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REDUCTION OF THE CODIMENSION OF ISOMETRIC IMMERSIONS IN SPACE FORMS

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§1. INTRODUCTION

WE CONSIDER C^∞ immersions $f: M^n \to Q_c^N$ of an n-dimensional connected manifold M, into an N-dimensional simply connected complete space form Q_c^N, $N > n$, of constant curvature c. The *codimension* of the immersion *can be reduced* to r, if there exists a totally geodesic, $(n+r)$-dimensional submanifold L of Q_c^N such that $f(M) \subset L$. Let $x_1, \ldots, x_n$ be local coordinates in M. The space generated by the derivatives of f, of all orders up to k, at a point $p \in M$, is the *k-th order osculating space* of f at p and it is denoted by $\overset{k}{\mathrm{Osc}}_p$. In particular, $\overset{1}{\mathrm{Osc}}_p$ is the tangent space $T_p M$, of M at p, and $\overset{2}{\mathrm{Osc}}_p$ is the direct sum of $T_p M$ and the subspace generated by the vectors $\alpha(x, y)$, $x, y \in T_p M$, where α is the second fundamental form of the immersion. Higher order osculating spaces were introduced by E. Cartan [2] and studied in [1], [6], [7], [9] and [13].

We want to establish sufficient conditions, on the osculating spaces, for reducing the codimension of an immersion. The simplest result in this direction is the classical property of curves in Euclidean space. Namely, if β is a regular curve in R^N, whose curvatures $k_1, \ldots, k_{j-1}$ do not vanish and k_j is identically zero, then β is contained in an affine j-dimensional subspace of R^N. Equivalently, if at every point p of the curve $\dim \overset{i}{\mathrm{Osc}}_p = i$, for each i, $1 \le i \le j-1$ and $\dim \overset{j}{\mathrm{Osc}}_p = j-1$, then we can reduce the codimension to $j-2$.

Our main theorems are generalizations of results contained in [14]. Our first theorem shows that for an immersion of a compact manifold M^n, $n \ge 2$, if the dimension of the k-th order osculating space, $k \ge 2$, is a constant less than $n+k$, then we can reduce the codimension.

THEOREM 1. *Let $f: M^n \to Q_c^N$ be an isometric immersion of a compact manifold M^n, $n \ge 2$. Suppose there exist integers k, r, $k \ge 2$ and $0 \le r \le k-1$, such that* $\dim \overset{k}{\mathrm{Osc}}_p = n + r$, *for every $p \in M$. Then, the codimension can be reduced to r. If $c > 0$, M needs only to be complete.*

The above result does not hold for curves. In fact consider the following example.

Example 1. Let S^1 be the unit circle and $\beta: S^1 \to R^3$ the immersion that to each point $(\cos t, \sin t)$ of S^1, associates

$$\beta(t) = (\cos(\cos t), \sin(\cos t), \sin t).$$

It is easy to see that $\dim \overset{2}{\mathrm{Osc}} \equiv 2$. However, we cannot reduce the codimension.

If the ambient space Q_c^N is such that $c \le 0$, then the hypothesis in theorem 1, on the compacity of M, can be relaxed by considering M complete with bounded image $f(M)$ in Q_c^N. The following example shows that we really need this extra condition.

Example 2. Let $\beta: R \to R^{N-1}$ be a regular curve which is not contained in any affine hyperplane. We consider $f: R \times R \to R^N$, defined by $f(s, t) = (\beta(s), t)$. Then $\dim \overset{2}{\mathrm{Osc}} \equiv 3$, but we cannot reduce the codimension.

If the ambient space is Euclidean and M is a complete manifold with non-negative Ricci curvature, then we prove the following

THEOREM 2. *Let $f: M^n \to R^N$, $n \geq 2$, be an isometric immersion of a complete manifold M, with non-negative Ricci curvature. Suppose there exist integers k and r, $k \geq 2$ and $0 < r \leq k-1$ such that* $\dim \overset{k}{\mathrm{Osc}}_p = n + r$, *for every p in M. Then, either*

(i) *M is isometric to a cylinder over a curve; or*
(ii) *the codimension can be reduced to r.*

It is not difficult to see that theorem 2 does not hold if we eliminate the condition on the curvature [14]. Moreover, both theorems are the best possible, in the sense that the constant dimension of the k-th order osculating space cannot be increased to $n+k$. In fact, consider the following examples.

Example 3. Let $f: S^1 \times S^1 \to R^6$ be defined by $f(s, t) = (\beta(s), \beta(t))$, where β is the function of example 1. This is an immersion of a compact manifold whose second osculating space is 4-dimensional. However, the codimension cannot be reduced.

Example 4. Let $f: S^1 \times S^1 \times R \to R^7$ be defined by $f(s, t, u) = (\beta(s), \beta(t), u)$, where β is the function of example 1. This is an immersion of a complete manifold, with Ricc $\equiv 0$ and $\dim \overset{2}{\mathrm{Osc}} \equiv 5$. However, the manifold is not isometric to a cylinder over a curve and the codimension cannot be reduced.

The above theorems, which provide local criteria for the global problem of reducing the codimension, generalize the corresponding results for $k = 2$ obtained in [14]. Refining the arguments used in [14], the theorems are proved by showing that $\overset{k}{\mathrm{Osc}}$ is parallel in M. A result analogous to theorem 1, when the ambient space is a complete (not necessarily simply connected) space form, is considered in Remark 4.

§2. PRELIMINARIES

In this section we will consider basic results which will be used in the proofs of the main theorems. Let M^n be a connected manifold immersed into an N-dimensional simply connected, complete, space form Q_c^N of constant curvature c. We consider M with the metric induced by the immersion. The Riemannian connections of Q_c^N and M^n are denoted by $\bar{\nabla}$ and ∇ respectively and $\nabla^\perp$ denotes the connection of the normal bundle $TM^\perp$. The second fundamental form of the immersion is denoted by α.

The k-th osculating space of the immersion at $p \in M$, $\overset{k}{\mathrm{Osc}}_p$, is generated by $X(p)$, $(\bar{\nabla}_{X_1}(\bar{\nabla}_{X_2} \ldots (\bar{\nabla}_{X_{r-1}} X_r) \ldots)_p$, where $X, X_1, \ldots, X_r$ are tangent vector fields on M and r assumes all integer values from 2 to k, whenever $k \geq 2$. If U is an open subset of M, where the dimension of $\overset{k}{\mathrm{Osc}}_p$ is independent of $p \in U$, then we have a vector bundle over U, which we denote by $\overset{k}{\mathrm{Osc}}\, U$.

In the proofs of our main results we show that $\overset{k}{\mathrm{Osc}}\, M$ is parallel in M, i.e. $(\bar{\nabla}_X \xi)_p \in \overset{k}{\mathrm{Osc}}_p$ for every smooth section ξ in $\overset{k}{\mathrm{Osc}}\, M$ and every X in TM. The reduction of the codimension is obtained as a consequence of the following well known result [8, 15].

THEOREM A. *Let M^n be a connected submanifold immersed in a simply connected, complete, space form Q_c^N. Let D be a j-dimensional distribution along M, such that $T_p M \subset D(p)$, for all*

$p \in M$. Suppose that D is parallel along every curve in M. Then M lies in some j-dimensional totally geodesic submanifold of Q.

In general, given an immersion $f: M^n \to \bar{M}^N$, the dimension of the osculating space $\overset{j}{\mathrm{Osc}}_p$, $j \geq 2$, depends on the point $p \in M$. The following lemma provides open and dense subsets of M, whose connected components have constant dimensional osculating spaces.

LEMMA 1. *Let $f: M^n \to \bar{M}^N$ be an immersion. For any integer $k \geq 2$ consider*

$$B_i^j = \left\{ p \in M; \dim \overset{j}{\mathrm{Osc}}_p = n + i \right\},$$

$$A^j = \bigcup_{i=0}^{N-n} \operatorname{int} B_i^j,$$

$$A = \bigcap_{j=2}^{k} A^j,$$

where $0 \leq i \leq N - n$ and $j \geq 2$. Then, A^j and A are open and dense subsets of M. Moreover, for any connected component C of A the dimension of $\overset{j}{\mathrm{Osc}}_p$ is independent of the point $p \in C$ for each $j \leq k$.

Proof. We need to show that for any open subset V of M we have $V \cap A^j \neq \emptyset$. Denote by f^j the function defined on V by $f^j(p) = \dim \overset{j}{\mathrm{Osc}}_p$, $p \in V$. Since $\dim \overset{j}{\mathrm{Osc}}_p \leq N$, the image of f^j is a finite set of integers, therefore there exists an integer $n + i_0$, which is the maximum of f^j. Let $q_0 \in V$ be such that $\dim \overset{j}{\mathrm{Osc}}_{q_0} = n + i_0$. Then there exists a neighborhood V_0 of q_0, $V_0 \subset V$, such that $\dim \overset{j}{\mathrm{Osc}}_p \geq n + i_0$, for all $p \in V_0$. Since $u + i_0$ is the maximum value of f^j on V, we get $\dim \overset{j}{\mathrm{Osc}}\, V_0 = n + i_0$. Hence $V_0 \subset \operatorname{int} B_{i_0}^j$. Therefore, $V_0 \subset V \cap A^j$ i.e. A^j is an open and dense subset of M. Hence we conclude that A is also an open and dense subset of M.

The fact that the connected components of A have constant dimensional osculating spaces follows from the construction of A. q.e.d.

As a consequence of the above lemma, we have the following result, which will be very useful in the next section.

LEMMA 2. *Let $f: M^n \to \bar{M}^N$ be an immersion. Suppose there exist integers $k > 2$ and $r > 0$, such that $\dim \overset{k}{\mathrm{Osc}}\, M = n + r$.*

(a) *If $r = 1$, then $\overset{k}{\mathrm{Osc}}$ is parallel in M.*

(b) *If $1 < r \leq k - 1$, then $\overset{k}{\mathrm{Osc}}$ is parallel in the set*

$$U = \{ p \in M; \dim \overset{2}{\mathrm{Osc}}_p > n + 1 \}.$$

Proof. We will use the notation introduced in Lemma 1. We observe that since $\dim \overset{k}{\mathrm{Osc}}\, M = n + r$, where $k > 1$ and $r > 0$, it follows that B_0^2 has empty interior.

(a) If $r = 1$, then from Lemma 1 we have that $A^2 = \operatorname{int} B_1^2$ is an open and dense subset of M. Since $\dim \overset{k}{\mathrm{Osc}}\, M = n + 1$ and $k > 2$, it follows that $\overset{2}{\mathrm{Osc}}\, A^2 = \ldots = \overset{k}{\mathrm{Osc}}\, A^2$. Hence, $\overset{k}{\mathrm{Osc}}$ is parallel in the dense subset A^2 of M, and therefore is parallel in M.

(b) We consider f restricted to U and we apply Lemma 1 to this restriction. Then the set

$$A = \bigcap_{j=2}^{k} \bigcup_{i=2}^{r} \operatorname{int}\{ p \in U; \dim \overset{j}{\mathrm{Osc}}_p = n + i \}$$

is an open and dense subset of U. Moreover, for any connected component C of A and any integer j, $2 \leq j \leq k$, $\dim \overset{j}{\mathrm{Osc}}_p$ is a constant independent of $p \in C$. In particular

$$n+2 \leq \dim \overset{2}{\mathrm{Osc}}\, C \leq n+r.$$

Therefore, since $r \leq k-1$ and $\dim \overset{k}{\mathrm{Osc}}\, C = n+r$, for each component C, there exists an integer i, $i \leq k-1$, such that

$$\overset{i}{\mathrm{Osc}}\, C = \ldots = \overset{k-1}{\mathrm{Osc}}\, C = \overset{k}{\mathrm{Osc}}\, C.$$

Hence, $\overset{k}{\mathrm{Osc}}$ is parallel in each connected component of A. Since A is dense in U, it follows that $\overset{k}{\mathrm{Osc}}$ is parallel in U. q.e.d.

The following lemma can be found in ([14] lemma 2).

Lemma 3. *Let $f: M^n \to Q_c^N$ be an isometric immersion and W an open subset of M such that $\dim \overset{2}{\mathrm{Osc}}\, W = n+1$. If at each point of W there exists some sectional curvature $K \neq c$, then $\overset{2}{\mathrm{Osc}}$ is parallel in W.*

As a consequence of Lemma 3 we obtain

Lemma 4. *Let $f: M^n \to Q_c^N$ be an isometric immersion and W an open subset of M where $\dim \overset{2}{\mathrm{Osc}}_p \leq n+1$, for any $p \in W$. If there exists an integer $k > 2$ such that $\dim \overset{k}{\mathrm{Osc}}_p > n+1$, for any $p \in W$, then the sectional curvature K in W is constant equal to c.*

Proof. Consider $W = W_0 \cup W_1$, where

$$W_i = \left\{ p \in W;\, \dim \overset{2}{\mathrm{Osc}}_p = n+i \right\}, \quad i = 0, 1.$$

Then W_1 is an open set of M and W_0 has empty interior, since $\dim \overset{k}{\mathrm{Osc}}_p > n+1$, for any $p \in W$.

Suppose that the set

$$D = \{ p \in W_1;\, K(p) \not\equiv c \}$$

is non empty. Then it follows from Lemma 3 that $\overset{2}{\mathrm{Osc}}$ is parallel in D. Hence $\overset{2}{\mathrm{Osc}}\, D = \overset{k}{\mathrm{Osc}}\, D$, which contradicts the hypothesis. Therefore, $K \equiv c$ in W_1. Since W_0 has empty interior, we conclude that $K \equiv c$ in W. q.e.d.

We recall that the relative nullity space of an immersion at a point $p \in M$ is the set of tangent vectors $X \in T_p M$ such that $\alpha(X, Y) = 0$, for all $Y \in T_p M$. The relative nullity index $\nu(p)$ is the dimension of the relative nullity space at p. The following fact about ν will be used in the proofs of the main theorems.

Remark 1. Let $f: M \to Q_c^N$ be an isometric immersion. If the relative nullity index ν is a constant l in an open set U of M, then the relative nullity distribution in U is involutive and the leaves of the foliation are totally geodesic in Q_c^N [10, 12]. Moreover, generalizing Lemma 3.1 in [11] and Lemma 2 in [4], one can show that every boundary point $\tilde{p}$ of U, which is also a limit point of a leaf, has relative nullity index $\nu(\tilde{p}) = l$.

Remark 2. Let $f: M^n \to Q_c^N$ be an isometric immersion and V be an open subset of M, where $\dim \overset{2}{\mathrm{Osc}}\, V = n+1$. Then, the sectional curvature at $p \in V$, $K(p)$ is not identically equal to c, if and only if, the relative nullity index $\nu(p) < n-1$. Moreover, if $K(p) \equiv c$ then $\nu(p) = n-1$.

Our next result is necessary for the proof of Lemma 6.

LEMMA 5. *Let $f: M^n \to Q_c^N$ be an isometric immersion such that the dimension of* $\overset{k}{\mathrm{Osc}}_p$ *and* $\overset{k+1}{\mathrm{Osc}}_p$ *is independent of $p \in M$, for some $k \geq 2$. Suppose that for any smooth section η of* $\overset{k}{\mathrm{Osc}}\, M$ *and each vector X of the relative nullity space D_p at $p \in M$, $(\bar{\nabla}_X \eta)_p \in \overset{k}{\mathrm{Osc}}_p$. Then, for any smooth section of* $\overset{k+1}{\mathrm{Osc}}\, M$, $(\bar{\nabla}_X \xi)_p \in \overset{k+1}{\mathrm{Osc}}_p$.

Proof. For $p \in M$, let N_p^k be the k-th normal space of M at p, i.e. the orthogonal complement of $\overset{k}{\mathrm{Osc}}_p$ in $\overset{k+1}{\mathrm{Osc}}_p$. It follows from the hypothesis that N^k is a vector bundle over M.

In order to prove the lemma, it suffices to show that for any vector field ξ in N^k and $X \in D_p$, $(\nabla_X^\perp \xi)_p \in \overset{k+1}{\mathrm{Osc}}_p$. Since ξ is locally a sum of terms $f\nabla_Y^\perp \eta$, where $\eta \in \overset{k}{\mathrm{Osc}}\, M\, Y \in TM$ and f is a smooth function, we may consider $\xi = \nabla_Y^\perp \eta$.

Let X be a vector in the relative nullity space D_p. Extend X to a vector field in a neighborhood of p. Then

$$\nabla_X^\perp \xi = \nabla_X^\perp \nabla_Y^\perp \eta = R^\perp(X, Y)\eta + \nabla_Y^\perp \nabla_X^\perp \eta + \nabla_{[XY]}^\perp \eta.$$

Since $X \in D_p$, it follows from Ricci equation that $R^\perp(X, Y)\eta = 0$ at the point p. Therefore,

$$\nabla_X^\perp \xi = \nabla_Y^\perp \nabla_X^\perp \eta + \nabla_{[X,Y]}^\perp \eta.$$

Now, by hypothesis $(\nabla_X^\perp \eta)_p \in \overset{k}{\mathrm{Osc}}_p$ for each $p \in M$. Therefore, $(\nabla_Y^\perp \nabla_X^\perp \eta)_p$ and $(\nabla_{[X,Y]}^\perp \eta)_p$ are vectors in $\overset{k+1}{\mathrm{Osc}}_p$. We conclude that $(\nabla_X^\perp \xi)_p \in \overset{k+1}{\mathrm{Osc}}_p$. q.e.d.

Our last lemma is a generalization of ([14], lemma 4).

LEMMA 6. *Let $f: M^n \to Q_c^N$ be an isometric immersion, such that the dimension of* $\overset{k}{\mathrm{Osc}}_p$ *is independent of $p \in M$, for some $k \geq 2$. Let U be an open subset of M where the relative nullity index is a constant $l > 0$. Let γ: $[0, a] \to M$ be a geodesic segment contained in a leaf of the relative nullity foliation of U. If* $\overset{k}{\mathrm{Osc}}$ *is parallel at $\gamma(a)$, then it is parallel for all $t \in [0, a)$.*

Proof. Let $X_1, \ldots, X_n$ be tangent frame fields, defined in a neighborhood of $\gamma(t)$ in U, such that $X_1, \ldots, X_l$ span the relative nullity space D and $X_1 = \gamma'(t)$. Since the leaf of the relative nullity foliation, which contains γ, is totally geodesic we may suppose $\nabla_{\gamma'(t)} X_i(t) = 0$, $1 \leq i \leq n$.

(a) First we show that for any point $p \in U$ and any smooth section ξ of $\overset{k}{\mathrm{Osc}}\, M$, we have $(\bar{\nabla}_X \xi)_p \in \overset{k}{\mathrm{Osc}}_p$, when X is a vector of the relative nullity space D_p.

Let A be an open and dense subset of U, such that for any connected component C of A the dimension of $\overset{j}{\mathrm{Osc}}_p$ is independent of the point $p \in C$, for each $j \leq k$ (see Lemma 1). We consider any such connected component C. Let v be a normal vector field defined on a neighborhood of a point $p \in C$, such that v is orthogonal to $\overset{2}{\mathrm{Osc}}\, C$. Then,

$$0 = X_i \langle \alpha(X_r, X_s), v \rangle = \langle \nabla_{X_i}^\perp \alpha(X_r, X_s), v \rangle + \langle \alpha(X_r, X_s), \nabla_{X_i}^\perp v \rangle,$$

for any $1 \leq r, s \leq n$ and $1 \leq i \leq l$. Since

$$\langle \alpha(X_r, X_s), \nabla_{X_i}^\perp v \rangle = \langle \alpha(X_i X_s), \nabla_{X_r}^\perp v \rangle$$

and X_i is in the relative nullity, it follows from the above equality that

$$\langle \nabla_{X_i}^\perp \alpha(X_r, X_s), v \rangle = 0.$$

Therefore, for any vector field η in $\overset{2}{\mathrm{Osc}}\, C$, $p \in C$ and $X \in D_p$, we have $(\bar{\nabla}_X \eta)_p \in \overset{2}{\mathrm{Osc}}\, p$. Using Lemma 5 inductively, we obtain that $(\bar{\nabla}_X \xi)_p \in \overset{k}{\mathrm{Osc}}_p$, for any smooth section ξ of $\overset{k}{\mathrm{Osc}}\, C$, $p \in C$ and $X \in D_p$. The proof of (a) is completed as a consequence of A being dense in U.

(b) Let $e_1, \ldots, e_{N-n}$ be an orthonormal normal frame field defined along the geodesic

$\gamma(t)$, such that $X_1, \ldots, X_n, e_1, \ldots, e_m$ span $\overset{k}{\mathrm{Osc}}$ and $\nabla^{\perp}_{\gamma'} e_I = 0$, for all $1 \leq I \leq N-n$. Since $\bar{\nabla}_{\gamma'} \overset{k}{\mathrm{Osc}} \subset \overset{k}{\mathrm{Osc}}$, such frame exists. Now, for each $r \leq m$ and $\delta > m$ we consider the function

$$f_i(t) = \langle \nabla^{\perp}_{X_i} e_r, e_\delta \rangle. \tag{2}$$

We want to show that for any $1 \leq i \leq n$, $f_i(t) = 0$ if $f_i(a) = 0$.

For $i \leq l$, this follows from part (a). Therefore, we only need to consider $i > l$. Differentiating (2) along γ, we obtain

$$\begin{aligned} f_i'(t) &= X_1 \langle \nabla^{\perp}_{X_i} e_r, e_\delta \rangle \\ &= \langle R^{\perp}(X_1, X_i) e_r, e_\delta \rangle + \langle \nabla^{\perp}_{X_i} \nabla^{\perp}_{X_1} e_r, e_\delta \rangle \\ &\quad + \langle \nabla^{\perp}_{[X_1, X_i]} e_r, e_\delta \rangle. \end{aligned}$$

Since $X_1 \in D$ and $\nabla_{X_1} X_i(t) = 0$, the above equation reduces to

$$\begin{aligned} f_i'(t) &= \langle \nabla^{\perp}_{X_i} \nabla^{\perp}_{X_1} e_r, e_\delta \rangle - \langle \nabla^{\perp}_{\nabla_{X_i} X_1} e_r, e_\delta \rangle \\ &= X_i \langle \nabla^{\perp}_{X_1} e_r, e_\delta \rangle - \langle \nabla^{\perp}_{X_1} e_r, \nabla^{\perp}_{X_i} e_\delta \rangle - \sum_{j=1}^{n} \langle \nabla_{X_i} X_1, X_j \rangle \langle \nabla^{\perp}_{X_j} e_r, e_\delta \rangle. \end{aligned}$$

From the choice of the normal frame field we have $\nabla^{\perp}_{\gamma'} e_r(t) = 0$. Moreover, from part (a) $\langle \nabla^{\perp}_{X_j} e_r, e_\delta \rangle = 0$ for any $1 \leq j \leq l$. Hence the above expression reduces to

$$f_i'(t) = -\sum_{j=l+1}^{n} \langle \nabla_{X_i} X_1, X_j \rangle f_j(t), \quad l < i \leq n.$$

Since $f_i(a) = 0$, it follows from the uniqueness of solutions of such systems of equations that $f_i(t) = 0$ for $t \in [0, a)$. q.e.d.

§3. PROOF OF THE MAIN THEOREMS

Proof of theorem 1. We will only consider $k > 2$, since the case $k = 2$, and therefore $r = 1$, was proved in [14].

If $k > 2$ and $r = 1$, then it follows from Lemma 2 (a) that $\overset{k}{\mathrm{Osc}}$ is parallel in M. Hence, using Theorem A we reduce the codimension to 1.

If $k > 2$ and $r > 1$, we consider the set

$$U = \left\{ p \in M; \dim \overset{2}{\mathrm{Osc}}_p > n+1 \right\}.$$

Since $1 < r \leq k-1$, it follows from Lemma 2 (b) that $\overset{k}{\mathrm{Osc}}$ is parallel in U and hence in the closure $\bar{U}$. Let $W = M - \bar{U}$. Using Lemma 4, we obtain that the sectional curvature in W is constant equal to c. Consider

$$W = W_0 \cup W_1,$$

where $W_i = \{p \in W;\ \dim \overset{2}{\mathrm{Osc}}_p = n+i\}$. Then, W_0 has empty interior and W_1 is an open subset of M. Moreover, from Remark 2, we get that the dimension of relative nullity index $\nu(p) = n-1$, for any $p \in W_1$. Therefore, W_1 is foliated by $(n-1)$-dimensional totally geodesic submanifolds of Q_c^N.

Now, for any point $p \in W_1$, we consider a geodesic γ, tangent to the relative nullity foliation, such that $\gamma(0) = p$. We claim that such a geodesic, which is also a geodesic of Q_c^N, cannot be extended indefinitely in W_1. In fact, if $c \leq 0$, this follows from the compactness of

M. If $c > 0$, the existence of a closed geodesic entirely contained in W_1 implies that the relative nullity $\nu \equiv n \bmod 2$, see [5]. This contradicts that $\nu \equiv n-1$ in W_1.

Hence γ must hit the boundary of W_1 at a point $\gamma(a)$. From Remark 1, we have that the relative nullity $\nu(\gamma(a)) = n-1$. Therefore $\gamma(a) \notin W_0$, since the relative nullity in W_0 is n. Now, the boundary of W_1 is the union of W_0 with the boundary of W. It follows that $\gamma(a)$ is a point in the boundary of W. Therefore, $\gamma(a)$ belongs to the boundary of U, where $\overset{k}{\mathrm{Osc}}$ is parallel. Using Lemma 6, we conclude that $\overset{k}{\mathrm{Osc}}$ is parallel at p. Since p is an arbitrary point in W_1, we obtain that $\overset{k}{\mathrm{Osc}}$ is parallel in W_1.

Finally, since W_0 has empty interior, we obtain that $\overset{k}{\mathrm{Osc}}$ is also parallel in W_0, and threrefore in M. Applying theorem A to the $(n+r)$-dimensional parallel distribution $\overset{k}{\mathrm{Osc}}$ in M, we reduce the codimension to r.

Remark 3. It is clear from the proof of theorem 1 that the compactness of M, for $c \leq 0$, can be replaced by the weaker assumption that M is complete and the image $f(M)$ is bounded in Q_c^N. Moreover, when $c > 0$, M needs only to be complete.

Proof of theorem 2. Using exactly the same arguments as in the previous theorem we obtain that W_1 is foliated by $(n-1)$-dimensional totally geodesic submanifolds of R^N.

Suppose there exists a point $p \in W^1$ such that any geodesic coming out from p, lying in the relative nullity foliation can be extended indefinitely. Then, the leaf of the relative nullity foliation passing through p is an $(n-1)$-dimensional affine plane in R^N. It follows from Cheeger's splitting theorem [3], that M is isometric to a cylinder over a curve.

Otherwise, for any $p \in W_1$, there exists a geodesic coming out from p, lying in the relative nullity foliation, which hits the boundary of W_1. In this case, as in the proof of theorem 1, we conclude that the codimension can be reduced to r. q.e.d.

Remark 4. If the ambient space is a complete (not necessarily simply connected) space form $\tilde{M}_c^N$ of constant curvature c then, as an immediate consequence of theorem 1, one can show the following.

THEOREM 1′. *Let $f: M^n \to \tilde{M}_c^N$ be an isometric immersion of a compact, simply connected manifold M^n, $n \geq 2$. Suppose there exist integers k, r, $k \geq 2$ and $0 \leq r \leq k-1$, such that $\dim \overset{k}{\mathrm{Osc}}_p = n+r$, for every $p \in M$. Then, there exists an $(n+r)$-dimensional manifold L^{n+r} and a totally geodesic isometric immersion $\pi: L \to \tilde{M}$ such that $f(M) \subset \pi(L)$. If $c > 0$, M needs only to be complete.*

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