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Groups with compact open subgroups and multiplier Hopf *-algebras

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Abstract

For a locally compact group G we look at the group algebras $C_0(G)$ and $C_r^*(G)$, and we let $f \in C_0(G)$ act on $L^2(G)$ by the multiplication operator $M(f)$. We show among other things that the following properties are equivalent:
1. G has a compact open subgroup.
2. One of the C^*-algebras has a dense multiplier Hopf *-subalgebra (which turns out to be unique).
3. There are non-zero elements $a \in C_r^*(G)$ and $f \in C_0(G)$ such that $aM(f)$ has finite rank.
4. There are non-zero elements $a \in C_r^*(G)$ and $f \in C_0(G)$ such that $aM(f) = M(f)a$.
If G is abelian, these properties are equivalent to:
5. There is a non-zero continuous function with the property that both f and $\widehat{f}$ have compact support.

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Introduction

The background for this article is the observation in [11, Section 3] that the algebra $C_c^\infty(G)$ of *smooth* functions on a totally disconnected locally compact group G is a *multiplier*

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*Hopf *-algebra* as defined in [24]. Therefore, it is natural to classify all multiplier Hopf *-algebras which are commutative or co-commutative; this is the same as answering the following question: When does $C_0(G)$ or $C_r^*(G)$ have a dense multiplier Hopf *-algebra? It is well known that the answer is yes if G is compact or discrete and the main results of Sections 3 and 5 are that in general the answer is yes if and only if G has a compact open subgroup. If so, such a multiplier Hopf *-algebra is unique and can be described explicitly as the algebra of polynomial functions on G.

We believe that multiplier Hopf *-algebras in general are the right framework for studying totally disconnected locally compact quantum groups (what ever that is) and it is therefore natural to first give a complete account for $C_0(G)$ and $C_r^*(G)$.

$C_0(G)$ is treated in Section 3, the main tool is in Corollary 1.3 where it is shown that the existence of functions satisfying some algebraic relations is equivalent to the existence of a compact open subgroup.

To get the same results for $C_r^*(G)$ is a little more tricky. For $C_0(G)$ one can quickly show that elements of a multiplier Hopf *-subalgebra must have compact support and are therefore integrable. It is not so easy to show that elements of a multiplier Hopf *-subalgebra of $C_r^*(G)$ automatically are integrable with respect to the Haar–Plancherel weight. However, when this is proved, the results for $C_r^*(G)$ follow from those of $C_0(G)$. In fact, we show that if $\mathscr{A}$ is the unique dense multiplier Hopf *-algebra of $C_0(G)$ and L denotes the left regular representation, then $\{L(f) | f \in \mathscr{A}\}$ is the unique dense multiplier Hopf *-algebra of $C_r^*(G)$.

For $C^*(G)$ the situation is different. Here, the existence of a multiplier Hopf *-subalgebra does not imply that G has a compact open subgroup. However, the corresponding uniqueness result is true.

The algebras $C_0(G)$ and $C_r^*(G)$ are dual as locally compact quantum groups and we shall also see that many properties of this duality are equivalent with the existence of a compact open subgroup.

It is well known that if $a \in C_r^*(G)$, $f \in C_0(G)$ and $M(f)$ is the corresponding multiplication operator on $L^2(G)$, then $aM(f)$ is compact. Since the finite-rank operators are dense in the algebra of compact operators it is natural to ask when $aM(f) \neq 0$ is of finite rank. We show this is possible if and only if G has a compact open subgroup.

It is a consequence of the Heisenberg relations that if $G = \mathbb{R}^n$ then a and $M(f)$ as above never commute unless one of them is zero. We show that in general $aM(f) = M(f)a \neq 0$ is possible if and only if G has a compact open subgroup.

Finally, as a bonus for the patient reader we look at the case where G is abelian and ask whether one can have $f \in C_c(G)$, $f \neq 0$ and $\widehat{f} \in C_c(\widehat{G})$. The answer should not surprise.

Some of the results here are probably folklore and known to those who have worked with representations of p-adic groups.

1. Preliminaries

We start with fixing our notation regarding function spaces on G.

Definition 1.1. As usual $C_0(G)$ are the continuous complex functions on G vanishing at ∞ and $L^p(G)$ is defined with respect to a fixed left Haar measure μ. We will often write just

$$\int f\,\mathrm{d}\mu = \int f(x)\,\mathrm{d}x.$$

The left and right action of G on such functions is given by

$$_x f(y) = f(x^{-1}y), \quad f_x(y) = f(yx). \tag{1.1}$$

However, for functions in $L^2(G)$ we will instead use the left and right regular representations given by

$$L_x f(y) = f(x^{-1}y), \quad R_x f(y) = \Delta_G(x)^{1/2} f(yx) \tag{1.2}$$

where Δ_G is the modular function on G.

Many of the arguments are based on the following lemma:

Lemma 1.2. *Suppose G is a locally compact group with a continuous action α by isometries on a Banach space A. For a fixed non-zero vector $a \in A$ and any subset $U \subset G$, denote by $F_U(a)$ the linear span of the set $\{\alpha_x(a) \mid x \in U\}$.*

- (i) *Suppose that $F_U(a)$ is finite dimensional for a neighborhood U of e. Then G has an open subgroup H s.t. $F_H(a)$ is also finite dimensional.*
- (ii) *The same conclusion holds if we have a non-negligible set C s.t. $F_C(a)$ is finite dimensional.*
- (iii) *If one further assumes that the functions $x \mapsto \langle \alpha_x(a), \phi \rangle$ are in $C_0(G)$ for $a \in A$, $\phi \in A^*$, then the subgroup H is also compact.*

Proof. Clearly $V \subset U$ implies that $F_V(a) \subset F_U(a)$. To prove (i), take a neighborhood U s.t. $F_U(a)$ has minimal, positive dimension. Take a neighborhood V of e s.t. $V = V^{-1}$ and $V^2 \subset U$. Then $F_V(a) = F_U(a)$ is invariant by the open subgroup H generated by V and therefore $F_V(a) = F_H(a)$.

For (ii), take a measurable set C with finite Haar measure $\mu(C) > 0$ s.t. $F_C(a)$ has minimal, positive dimension. Clearly $U = \{y \in G \mid \mu(y^{-1}C \cap C) > 0\}$ is a neighborhood of e which satisfies

$$y \in U \Rightarrow F_C(a) = F_{y^{-1}C\cap C}(a).$$

Therefore, the non-trivial linear space $F_C(a)$ is invariant by the open subgroup H generated by U and also here $F_C(a) = F_H(a)$ is finite dimensional.

As for (iii), let $\{b_i\}$ be a basis for $F_H(a)$, let $\phi_j \in A^*$ s.t. $\langle b_i, \phi_j \rangle = \delta_{ij}$ and define $\psi_{ji}(z) = \langle \alpha_z(b_i), \phi_j \rangle$. Then $\psi_{ji} \in C_0(G)$ and for $y, z \in H$:

$$\alpha_z(b_i) = \sum_k \psi_{ki}(z) b_k,$$

$$\psi_{ji}(yz) = \sum_k \psi_{ki}(z)\langle \alpha_y(b_k), \phi_j\rangle$$
$$= \sum_k \psi_{jk}(y)\psi_{ki}(z).$$

So $1 = \psi_{11}(yy^{-1}) = \sum_k \psi_{1k}(y)\psi_{k1}(y^{-1})$ is constant on H and in $C_0(G)$, so H must be compact. □

Corollary 1.3. *If $f, g, f_i, g_i \in C_0(G)$ are non-zero functions s.t.*

$$f(xy)g(y) = \sum_1^n f_i(x)g_i(y) \quad \textit{for all } x, y \in G, \tag{1.3}$$

then G has a compact open subgroup H and there are functions $f'_j \in C_0(G)$ and $g'_j \in C(H)$ s.t.

$$f(xy) = \sum_1^n f'_j(x)g'_j(y) \quad \textit{for all } x \in G, \quad y \in H. \tag{1.4}$$

Proof. Pick a non-empty open subset U of G with $g(y) \neq 0$ for $y \in U$, and we may assume $e \in U$. We then have

$$f_y = \sum h_i(y) f_i, \quad \text{where } h_i(y) = g_i(y)/g(y) \quad \text{for } y \in U,$$

so the result follows from Lemma 1.2 by taking $A = C_0(G)$ and the action given by $\alpha_y(f) = f_y$. □

Lemma 1.4. *Suppose that we have a continuous action of G as in Lemma* 1.2, *that we have a finite set of non-zero elements $a, a_i \in A$, and functions g, g_i s.t.*

$$g(y)\alpha_y(a) = \sum g_i(y)a_i \quad \textit{for } y \in G. \tag{1.5}$$

Then g has compact support.

Proof. We may assume that $\{a_i\}$ are linearly independent, so pick $v_i \in A^*$ s.t. $\langle a_i, v_j\rangle = \delta_{ij}$. Then,

$$g_i(y) = g(y)\langle \alpha_y(a), v_i\rangle,$$
$$g(y)\alpha_y(a) = g(y)\sum \langle \alpha_y(a), v_i\rangle a_i,$$
$$g(y)a = g(y)\sum \langle \alpha_y(a), v_i\rangle \alpha_{y^{-1}}(a_i).$$

Pick $v_0 \in A^*$ with $\langle a, v_0\rangle = 1$, then

$$g(y) = g(y)\sum \langle \alpha_y(a), v_i\rangle \langle \alpha_{y^{-1}}(a_i), v_0\rangle = g(y)c(y),$$

with $c \in C_0(G)$, which is possible only if g has compact support. □

Corollary 1.5. *With the assumptions in Corollary* 1.3, $g \in C_c(G)$.

We shall also need the following:

Lemma 1.6. *Suppose G contains a subgroup H, that A is a vector space and that we have functions $g, g_i : G \mapsto A$ and $f_i : G \mapsto \mathbb{C}$ s.t.*

$$\chi_H(xy)g(y) = \sum_1^n f_i(x)g_i(y) \quad \textit{for all } x, y \in G. \tag{1.6}$$

Then g has finite support in G/H, i.e. there is a finite set F s.t. $g(y) = 0$ for $y \notin FH$.

Proof. We may assume that the set $\{f_i\}$ is linearly independent, so by [9, (28.14)] there is a finite set $F_1 = \{x_i\}$ s.t. the matrix $\{f_i(x_j)\}$ is invertible. Then $y \notin F_1^{-1}H \Rightarrow g_i(y) = 0$ and therefore also $g(y) = 0$. So we can take $F = F_1^{-1}$. □

Corollary 1.7. *Suppose the functions $f, g, f_i, g_i \in L^2(G)$ satisfy*

$$f(x)g(x^{-1}y) = \sum_1^n f_i(x)g_i(y) \quad \textit{for almost all } x, y \in G. \tag{1.7}$$

Then G has a compact open subgroup H and there are functions $f'_j \in C(H)$ and $g'_j \in L^2(G)$ s.t.

$$g(x^{-1}y) = \sum_1^m f'_j(x)g'_j(y) \quad \textit{for } x \in H, \quad y \in G. \tag{1.8}$$

Proof. Pick a set C such that $0 < \mu(C) < \infty$ and $f(x) \neq 0$ for $x \in C$. Then divide by $f(x)$ and apply part (ii) of Lemma 1.2 with $A = L^2(G)$ and the action given by L_x. □

Remark 1.8. The Kac–Takesaki operator on $L^2(G \times G)$ is defined by $Wf(x, y) = f(x, x^{-1}y)$. Therefore, (1.8) is equivalent to $W(f \otimes g) = \sum_1^n f_i \otimes g_i$. The reader may want to compare this with [2, Definition 1.8].

Remark 1.9. Our main results so far have been that the existence of certain functions satisfying (1.3) is possible only if G has a compact open subgroup. Note, however, that this conclusion is possible only with some restriction on the functions involved. On $\mathbb{R}$ or matrix groups like $GL(n, \mathbb{C})$ one clearly has unbounded functions satisfying (1.3), but there are no compact open subgroups.

Remark 1.10. As we shall see later, the conditions studied here are in fact equivalent to the existence of a compact open subgroup H. For the opposite implication, just take $f = g = \chi_H$.

2. Multiplier Hopf *-algebras

Multiplier Hopf *-algebras were introduced in [26], in this section we shall recall some of the main definitions and results and refer to [24–26] or [13, Appendix] for more precise statements.

Let A be a *-algebra over $\mathbb{C}$, with or without identity, but with a non-degenerate product. The *multiplier algebra* $M(A)$ can be characterized as the largest algebra with identity in which A sits as an essential two-sided ideal. We always let $A\otimes A$ denote the algebraic tensor product. A *comultiplication* (or a *coproduct*) on A is a non-degenerate *-homomorphism $\Delta : A \to M(A \otimes A)$ such that $\Delta(a)(1 \otimes b)$ and $(a \otimes 1)\Delta(b)$ are elements of $A \otimes A$ for all $a, b \in A$. It is assumed to be *coassociative* in the sense that $(\Delta \otimes \iota) \circ \Delta(a) = (\iota \otimes \Delta) \circ \Delta(a)$ inside $M(A \otimes A \otimes A)$; where ι denotes the identity map, see [24] for a more precise definition.

Definition 2.1. A pair (A, Δ) of a *-algebra A over $\mathbb{C}$ with a non-degenerate product and a comultiplication Δ on A is called a *multiplier Hopf* **-algebra* if the linear maps from $A \otimes A$ defined by

$$a \otimes b \to \Delta(a)(1 \otimes b), \tag{2.1}$$

$$a \otimes b \to (a \otimes 1)\Delta(b) \tag{2.2}$$

are injective with range equal to $A \otimes A$.

For any multiplier Hopf *-algebra, there is a *counit* $\varepsilon : A \mapsto \mathbb{C}$ which is the unique *-homomorphism satisfying

$$(\varepsilon \otimes \iota)(\Delta(a)(1 \otimes b)) = ab, \tag{2.3}$$

$$(\iota \otimes \varepsilon)((a \otimes 1)\Delta(b)) = ab \tag{2.4}$$

for all $a, b \in A$. There is also an *antipode* which is the unique linear map $S : A \mapsto A$ satisfying

$$m(S \otimes \iota)(\Delta(a)(1 \otimes b)) = \varepsilon(a)b, \tag{2.5}$$

$$m(\iota \otimes S)((a \otimes 1)\Delta(b)) = \varepsilon(b)a, \tag{2.6}$$

where m denotes multiplication defined as a linear map from $A \otimes A$ to A. The antipode is an injective anti-homomorphism and satisfies the relation $S(a^*) = S^{-1}(a)^*$ for all $a \in A$.

Definition 2.2. A *right integral* on A is a linear functional μ s.t.

$$(\mu \otimes \iota)(\Delta(a)(1 \otimes b)) = \mu(a)b.$$

In general a right integral may not exist, but if it does there is also a left integral (defined in a similar way). This will always be true in the cases we are studying. A multiplier Hopf *-algebra with a positive right integral is called an *algebraic quantum group*, (which should not be confused with a quantization of an algebraic group).

A multiplier Hopf *-algebra has local units in the following sense:

Lemma 2.3. *If $\mathscr{A}$ is a multiplier Hopf *-algebra and F is a finite subset of $\mathscr{A}$, then there is $b \in \mathscr{A}$ s.t. $ab = a$ for all $a \in F$.*

Proof. See [6, Proposition 2.2]. □

3. Multiplier Hopf *-algebras in $C_0(G)$

We start with $C_0(G)$ where the comultiplication, antipode and counit is given by

$$\Delta(f)(x, y) = f(xy), \quad S(f)(x) = f(x^{-1}), \quad \varepsilon(f) = f(e)$$

and a Haar integral is given by $f \mapsto \int f \, d\mu$ where μ as before is a left Haar measure on G.

Standing Hypothesis 3.1. Let $\mathscr{A}$ be a *-subalgebra of $C_0(G)$ which is also invariant under the antipode S. We also assume that

$$\text{span}\{\Delta(f)(1 \otimes g) \mid f, g \in \mathscr{A}\} = \mathscr{A} \otimes \mathscr{A}.$$

It can then be shown that $\mathscr{A}$ is a multiplier Hopf *-algebra with the coproduct inherited from $C_0(G)$. We call $\mathscr{A}$ *a multiplier Hopf*-subalgebra* of $C_0(G)$. It is actually not necessary to assume that $\mathscr{A}$ is invariant under the antipode S, for details on all this see [5].

The main results in this section is that such a multiplier Hopf *-subalgebra $\mathscr{A}$ of $C_0(G)$ exists only if G has a compact open subgroup. If $\mathscr{A}$ also is dense in $C_0(G)$, it is unique and can be described.

Definition 3.2. If H is a compact group, define $\mathscr{P}(H)$= all polynomial functions on H, so $f \in \mathscr{P}(H) \Leftrightarrow \exists f_i, g_i \in C(H)$ s.t.

$$f(hk) = \sum_1^n f_i(h) g_i(k) \quad \text{for } h, k \in H. \tag{3.1}$$

Thus $\mathscr{P}(H)$ equals the matrix functions corresponding to finite-dimensional (unitary) representations of H.

Suppose G is a locally compact group and that H is a compact open subgroup. Then $C(H) \subset C_c(G)$ in an obvious way, this is used next.

Lemma 3.3. *The following are equivalent*:

(i) $f \in \text{span}\{_x\phi \mid x \in G, \ \phi \in \mathscr{P}(H)\}$.
(ii) $\exists f_i \in C_c(G), \ \phi_i \in C(H)$ *s.t.* $f(xh) = \sum_1^n f_i(x)\phi_i(h)$ *for* $x \in G, h \in H$.

Proof. (i) ⇒ (ii): If $\phi \in \mathscr{P}(H)$ satisfies (3.1) and $f = {}_x\phi$, then for $y \in G, h \in H$:

$$f(yh) = \chi_{xH}(yh)\phi(x^{-1}yh) = \chi_{xH}(y) \sum f_i(x^{-1}y) g_i(h). \tag{3.2}$$

So f satisfies (ii).

(ii) $\Rightarrow$ (i): If f satisfies (ii) and has support in $\bigcup_1^m x_i H$ we have $f = \sum \chi_{x_i H} f$. So we may suppose $f = {}_x\phi$ and have to show that $\phi \in \mathscr{P}(H)$. But this follows from $\phi(hk) = f(xhk) = \sum f_i(xh) g_i(k)$. $\square$

Lemma 3.4. *If f satisfies* (ii) *above, f_i, ϕ_i can be chosen s.t. also f_i satisfies* (ii) *and $\phi_i \in \mathscr{P}(H)$.*

Proof. First note that we can assume that $\{\phi_i\}$ is an orthonormal set in $L^2(H)$, so

$$f_i(x) = \int_H f(xh)\overline{\phi_i(h)}\,\mathrm{d}h,$$

$$f_i(xk) = \int_H f(xkh)\overline{\phi_i(h)}\,\mathrm{d}h = \sum_j f_j(x) \int_H \phi_j(kh)\overline{\phi_i(h)}\,\mathrm{d}h,$$

therefore f_i satisfies (ii). Since span$\{f_i\}$ is R_H-invariant and ϕ_i are the matrix functions with respect to this finite-dimensional representation, it follows that $\phi_i \in \mathscr{P}(H)$. $\square$

Lemma 3.5. *Suppose H, K are two compact open subgroups of G. Then f satisfies the conditions of Lemma* 3.3 *with respect to H if and only if it does for K.*

Proof. It is enough to show that the statement is true if $H \subset K$. Using Lemma 3.3 it further suffices to show that under the natural embedding of $C(H)$ into $C(K)$ we have $\mathscr{P}(H)$ mapped into $\mathscr{P}(K)$. If $K = \bigcup_1^n k_i H$ we have

$$\chi_H(kl) = \sum_1^n \chi_H(kk_i)\chi_H(k_i^{-1}l) \quad \text{for } k, l \in K. \tag{3.3}$$

Now suppose $f \in \mathscr{P}(H)$, so $f(hk) = \sum_1^m f_i(h) g_i(k)$ for $h, k \in H$. Then we have for $k, l \in K$ that

$$f(kl) = \chi_H(kl) f(kl) = \sum \chi_H(kk_i)\chi_H(k_i^{-1}l) f(kk_ik_i^{-1}l)$$

$$= \sum \chi_H(kk_i)\chi_H(k_i^{-1}l) f_j(kk_i) g_j(k_i^{-1}l)$$

which shows that $f \in \mathscr{P}(K)$. $\square$

Definition 3.6. *The polynomial functions* on G is the space $\mathscr{P}(G)$ of all functions $f \in C_c(G)$ satisfying the conditions of Lemma 3.3 for some (hence all) compact open subgroups of G.

Theorem 3.7. *Suppose G has a compact open subgroup. Then $\mathscr{P}(G)$ is a multiplier Hopf *-subalgebra of $C_0(G)$ separating points of G which is invariant under the left and right action given by $f \mapsto {}_x f$ and $f \mapsto f_x$.*

Proof. If $f, g \in \mathscr{P}(G)$ there is a compact open subgroup s.t. the conditions of Lemma 3.3 hold for both. The same subgroup then holds for both $f + g$ and fg.

The antipode in $C_0(G)$ is given by $S(f)(y) = f(y^{-1})$. If $f={}_x\phi$ with $\phi \in \mathscr{P}(H)$, then $S(f)={}_{x^{-1}}\psi$ where $\psi \in \mathscr{P}(xHx^{-1})$ is given by $\psi(y) = \phi(x^{-1}y^{-1}x)$. So $\mathscr{P}(G)$ is S-invariant. $\mathscr{P}(G)$ is obviously invariant under $f \mapsto {}_xf$, and if f satisfies Lemma 3.3 with respect to H, then f_x satisfies Lemma 3.3 with respect to xHx^{-1}.

We next have to show that $\Delta(\mathscr{P}(G))(\mathscr{P}(G)\otimes 1) = \mathscr{P}(G)\otimes\mathscr{P}(G)$ etc. If $f={}_x\phi$ and $g={}_y\psi$ with $\phi, \psi \in \mathscr{P}(H)$, then the function

$$h(s,t) := \Delta(f)(g\otimes 1)(s,t) = \phi(x^{-1}st)\psi(y^{-1}s) \tag{3.4}$$

has support inside $yH \times Hy^{-1}xH$. By compactness we get $Hy^{-1}xH = \bigcup_1^m h_i y^{-1}xH$, take $z_i = h_i y^{-1}x$ and $K = H\bigcap_i z_i H z_i^{-1}$. Suppose $\Delta(\phi) = \sum_1^n \alpha_j \otimes \beta_j$, then

$$\begin{aligned} h(s,t) &= \sum_i \chi_{z_iH}(t)\phi(x^{-1}st)\psi(y^{-1}s) \\ &= \sum_i \chi_H(z_i^{-1}t)\chi_H(x^{-1}st)\phi(x^{-1}st)\psi(y^{-1}s) \\ &= \sum_i \chi_H(z_i^{-1}t)\chi_H(x^{-1}sz_i)\phi(x^{-1}sz_iz_i^{-1}t)\psi(y^{-1}s) \\ &= \sum_{i,j} \chi_H(z_i^{-1}t)\chi_H(x^{-1}sz_i)\alpha_j(x^{-1}sz_i)\beta_j(z_i^{-1}t)\psi(y^{-1}s). \end{aligned}$$

From this it follows that $h \in C_0(G)\otimes C_0(G)$. An easy computation shows that $h \in \mathscr{P}(G)\otimes\mathscr{P}(G)$ with respect to the subgroup $K\times K$. So we have therefore proved that $\Delta(\mathscr{P}(G))(\mathscr{P}(G)\otimes 1) \subset \mathscr{P}(G)\otimes\mathscr{P}(G)$. Since the inverse of the map $a\otimes b \mapsto (a\otimes 1)\Delta(b)$ is given by $a\otimes b \mapsto (a\otimes 1)(S\otimes\iota)\Delta(b)$ it follows that $\Delta(\mathscr{P}(G))(\mathscr{P}(G)\otimes 1)=\mathscr{P}(G)\otimes\mathscr{P}(G)$. We leave other details to the reader. □

Proposition 3.8. *$f \in \mathscr{P}(G)$ if and only if $f \in C_0(G)$ and there are non-zero functions $g, h, f_i, g_i, f'_j, g'_j \in C_0(G)$ s.t. for all $x, y \in G$*

(i) $f(xy)g(y) = \sum_1^m f_i(x)g_i(y)$ *and*
(ii) $f(y)h(xy) = \sum_1^n f'_j(x)g'_j(y)$.

Proof. If $f \in \mathscr{P}(G)$ it follows from (3.1) that (i) and (ii) hold with $g=h=\chi_H$. Conversely if (i) holds, it follows from Corollary 1.3 that there is a compact open subgroup H and functions $h_i \in C_0(G)$ and $k_i \in C(H)$ s.t. $f(xy) = \sum_1^n h_i(x)k_i(y)$ for $y \in H$. Finally, from Corollary 1.5 we have $f \in C_c(G)$, hence $f \in \mathscr{P}(G)$. □

The next characterization $\mathscr{P}(G)$ will also be useful.

Proposition 3.9. *Suppose G has a compact open subgroup H. Then $f \in \mathscr{P}(G)$ if and only if there are finitely many functions $f_i, g_i, f'_j, g'_j \in C_0(G)$ s.t.*

(i) $f(xy)\chi_H(y) = \sum_1^m f_i(x)g_i(y)$ *and*
(ii) $f(y)\chi_H(xy) = \sum_1^n f'_j(x)g'_j(y)$.

Proof. If f satisfies (ii) it follows from Lemma 1.6 that $f = \sum_1^n {}_{x_i}\phi_i$ with $\phi_i \in C(H)$ and the sets $\{x_i H\}$ disjoint. We want to show that (i) implies that $\phi_i \in \mathscr{P}(H)$:

$$\phi_i(hk) = f(x_i^{-1}hk) = \sum_1^n f_j(x_i^{-1}h)g_j(k) \quad \text{for all } h, k \in H, \tag{3.5}$$

so $\phi_i \in \mathscr{P}(H)$ and $f \in \mathscr{P}(G)$ by Lemma 3.3.

Conversely, if $f = {}_z\phi$ with $\phi \in \mathscr{P}(H)$ one checks that f satisfies (i) and (ii), so again by Lemma 3.3 this is true for any $f \in \mathscr{P}(G)$. □

Remark 3.10. Note that both (i) and (ii) are needed in general to characterize $\mathscr{P}(G)$: If G is discrete and $H = \{e\}$ then (i) is automatic, if G is compact and $H = G$ then (ii) is automatic.

Theorem 3.11. *Suppose $\mathscr{A}$ is a multiplier Hopf*-subalgebra of $C_0(G)$ separating points. Then G contains a compact open subgroup H and $\mathscr{A} = \mathscr{P}(G)$.*

Proof. It follows from Corollary 1.3 and Proposition 3.8 that G contains a compact open subgroup H and that $\mathscr{A} \subset \mathscr{P}(G)$.

Claim 1. *If v is a measure on G with compact support and $f \in \mathscr{A}$, then $f * v \in \mathscr{A}$.*

Let C be the support of v. Since $\mathscr{A}$ separates points in G there is $g \in \mathscr{A}$ s.t. $g(y) > 0$ for $y \in C$. There are functions $f_i, g_i \in \mathscr{A}$ s.t. for $y \in C$ we have

$$f(xy^{-1})g(y) = \sum_1^n f_i(x)g_i(y),$$

$$f(xy^{-1}) = \sum_1^n f_i(x)g_i(y)/g(y),$$

$$f * v(x) = \sum_1^n f_i(x)v(g_i/g).$$

So $f * v \in \mathscr{A}$. In particular this means that $\mathscr{A}$ is invariant under $f \mapsto f_x$ and therefore also under $f \mapsto {}_x f$. Moreover, it follows that if $f \in \mathscr{P}(G)$ and $g \in \mathscr{A}$ then $f * g \in \mathscr{A}$.

Claim 2. $\chi_H \in \mathscr{A}$.

By Stone–Weierstrass $\|f - \chi_H\|_\infty < \varepsilon < 1/2$ for some positive function $f \in \mathscr{A}$. Then by Claim 1 $g = f * \chi_H \in \mathscr{A} \cap C_c(G/H)$ and $\|g - \chi_H\|_\infty < \varepsilon$. The support of g equals $\bigcup_{i=0}^N x_i H$ with $x_0 = e$. Take $\alpha_i = g(x_i)$ and define

$$\phi(x) = g(x)\prod_{i=1}^n[\alpha_0 g(x) - \alpha_i g(x^{-1}x_i)]. \tag{3.6}$$

Then $\phi \in \mathscr{A}$, we have $\phi(x_i) = 0$ for $i \neq 0$, $\phi(e) = \alpha_0\prod_1^n[\alpha_0^2 - \alpha_i^2] \neq 0$. So $\phi = \phi(e)\chi_H$, hence $\chi_H \in \mathscr{A}$.

Claim 3. *If $f \in \mathscr{P}(H)$ there is $g \in \mathscr{A}$ s.t. $f = g\,|_H$.*

By taking a minimal decomposition with $f_i, g_i \in C(H)$ s.t.

$$f(hk) = \sum_1^n f_i(h)g_i(k) \quad \text{for } h, k \in H,$$

we may assume that $\{g_i\}$ is orthonormal and that $\{f_i\}$ is linearly independent. Since $\mathscr{A}$ is dense in $C_0(G)$ there are $h_i \in \mathscr{A}$ s.t.

$$\int_H g_i(k)h_j(k^{-1})\,\mathrm{d}k = \delta_{ij}.$$

Then $f * h_i$ is in $\mathscr{A}$ and for $h \in H$

$$\begin{aligned} f * h_i(h) &= \int_H f(hk)h_i(k^{-1})\,\mathrm{d}k \\ &= \int_H \sum_j f_j(h)g_j(k)h_i(k^{-1})\,\mathrm{d}k = f_i(h). \end{aligned}$$

From this it follows that

$$f(h) = \sum_j f_j(h)g_j(e) = \sum_j f * h_j(h)g_j(e)$$

which proves the claim.

Finally it follows from Claim 2 + 3 that $\mathscr{P}(H) \subset \mathscr{A}$, and then from Lemma 3.3 that $\mathscr{P}(G) \subset \mathscr{A}$. $\square$

4. Totally disconnected groups

It is natural now to look these groups since they have a basis of neighborhoods of e consisting of compact open subgroups. In addition, it was our discovery that the *smooth* functions on G is a multiplier Hopf *-algebra that started this work.

Definition 4.1. If G is a totally disconnected group, define the *smooth* functions on G by

$$\begin{aligned} C_c^\infty(G) &= \cup\{C_c(G/H)\,|\,H \text{ a compact open subgroup}\} \\ &= \operatorname{span}\{\chi_{xH}\,|\,x \in G,\ H \text{ a compact open subgroup}\} \\ &= \operatorname{span}\{\chi_{xHy}\,|\,x, y \in G,\ H \text{ a compact open subgroup}\}. \end{aligned}$$

Theorem 4.2. *If G is a totally disconnected group, $C_c^\infty(G) = \mathscr{P}(G)$.*

Proof. If H is a compact open subgroup then $\chi_H \in \mathscr{P}(G)$, and since both $C_c^\infty(G)$ and $\mathscr{P}(G)$ are translation invariant $C_c^\infty(G) \subset \mathscr{P}(G)$.

To prove the converse, for the same reason it suffices to show that $\mathscr{P}(H) \subset C_c^\infty(G)$. So suppose $f \in \mathscr{P}(H)$ satisfies

$$f(hk) = \sum_1^n f_j(h) g_j(k) \quad \text{for all } h, k \in H \tag{4.1}$$

and we may assume that $\{g_j\}$ is an orthonormal set in $L^2(H)$. Then as in the proof of Lemma 3.4 we get that $\{f_j\}$ is R_x-invariant for $x \in H$. This way we get a finite-dimensional representation of H on $X = \text{span}\{f_j\}$. Since H is totally disconnected, by [9, (28.19)] there is a compact open subgroup K s.t. $R_k = I$ on X for $k \in K$. This means that f_j and therefore also $f \in C(H/K) \subset C_c^\infty(G)$. □

Remark 4.3. Note that if G is totally disconnected $C_c^\infty(G)$ equals the space of regular functions as defined by Bruhat in [3], but in general these spaces are different. For more about functions on totally disconnected groups, see also [18, Chapter 1.1].

5. Multiplier Hopf *-algebras in $C_r^*(G)$

Definition 5.1. We have already defined the left and right regular representations of G on $L^2(G)$ in Definition 1.1. For $f \in L^1(G)$ let

$$L_f = \int f(x) L_x \, \mathrm{d}x, \quad R_f = \int f(x) R_x \, \mathrm{d}x.$$

Then $C_r^*(G)$ is defined as the norm closure of $\{L_f \mid f \in L^1(G)\}$. It is standard that $L_x \in M(C_r^*(G))$ and we shall often identify an element $x \in G$ with L_x. We shall also need the weak closures

$$\mathscr{L}(G) := \{L_g \mid g \in G\}'' \quad \text{and} \quad \mathscr{R}(G) := \{R_g \mid g \in G\}''.$$

The comultiplication on $C_r^*(G)$ is defined by

$$\Delta(L_f) = \int f(x)(L_x \otimes L_x) \, \mathrm{d}x$$

for $f \in L^1(G)$ and can be extended to a non-degenerate *-homomorphism $C_r^*(G) \mapsto M(C_r^*(G) \otimes C_r^*(G))$, see [23, Proposition 4.3] or (in a more general setting) [12, (3.2)].

The antipode and counit are given by

$$S(L_f) = \int \Delta_G(x^{-1}) f(x^{-1}) L_x \, \mathrm{d}x, \quad \varepsilon(L_f) = \int f(x) \, \mathrm{d}x,$$

where Δ_G is the modular function of G. A left Haar integral is given by $w_G(L_f) = f(e)$.

The antipode S can be extended to $C_r^*(G)$, but not the counit ε. There is an extension of w_G to an (unbounded) weight on $C_r^*(G)$. For more details we refer to [14, Chapter 7.2].

We shall also need the modular automorphism group corresponding to this weight, it will satisfy

$$\sigma_t(L_f) = \int \Delta_G(x)^{it} f(x) L_x \,\mathrm{d}x.$$

As usual we also use the notation

$$\mathcal{N}_{w_G} = \{a \mid w_G(a^*a) < \infty\}, \quad \mathcal{M}_{w_G} = \mathrm{span}\{a^*b \mid a, b \in \mathcal{N}_{w_G}\}.$$

Standing Hypothesis 5.2. Let $\mathcal{A}$ be a *-subalgebra of $C_r^*(G)$ which is also invariant under the antipode S. We also here assume that

$$\mathrm{span}\{\Delta(a)(1 \otimes b) \mid a, b \in \mathcal{A}\} = \mathcal{A} \otimes \mathcal{A}.$$

It follows that $\mathcal{A}$ is a multiplier Hopf *-algebra with the coproduct inherited from $C_r^*(G)$. We call $\mathcal{A}$ *a multiplier Hopf* **-subalgebra* of $C_r^*(G)$. As in 3.1 it is actually not necessary to assume that $\mathcal{A}$ is invariant under the antipode S, for details see [5].

First, we address some properties which are not so easy to prove as for $C_0(G)$. We saw in Section 1 that elements of a multiplier Hopf *-subalgebra of $C_0(G)$ must have compact support and are therefore automatically integrable with respect to Haar measure. We shall see that the similar result is somewhat more complicated in $C_r^*(G)$.

Proposition 5.3. *Let $\mathcal{A}$ be a multiplier Hopf*-subalgebra of $C_r^*(G)$. Then $\mathcal{A}$ is σ-invariant and every element $a \in \mathcal{A}$ is analytic with respect to the modular automorphism group σ_t of the weight w_G.*

Proof. For $a, b \in \mathcal{A}$ we have elements $a_i, b_i \in \mathcal{A}$ s.t.

$$a \otimes b = \sum_1^n \Delta(a_i)(1 \otimes b_i). \tag{5.1}$$

Since $\sigma_t(L_x) = \Delta_G(x)^{it} L_x$, we have $(\sigma_t \otimes \sigma_{-t}) \circ \Delta = \Delta$ and

$$\sigma_t(a) \otimes \sigma_{-t}(b) = \sum_1^n \Delta(a_i)(1 \otimes \sigma_{-t}(b_i)). \tag{5.2}$$

Multiply with $1 \otimes b^*$ to get

$$\sigma_t(a) \otimes b^*\sigma_{-t}(b) = \sum_1^n (1 \otimes b^*)\Delta(a_i)(1 \otimes \sigma_{-t}(b_i)). \tag{5.3}$$

Since $a_i, b_i \in \mathcal{A}$, we have $(1 \otimes b^*)\Delta(a_i) = \sum c_{ij} \otimes d_{ij}$, where the sum is finite and the set $\{c_{ij}\}$ is linearly independent. Take $V_0 = \mathrm{span}\{c_{ij}\}$, then $(1 \otimes b^*)\Delta(a_i) \in V_0 \otimes \mathcal{A}$ and also $\sigma_t(a) \otimes b^*\sigma_{-t}(b) \in V_0 \otimes C_r^*(G)$. With $b \neq 0$ we see that there is $\varepsilon > 0$ *s.t.* $\sigma_t(a) \in V_0$ for $|t| < \varepsilon$. From part (i) of Lemma 1.2 we see that $\sigma_t(c_{ij}) \in V_0$ for all $t \in \mathbb{R}$.

Take

$$V_1 = \text{span}\{\sigma_t(a) \mid t \in \mathbb{R}\} \quad \text{and}$$
$$V_2 = \text{span}\left\{\int_{-\infty}^{\infty} \mathrm{e}^{-k(t-t_0)^2}\sigma_t(a)\,\mathrm{d}t \mid t_0 \in \mathbb{R},\ k > 0\right\}.$$

If α is a linear functional on V_1 which is zero on V_2, we have

$$\int_{-\infty}^{\infty} \mathrm{e}^{-k(t-t_0)^2}\alpha(\sigma_t(a))\,\mathrm{d}t = 0 \tag{5.4}$$

for all t_0 and $k > 0$. This is only possible if $\alpha(\sigma_t(a)) \equiv 0$, so $\alpha = 0$ on V_1. It follows that $V_1 = V_2$, and from [21, Lemma 2.3] that a is analytic with respect to the modular automorphism group σ_t. □

Proposition 5.4. *Let $\mathscr{A}$ be a multiplier Hopf *-subalgebra of $C_r^*(G)$. Then all elements of $\mathscr{A}$ are integrable with respect to the modular automorphism group σ_t and $\mathscr{A} \subset \mathscr{N}_{w_G} \cap \mathscr{M}_{w_G}$.*

Proof. Take $a \in \mathscr{A}$, we just proved that there is a finite-dimensional subspace V_0 s.t. $\sigma_t(a) \in V_0$ for all t. By Lemma 2.3 there is $e \in \mathscr{A}$ s.t. $ex = x$ for all $x \in V_0$. Now take $z \in \mathscr{N}_{w_G}$ s.t. $\|e - z\| < (4\,\|e\| + 2)^{-1}$ and $y = z^*z$. Then $\|\mathrm{e}^*e - y\| < \frac{1}{2}$ and

$$a^*a = a^*(e^*e - y)a + a^*ya \leqslant \tfrac{1}{2}a^*a + a^*ya, \tag{5.5}$$

so $a^*a \leqslant 2a^*ya$. Since a is analytic with respect to w_G, it follows from [21, Lemma 2.4] that $w_G(a^*ya) < \infty$. So $w_G(a^*a) < \infty$ and $a \in \mathscr{N}_{w_G}$. Since $\mathscr{A}^2 = \mathscr{A}$ (as remarked above), we also have $a \in \mathscr{M}_{w_G}$. □

Remark 5.5. Actually $\mathscr{A}$ is contained in the Pedersen ideal of $C_r^*(G)$, but is in general a proper subset. We shall not need this, but the reader may recognize a main ingredient of [14, p. 175] in the above proof.

We now come to the first main result about $C_r^*(G)$:

Theorem 5.6. *Suppose $C_r^*(G)$ contains a multiplier Hopf *-subalgebra $\mathscr{A}$. Then G has a compact open subgroup and every element of $\mathscr{A}$ is of the form L_ϕ with $\phi \in C_c(G)$.*

Proof. By assumption we have $a, b, a_i, b_i \in \mathscr{A}$ with

$$\Delta(a)(1 \otimes b) = \sum_1^n a_i \otimes b_i \neq 0. \tag{5.6}$$

Then for all $x, y \in G$ we have

$$(1 \otimes yx^{-1})\Delta(xa)(1 \otimes b) = \sum_1^n xa_i \otimes yb_i. \tag{5.7}$$

We have $a, b \in \mathcal{N}_{w_G}$ and there are $\xi, \eta \in L^2(G)$ such that the following expression is not identically zero:

$$\langle xa, w_G \rangle \langle yx^{-1}b\xi \,|\, \eta \rangle = \sum_1^n \langle xa_i, w_G \rangle \langle yb_i\xi \,|\, \eta \rangle. \tag{5.8}$$

These functions are in $C_0(G)$ and satisfy Corollary 1.3, so G has a compact open subgroup and the functions are in fact in $C_c(G)$ by Corollary 1.5. With $\widehat{a}(x) = \phi(x^{-1}a)$ we then have $a = \int \widehat{a}(x) L_x \, \mathrm{d}x$. □

As in Section 3 we expect that $C_r^*(G)$ has a unique dense multiplier Hopf *-subalgebra, and this is true:

Theorem 5.7. *Suppose G has a compact open subgroup H and that $\mathcal{A}$ is a dense multiplier Hopf*-subalgebra of $C_r^*(G)$. Then*

$$\mathcal{A} = \{L_\phi \,|\, \phi \in \mathcal{P}(G)\}.$$

Proof. We just saw that $\mathcal{A} \subset \{L_\phi \,|\, \phi \in C_c(G)\}$. Let $\widehat{\mathcal{A}} = \{\widehat{a} \,|\, a \in \mathcal{A}\}$. We want to prove that this is a dense multiplier Hopf*-subalgebra of $C_0(G)$. It follows from our computations in Theorem 5.6 that

$$\mathrm{span}\{\Delta(\widehat{a})(1 \otimes \widehat{b}) \,|\, a, b \in \mathcal{A}\} = \widehat{\mathcal{A}} \otimes \widehat{\mathcal{A}}.$$

We have $a \otimes b = \sum \Delta(c_i)(1 \otimes d_i)$ so $\widehat{a}(x)\widehat{b}(x) = \sum \widehat{c_i}(x)\widehat{d_i}(e)$, and therefore $\widehat{\mathcal{A}}$ is an algebra under pointwise multiplication. With $b = S(a^*)$ we have $\widehat{b}(x) = \overline{\widehat{a}(x)}$, so $\widehat{\mathcal{A}}$ is conjugation invariant. By repeating such computations in various forms, the reader should be convinced that $\{\widehat{a} \,|\, a \in \mathcal{A}\}$ is a multiplier Hopf *-subalgebra of $C_0(G)$. The conclusion now follows from Theorem 3.11. □

Remark 5.8. In the last part of this section we show that if G has a compact open subgroup H, the unique dense multiplier Hopf *-subalgebra $\mathcal{A}$ of $C_r^*(G)$ can be characterized using the conditional expectation $E : C_r^*(G) \mapsto C_r^*(H)$. We believe this is useful for generalizations.

Next we shall give an alternate description of $\mathcal{A}$ which is the dual of Proposition 3.9. Two tools are needed: the projection

$$p_H = \int_H L_h \, \mathrm{d}h \tag{5.9}$$

(we assume the Haar measure is normalized such that $\mu(H) = 1$) and the conditional expectation $E : C_r^*(G) \mapsto C_r^*(H)$ given by

$$E(a) = (\iota \otimes \tau)\Delta(a) = (\tau \otimes \iota)\Delta(a),$$

where τ is the vector state given by $\tau(a) = \langle a\chi_H, \chi_H \rangle$. Note that

$$\Delta \circ E = (E \otimes i) \circ \Delta = (i \otimes E) \circ \Delta$$

and that for $b \in C_r^*(H)$:

$$bp_H = \tau(b)p_H, \quad \Delta(b)(1 \otimes p_H) = b \otimes p_H.$$

Lemma 5.9. *Suppose* $a, a_i, b_i \in C_r^*(G)$ *satisfy*

$$\Delta(a)(1 \otimes p_H) = \sum_1^n a_i \otimes b_i. \tag{5.10}$$

Then there is a finite set F *s.t.* $E(x^{-1}a) = 0$ *for* $x \notin FH$ *and* $a = \sum_{x \in F} x E(x^{-1}a)$.

Proof. By multiplying (5.10) to the left with $x^{-1} \otimes y$ and applying $E \otimes \tau$ we get

$$\chi_H(yx)E(x^{-1}a) = \sum \tau(yb_i)E(x^{-1}a_i).$$

Now Lemma 1.6 gives a finite set F s.t. $E(x^{-1}a) = 0$ for $x \notin FH$.

To prove the last claim, choose F s.t. $FH = \bigcup_{x \in F} xH$ is a disjoint union and take $b = \sum_{x \in F} xE(x^{-1}a)$. Then $E(y^{-1}b) = E(y^{-1}a)$ for all $y \in G$ (look at $y \in FH$ and $y \notin FH$, separately).

So $E(cy^{-1}b) = E(cy^{-1}a)$ for all $y \in G$ and $c \in C_r^*(H)$. Since $\cup y C_r^*(H)$ is dense in $C_r^*(G)$ and E is faithful it follows that $a = b$. □

Lemma 5.10. *Suppose* $a = \sum_1^n x_i a_i$ *with* $a_i \in C_r^*(H)$, $x_i \in G$, $x_j^{-1} x_i \notin H$ *for* $i \neq j$ *and that*

$$(a \otimes 1)\Delta(p_H) = \sum_1^m b_k \otimes c_k. \tag{5.11}$$

Then also each a_i *satisfies* (5.11), *in fact*

$$(a_i \otimes 1)\Delta(p_H) = \sum_k E(x_i^{-1} b_k) \otimes c_k. \tag{5.12}$$

Proof. Just use the map $b \otimes c \mapsto E(x_i^{-1}b) \otimes c$ on

$$\sum (x_i a_i \otimes 1)\Delta(p_H) = \sum b_k \otimes c_k. \qquad \square \tag{5.13}$$

Lemma 5.11. *Suppose* H *is compact and that* $a, b_i, c_i \in C_r^*(H)$ *satisfies*

$$(a \otimes 1)\Delta(p_H) = \sum_1^n b_i \otimes c_i. \tag{5.14}$$

Then there is $f \in \mathscr{P}(H)$ *s.t.* $a = L_f$.

Proof. We may assume that $\{c_i\}$ is linearly independent, so there is a central projection $e_0 \in C_r^*(H)$ s.t. also $\{c_i e_0\}$ is linearly independent. Choose $\psi_j \in (C_r^*(H)e_0)^*$ s.t. $\psi_j(c_i e_0) = \delta_{ij}$ and note that ψ_j can be considered an element of $\mathscr{P}(H)$. Use $i \otimes \psi_j$ on (5.14) to obtain $b_j = aL_{\psi_j} \in C_r^*(H)S(e_0)$. So $b_i = L_{f_i}$ for some $f_i \in \mathscr{P}(H)$, and $(a \otimes 1)\Delta(p_H) = \sum_1^n L_{f_i} \otimes c_i$. By (2.6) $a = \sum_1^n L_{f_i} S(c_i)$ and since $\{L_f \mid f \in \mathscr{P}(H)\}$ is an ideal in $C_r^*(H)$, we have $a \in \{L_f \mid f \in \mathscr{P}(H)\}$. □

Theorem 5.12. *If G has a compact open subgroup H and $a \in C_r^*(G)$ the following are equivalent*:

(i) $a = L_\phi$ *with* $\phi \in \mathscr{P}(G)$
(ii) *There are finitely many* $b_i, c_i, b'_j, c'_j \in C_r^*(G)$ *s.t.*

$$\Delta(a)(1 \otimes p_H) = \sum b_i \otimes c_i \quad \textit{and} \quad (a \otimes 1)\Delta(p_H) = \sum b'_j \otimes c'_j.$$

Proof. That (i) implies (ii) is left to the reader (use Proposition 3.9, multiply with $L_x \otimes L_y$ and integrate). Conversely, if a satisfies (ii) it follows from the previous that $a = \sum_k L_{x_k} a_k$ with $x_k \in G$ and $a_k \in C_r^*(H)$. By Lemma 3.4 and Lemma 5.11 we get $\phi_k \in \mathscr{P}(H)$ s.t. $a_k = L_{\phi_k}$, so $f = \sum_k {}_{x_k}\phi_k$ is in $\mathscr{P}(G)$ and we have $a = L_f$. □

Remark 5.13. Note that as in Remark 3.10 both parts of (ii) are needed in general to characterize $\mathscr{A}$.

6. Multiplier Hopf *-algebras in $C^*(G)$

What happens if we look at $C^*(G)$ instead of $C_r^*(G)$? Here, $C^*(G)$ is the enveloping C^*-algebra of $L^1(G)$ and the maps $\Delta, S, \varepsilon, \sigma_t$ in Section 5 all extends to $C^*(G)$, cf. [10, Theorem 3.9] or [15] for an updated survey. If π_r is the natural map $C^*(G) \to C_r^*(G)$, we also get a weight on $C^*(G)$ by $a \mapsto \phi(\pi_r(a))$, but this weight is in general not faithful so $C^*(G)$ is not really a locally compact quantum group.

In Theorem 5.7 we showed that the existence of one finite set of elements in $C_r^*(G)$ satisfying (5.6) implies the existence of a compact open subgroup. However, this is not true for $C^*(G)$. Akemann and Walter proved (see [1] or [22]) that if G has property (T), then there is a central minimal projection $p_0 \in C^*(G)$ s.t. $\pi_0(p_0)=1$ for the trivial representation π_0 and $\pi(p_0) = 0$ for all other irreducible representations of G. Clearly,

$$\Delta(p_0)(1 \otimes p_0) = p_0 \otimes p_0,$$

but there are groups with property (T) – e.g. SL(3, $\mathbb{R}$) – which do not have compact open subgroups.

Note that if G has a compact open subgroup H the analogue of Theorem 5.12 can be proved the same way, since by [17, Proposition 1.2] there is a conditional expectation $E : C^*(G) \mapsto C^*(H) = C_r^*(H)$. The map τ is then defined by $\tau(a) = \langle E(a)\chi_H, \chi_H \rangle$, the proof of Theorem 5.12 can be repeated verbatim and we have:

Theorem 6.1. *If G has a compact open subgroup H*, $a \in C^*(G)$ *and U is the universal representation of G the following are equivalent*:

(i) $a = U_f$ *with* $f \in \mathscr{P}(G)$.
(ii) *There are finitely many* $b_i, c_i, b'_j, c'_j \in C^*(G)$ *s.t.*

$$\Delta(a)(1 \otimes p_H) = \sum b_i \otimes c_i \quad \textit{and} \quad (a \otimes 1)\Delta(p_H) = \sum b'_j \otimes c'_j.$$

7. Multiplication and Convolution Operators

The dual locally compact quantum groups $C_0(G)$ and $C_r^*(G)$ have both natural representations on $L^2(G)$ and we shall study properties of these representations which also turns out to be equivalent to the existence of a compact open subgroup. It is well known, see [19, Proposition 3.3] or [23, Lemme 5.2.8] (although the result is probably older) that if $a \in C_r^*(G)$ and $f \in C_0(G)$, then $aM(f)$ is a compact operator on $L^2(G)$. (See also [4] for a study of multiplication and convolution operators over $L^p(G)$.)

In this section we shall see that $aM(f)$ cannot be non-zero and of *finite rank* unless G has a compact open subgroup. We shall also see that $aM(f) = M(f)a \neq 0$ is possible only if G has a compact open subgroup. We first need the following two results:

Theorem 7.1. *For a closed subgroup H of G,*

(i) $C_0(G) \cap L^\infty(G/H) = C_0(G/H)$ *if H is compact and trivial otherwise.*
(ii) $C_r^*(G) \cap \mathscr{L}(H) = C_r^*(G) \cap L^\infty(H\backslash G)' = C_r^*(H)$ *if H is open and trivial otherwise.*

Proof. The first statement is obvious. It follows from the Takesaki–Nielsen–Rieffel commutant theorem [16, Theorem 2.6] that

$$\mathscr{L}(H) = \mathscr{L}(G) \cap L^\infty(H\backslash G)'. \tag{7.1}$$

Suppose $a \in C_r^*(G) \cap \mathscr{L}(H)$ with $a \gneq 0$. Then $b := M(\chi_U)aM(\chi_U) \neq 0$ for some open set U. So b is a compact operator $L^2(U) \mapsto L^2(U)$ and by the spectral theorem there is $\lambda \neq 0$ such that the eigenspace

$$H_\lambda := \{\xi \mid b\xi = \lambda\xi\} \tag{7.2}$$

is finite dimensional $\neq \{0\}$. For $\psi \in L^\infty(H\backslash G)$, $\xi \in H_\lambda$ then

$$bM(\psi)\xi = M(\psi)b\xi = \lambda M(\psi)\xi \tag{7.3}$$

so $M(\psi)H_\lambda \subset H_\lambda$. We therefore have a non-zero $\xi \in L^2(U)$ which is an eigenvector for all $M(\psi)$ with $\psi \in L^\infty(H\backslash G)$. Restricting to $\psi \in C_0(H\backslash G)$ one realizes that there is $x_0 \in G$ (not unique) s.t.

$$M(\psi)\xi = \psi(x_0)\xi \quad \text{for all } \psi \in C_0(H\backslash G). \tag{7.4}$$

Let $V = \{x \mid \xi_0(x) \neq 0\}$, so $\mu(V) > 0$ and $\psi(s) = \psi(x_0)$ for all $s \in V$, $\psi \in C_0(H\backslash G)$. For this it is necessary that $V \subset Hx_0$, so VV^{-1} is an open subset of H by [8, (20.17)] and therefore H is open. □

Theorem 7.2. *Suppose $a \in \mathscr{R}(G)$ and $f \in L^\infty(G)$ s.t. $M(f)a \neq 0$ has finite rank. Then G has a compact open subgroup.*

Proof. Pick a measurable set C with $0 < \mu(C) < \infty$ s.t. $M(\chi_C f)a \neq 0$, therefore we may assume that $f \in L^2(G)$. Pick $\xi_i, \eta_i \in L^2(G)$ s.t.

$$M(f)a\xi = \sum_1^n \xi_i \langle \xi \mid \eta_i \rangle, \quad \text{for all } \xi.$$

There is $\xi \in C_c(G)$ s.t. $M(f)a\xi \neq 0$, using that $aL_x = L_x a$ we get

$$M(f)aL_x\xi(y) = \sum_1^n \xi_i(y)\langle L_x\xi \,|\, \eta_i\rangle \quad \text{so}$$

$$f(y)a\xi(x^{-1}y) = \sum_1^n \xi_i(y)\langle L_x\xi \,|\, \eta_i\rangle.$$

The reader should check that $x \mapsto \langle L_x\xi \,|\, \eta_i\rangle$ is in $L^2(G)$, so by Corollary 1.3 we can conclude that G has a compact open subgroup. □

Remark 7.3. We clearly have the same result with $a \in \mathscr{L}(G)$ instead.

Theorem 7.4. *Suppose $a \in C_r^*(G)$ and $f \in C_0(G)$ are both non-zero s.t. $aM(f)=M(f)a$. Then G has a compact open subgroup.*

Proof. Fuglede's Theorem [7] implies that $a^*M(f) = M(f)a^*$, so

$$\mathscr{B} = \{g \in L^\infty(G) \,|\, M(g)a = aM(g)\}$$

is a weakly closed right invariant *-subalgebra of $L^\infty(G)$, so by [20, Theorem 2] $\mathscr{B} = L^\infty(H\backslash G)$ for some closed subgroup H of G. Since f is a non-zero element of $C_0(G) \cap L^\infty(H\backslash G)$, we get from part (i) of Theorem 7.1 that H is compact. Since a is a non-zero element of $C_r^*(G) \cap \mathscr{L}(H)$, part (ii) of the same theorem gives that H is open. □

The following description may also be useful.

Definition 7.5. A non-zero self-adjoint projection p in a multiplier Hopf *-algebra is called *group-like* (cf. [13] and [20, Theorem 10]) if

$$\Delta(p)(p \otimes 1) = \Delta(p)(1 \otimes p) = p \otimes p.$$

Proposition 7.6. *The following are equivalent*:

(i) *G has a compact open subgroup*,
(ii) *$C_0(G)$ has a group-like projection*,
(iii) *$C_r^*(G)$ has a group-like projection*.

Proof. If G has a compact open subgroup H, it is easy to check that χ_H is a group-like projection in $C_0(G)$ and that $p_H = L_{\chi_H}$ is a group-like projection in $C_r^*(G)$.

If p is a projection in $C_0(G)$, then $p = \chi_A$ for a compact open set A. It is easy to see that if p is group-like, then A is a subgroup of G. Finally, it follows from [20, Section 5] that if $p \in C_r^*(G)$ is group-like, then $p = L_{\chi_H}$ for some compact open subgroup H of G. □

Remark 7.7. Clearly (i–iii) above implies that $C^*(G)$ has a group-like projection. However, our remarks in Section 6 show that the reverse implication is false.

8. Abelian Groups

We close with a quick look at abelian groups. It is a basic fact of classical Fourier analysis that if we have a non-zero function $f \in C_c(\mathbb{R}^n)$, then its Fourier transform $\widehat{f}$ is analytic and therefore does not have compact support. For abelian groups in general we have the following:

Proposition 8.1. *If G is abelian, the following are equivalent:*

(i) *G has a compact open subgroup,*
(ii) *There is a non-zero $f \in C_c(G)$ with $\widehat{f} \in C_c(\widehat{G})$.*

Proof. If G has a compact open subgroup H, then $f = \chi_H \in C_c(G)$ and $\widehat{f} = \chi_{H^\perp} \in C_c(\widehat{G})$; so (i) implies (ii).

The opposite implication will in fact follow from [8, (24.30)], but we will give a proof that does not depend on the structure theory of locally compact abelian groups.

Suppose there is a non-zero $f \in C_c(G)$ with $\widehat{f} \in C_c(\widehat{G})$ and that U is a compact neigborhood of e. Then there is $g \in C_c(G)$ and $\phi \in C_c(\hat{G})$ with $gL_y f = L_y f$ and $\phi \widehat{L_y f} = \widehat{L_y f}$ for all $y \in U$.

Hence $L_{\widehat{\phi}} M(g)$ is a compact operator and $L_{\widehat{\phi}} M(g) L_y f = L_y f$ for all $y \in U$. This implies that span$\{L_y f \mid y \in U\}$ is finite dimensional and it follows from Lemma 1.2 that G has a compact open subgroup. □

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References

[1] C.A. Akemann, M.E. Walter, Unbounded negative definite functions, Can. J. Math. 33 (1981) 862–871.
[2] S. Baaj, G. Skandalis, Unitaires multiplicatifs et dualité pour les produits croisés de C^*-algébres, Ann. Sci. École Norm. Sup. 26 (4) (1993) 425–488.
[3] F. Bruhat, Distributions sur un groupe localement compact et applications à l'étude des représentations des groupes $\wp$-adiques, Bull. Soc. Math. France 89 (1961) 43–75.
[4] R.C. Busby, H.A. Smith, Product-convolution operators and mixed-norm spaces, Trans. Amer. Math. Soc. 263 (1981) 309–341.
[5] K. De Commer, A. Van Daele, Multiplier Hopf algebras imbedded in C^*- algebraic quantum groups, preprint, /math.OA/0611872 (2006) 1–24.
[6] B. Drabant, A. Van Daele, Y. Zhang, Actions of multiplier Hopf algebras, Comm. Algebra 27 (1999) 4117–4172.
[7] B. Fuglede, A commutativity theorem for normal operators, Proc. Nat. Acad. Sci. USA 36 (1950) 35–40.

[8] E. Hewitt, K.A. Ross, Abstract Harmonic Analysis. vol. I: Structure of Topological Groups. Integration Theory, Group Representations, Academic Press Inc., Publishers, New York, 1963.
[9] E. Hewitt, K.A. Ross, Abstract Harmonic Analysis. vol. II: Structure and Analysis for Compact Groups. Analysis on Locally Compact Abelian Groups, Springer, New York, 1970.
[10] V.B.M. Iório, Hopf C^*-algebras and locally compact groups, Pacific J. Math. 87 (1980) 75–96.
[11] S. Kaliszewski, M.B. Landstad, J. Quigg, Hecke C^*-algebras, Schlichting completions, and Morita equivalence, Proc. Edinburgh Math. Soc., to appear.
[12] M.B. Landstad, Duality theory for covariant systems, Trans. Amer. Math. Soc. 248 (1979) 223–267.
[13] M.B. Landstad, A. Van Daele, Compact and discrete subgroups of algebraic quantum groups I, preprint, /math.OA/0702458 (2007) 1–47.
[14] G.K. Pedersen, C^*-algebras and their automorphism groups, Academic Press Inc., London, 1979.
[15] I. Raeburn, Crossed products of C^*-algebras by coactions of locally compact groups, in: Operator Algebras and Quantum Field Theory (Rome, 1996), Internat. Press, Cambridge, MA, 1997, pp. 74–84.
[16] M.A. Rieffel, Commutation theorems and generalized commutation relations, Bull. Soc. Math. France 104 (1976) 205–224.
[17] M.A. Rieffel, Induced representations of C^*-algebras, Adv. Math. 13 (1974) 176–257.
[18] A.J. Silberger, Introduction to harmonic analysis on reductive p-adic groups, Princeton University Press, Princeton, 1979.
[19] H. Takai, On a duality for crossed products of C^*-algebras, J. Functional Analysis 19 (1975) 25–39.
[20] M. Takesaki, N. Tatsuuma, Duality and subgroups, Ann. of Math. 93 (2) (1971) 344–364.
[21] M. Takesaki, Theory of Operator Algebras. II, Springer, Berlin, 2003.
[22] A. Valette, Projections in full C^*-algebras of semisimple Lie groups, Mat. Ann. 294 (1992) 277–287.
[23] J.M. Vallin, C*-algèbres de Hopf et C*-algèbres de Kac, Proc. London Math. Soc. 50 (1985) 131–174.
[24] A. Van Daele, Multiplier Hopf algebras, Trans. Amer. Math. Soc. 342 (1994) 917–932.
[25] A. Van Daele, An algebraic framework for group duality, Advances in Mathematics 140 (1998) 323–366.
[26] A. Van Daele, Y. Zhang, A survey on multiplier Hopf algebras, Hopf Algebras and Quantum Groups (Brussels, 1998), Lecture Notes in Pure and Applied Mathematics, vol. 209, Dekker, New York, 2000.