscr{V})$, $\mathscr{N}$, Sp, $P_{\mathscr{O}}$
1.8. St(w), $\mathscr{V}_1(w)$
2.1. V^*, $S(V)$
2.2. R_M, f_M
2.3. r_M, p_M
2.4. $r_{\mathscr{V}}$, $p_{\mathscr{V}}$
2.5. $\mathbf{N}_\alpha$, $\mathbf{N}_\alpha^-$, $\mathbf{H}_\alpha$, $\mathbf{S}_\alpha$, $\mathfrak{s}_\alpha$, s_α
2.6. $\mathbf{B}_\alpha$, $\mathbf{B}_\alpha^-$
2.9. $\mathfrak{m}_\alpha$, $\mathfrak{m}_\alpha^-$
2.10. $W_{B'}$, $w_{B'}$, $\mathfrak{p}_{B'}$, $\mathfrak{m}_{B'}$, $\mathfrak{r}_{B'}$, $\mathfrak{n}_{B'}$, $\mathbf{G}_{B'}$, Ind $\mathscr{V}$, Ind $\mathscr{O}$
3.1. $\mathbf{P}_\alpha$, $\mathfrak{p}_\alpha$
4.4. $d(M)$
4.7. $\mathscr{R}$
5.1. $\tilde{p}_w$
5.2. e_ξ, $\bar{e}_\xi$, $J(w)$, $\mathscr{W}(w)$
5.5. $\mathscr{S}(M)$
6.2. C_α, $\mathscr{R}_{\hat{\mu}}$
6.6. $C(w)$, $C'(w)$
7.5. $\mathscr{V}_1(w)$, $\mathscr{V}_2(w)$, $\mathscr{V}(w)$, $S(w)$, $\mathfrak{n}_w^-$

8.4. X, Y
8.18. $l_\mu(w)$
9.1. $\mathscr{U}$, $\mathscr{U}_1(w)$, $\mathscr{U}_2(w)$, $\mathbf{Z}_G(u)$, $\mathbf{Z}_G^0(u)$, $\mathbf{A}(u)$
9.2. X_w, Y_w, Z_w
9.3. $\mathscr{B}$, $S(\mathscr{C})$, $S_w(\mathscr{C})$
9.9. τ_-, $\tau(w)$

Note added in proof. I believe that Refs (9, 28) may not now appear; but it is generally believed that the Jantzen conjecture holds even in the non-integral case.

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