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A Pseudoconservation Law for a Time-Limited Service Polling System with Structured Batch Poisson Arrivals

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Abstract—We consider a cyclic-service queueing system (polling system) with time-limited service, in which the length of a service period for each queue is controlled by a timer, i.e., the server serves customers until the timer expires or the queue becomes empty, whichever occurs first, and then proceeds to the next queue. The customer whose service is interrupted due to the timer expiration is attended according to the nonpreemptive service discipline. For the cyclic-service system with structured batch Poisson arrivals ($M^X/G/1$) and an exponential timer, we derive a pseudoconservation law and an exact mean waiting time formula for the symmetric system.

Keywords—Polling system, Structured batch arrival, Compound Poisson process, Time-limited service, Exponential timer, Exhaustive service, Nonpreemptive discipline.

1. INTRODUCTION

In cyclic-service queueing systems (polling systems) with nonzero switchover times, Watson [1] and Ferguson and Aminetzah [2] have first found an equation expressing a weighted sum of mean waiting times, called the pseudoconservation law. Using the stochastic decomposition of the workload in vacation systems, pseudoconservation laws have systematically been derived by Boxma and Groenendijk [3] for basic service disciplines, such as exhaustive, gated, one-limited (nonexhaustive), and one-decrementing (semiexhaustive) services. After them, these pseudoconservation laws have been extended to various service disciplines and a compound Poisson process with correlated arrivals as reviewed in survey articles by Takagi [4]. These pseudoconservation laws can be used to obtain simple and yet accurate approximations for the individual mean waiting times in asymmetric systems and also useful for optimization problems in flexible service disciplines with controllable parameters, e.g., [5]. On the other hand, time-limited service polling systems have gained much attention in view of both the applications and the theoretical analysis. The term time-limited service refers to the fact that the server serves a queue only up to an amount of time controlled by a timer during each service period, that is, the server serves waiting customers (messages or packets) until the timer expires or the queueing buffer becomes empty, whichever occurs first, and then proceeds to the next queue. The limited time is also called the

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maximum server attendance (MSA) time by Leung and Eisenberg [6]. The time-limited service disciplines are classified as exhaustive (or nongated) and gated time-limited services and furthermore, as nonpreemptive and preemptive-resume service disciplines with respect to the interrupted service caused by timer expiration as in [17]. The main merit of the time-limited service is that the MSA time can be arbitrarily adjusted. Such a flexible schedule is effective for the performance optimization, and has a potential applicability to communication systems with multiple grades of service requirements in multimedia broadband networks.

In this paper, we will derive a pseudoconservation law for a cyclic-service system with structured batch Poisson arrivals and an exponential time-limited service, which is an extension of the recent result for the $M/G/1$ polling system analyzed in [8]. The rest of the paper is organized as follows: in Section 2, we describe the model and notation in detail. After preliminaries in Section 3, we derive a pseudoconservation law and an exact mean waiting time formula for the symmetric polling system in Section 4. In Section 5, we give some remarks and provide a general pseudoconservation law combining the previous well-known results for the standard service disciplines.

2. MODEL AND NOTATION

We consider an $M^X/G/1$ cyclic-service queueing system with N infinite capacity buffers which are denoted by $Q_1, Q_2, \dots, Q_N$ and assume that the system has a Poisson arrivals process at rate λ such that each arrival contains G_i customers in Q_i, $i = 1, 2, \dots, N$, simultaneously. The generating functions (GFs) of the joint probability distribution (joint PD), $g(k_1, k_2, \dots, k_N) := \Pr\{G_1 = k_1,\ G_2 = k_2,\ \dots,\ G_N = k_N\}$, $k_i \geq 0$, $i = 1, 2, \dots, N$ and the GF of the marginal distribution are denoted by, respectively,

$$G(z_1, z_2, \dots, z_N) := \boldsymbol{E}\left[z_1^{G_1}, z_2^{G_2}, \dots, z_N^{G_N}\right] = G(\mathbf{z}),$$
$$G_i(x) := G(z_1 = 1,\ \dots,\ z_i = x,\ \dots,\ z_N = 1).$$

Some moments of the joint PD for $\{G_i\}$ are denoted by

$$g_i := \left[\frac{\partial G(\mathbf{z})}{\partial z_i}\right]_{\mathbf{z}=\mathbf{1}} = \boldsymbol{E}[G_i], \quad g_{i,j} := \left[\frac{\partial^2 G(\mathbf{z})}{\partial z_i \partial z_j}\right]_{\mathbf{z}=\mathbf{1}} = \boldsymbol{E}[G_i G_j], \qquad \text{for } i \neq j,$$
$$g_i^{(2)} := \left[\frac{\partial^2 G(\mathbf{z})}{\partial z_i^2}\right]_{\mathbf{z}=\mathbf{1}} = \boldsymbol{E}[G_i(G_i - 1)],$$

where the $\mathbf{z} = \mathbf{1}$ stands for $(z_1 = 1,\ \dots,\ z_i = 1,\ \dots,\ z_N = 1)$. The maximum length of a service period of a single server at Q_i, $i = 1, 2, \dots, N$ is limited by a given time T_i called the MSA time; in other words, the server serves the customers in Q_i until either the time limit expires, or the queue becomes empty, whichever occurs first, and then proceeds to Q_{i+1} mod (N), where customers arriving at currently in service can possibly be served in the same service period, i.e., exhaustive service discipline.

Furthermore, the service on the customer being served is completed during the current service period, i.e., nonpreemptive discipline. We assume that the MSA time T_i for Q_i, $i = 1, 2, \dots, N$ is exponentially distributed with mean $\bar{T}_i := 1/\alpha_i$. The Laplace-Stieltjes transform (LST) and the distribution function (DF) of the MSA time T_i, $i = 1, 2, \dots, N$ are denoted by $T_i^*(s) := \alpha_i/(s + \alpha_i)$ and $T_i(t)$, respectively. The time-limited schedule can be parameterized by a vector of $(\bar{T}_1, \bar{T}_2, \dots, \bar{T}_N)$. The LST of the DF, the mean, and the second moment of the service time H_i, $i = 1, 2, \dots, N$ of a customer at Q_i are denoted by $H_i^*(s)$, h_i, and $h_i^{(2)}$, respectively. Each arrival is also considered as a supercustomer whose service time (B) has the LST $B^*(s)$ of the DF, the mean, and the second moment given by, respectively,

$$B^*(s) := (H_1^*(s), H_2^*(s), \ldots, H_N^*(s)),$$
$$b := \sum_{i=1}^{N} g_i h_i,$$
$$b^{(2)} := \sum_{i=1}^{N} \left(g_i h_i^{(2)} + g_i^{(2)} h_i^2 \right) + 2 \sum_{i=2}^{N} h_i \sum_{j=1}^{i-1} g_{i,j} h_j.$$

The total load offered to the system is then given by

$$\hat{\rho} := \lambda b = \sum_{i=1}^{N} \rho_i,$$
$$\rho_i := \lambda_i h_i, \quad \lambda_i := \lambda g_i, \qquad i = 1, 2, \ldots, N.$$

The LST of the DF, the mean, and the second moment of the switchover time D_i, $i = 1, 2, \ldots, N$ needed by the server to switch from Q_i to Q_{i+1} are denoted by $D_i^*(s)$, d_i, and $d_i^{(2)}$, respectively. The switchover times are independent of the arrival and service processes. The mean and the variance of the total switchover time during a cycle of the server are then given by, respectively,

$$\bar{D} := \sum_{i=1}^{N} d_i, \qquad \sigma_D^2 := \sum_{i=1}^{N} \left(d_i^{(2)} - d_i^2 \right).$$

We refer to an instant the server arrives at Q_i from Q_{i-1} as a *polling instant* of Q_i. Furthermore, we define the *polling cycle time* (C) as the time between the server's visit to the same queue in successive cycles, the *service period* (S_i) of Q_i as the time between the arrival of the server at Q_i and his subsequent departure from Q_i, and the *intervisit time* (I_i) for Q_i as the time between the server's departure from Q_i and the next polling instant of Q_i.

REMARK 2.1. We do not consider a batch that contains no customers at all, i.e., $g(0, 0, \ldots, 0) = 0$. Some special cases are as follows; if each arrival contains only customers for a single queue, we have $g_{i,j} = 0$ for $i \neq j$, $N \geq i$, $j \geq 1$. Furthermore, if each arrival contains a single customer, we have $g_i = 1$ and $g_i^{(2)} = 0$ for $N \geq i \geq 1$. If the number of customers contained in each arrival is independent for different queues, we have $g_{i,j} = g_i g_j$ for $i \neq j$, $N \geq i$, $j \geq 1$, see [9].

3. PRELIMINARIES

The nonpreemptive, time-limited schedule is closely related to the Bernoulli schedule with parameters $(p_1, p_2, \ldots, p_N)$. From correspondence of the time-limited service to the Bernoulli schedule, we have the following results:

$$p_i := \Pr\{T_i > H_i\} = H_i^*(\alpha_i), \qquad \bar{p}_i := 1 - H_i^*(\alpha_i). \tag{1}$$

We define the following LSTs and GFs for $i = 1, 2, \ldots, N$:

$$\begin{aligned} F_i^*(s) &:= \boldsymbol{E}\left[e^{-sH_i} \mid T_i > H_i\right] = \frac{H_i^*(s + \alpha_i)}{H_i^*(\alpha_i)}, \\ \bar{F}_i^*(s) &:= \boldsymbol{E}\left[e^{-sH_i} \mid T_i \leq H_i\right] = \frac{H_i^*(s) - H_i^*(s + \alpha_i)}{1 - H_i^*(\alpha_i)}, \\ Q_{F_i}(x) &:= F_i^*(\lambda - \lambda G_i(x)), \qquad Q_{\bar{F}_i}(x) := \bar{F}_i^*(\lambda - \lambda G_i(x)), \\ Q_{f_i}(x) &:= p_i F_i^*(\lambda - \lambda G_i(x)), \qquad Q_{\bar{f}_i}(x) := \bar{p}_i \bar{F}_i^*(\lambda - \lambda G_i(x)), \\ Q_{H_i}(x) &:= H_i^*(\lambda - \lambda G_i(x)). \end{aligned} \tag{2}$$

REMARK 3.1. One sees that $H_i^*(s) = p_i F_i^*(s) + \bar{p}_i \bar{F}_i^*(s)$ and $F_i^*(s)(\bar{F}_i^*(s)) = H_i^*(s)$ only for $H_i^*(s) = e^{-sh_i}$ (i.e., $H_i(t)$ is a unit distribution with the mean h_i), that is, $F_i^*(s)(\bar{F}_i^*(s))$ has a bias to $H_i^*(s)$, and p_i ($\bar{p}_i$) is not independent of H_i.

Further, we define the following random variables for $i = 1, 2, \dots, N$:

K_i := the number of customers at the polling instant at Q_i,

N_i := the number of customers served during S_i,

L_i := the number of remaining customers in Q_i when the server leaves Q_i.

Then, we have

$$\boldsymbol{E}(L_i) = \boldsymbol{E}(K_i) - (1 - \rho_i)\boldsymbol{E}(N_i), \tag{3}$$

which can be derived from the following known equations in [10]:

$$\begin{aligned} &\boldsymbol{E}(K_i) + \lambda_i \boldsymbol{E}(S_i) = \boldsymbol{E}(N_i) + \boldsymbol{E}(L_i), \qquad \boldsymbol{E}(S_i) = h_i \boldsymbol{E}(N_i), \\ &\boldsymbol{E}(S_i) = \rho_i \boldsymbol{E}(C), \qquad \boldsymbol{E}(I_i) = (1 - \rho_i)\boldsymbol{E}(C), \qquad \boldsymbol{E}(C) = \frac{\bar{D}}{(1 - \hat{\rho})}. \end{aligned} \tag{4}$$

In the next section, we will use Lemma 1 on the following GFs defined by: for $m \geq 0$,

$P_i(x)$:= GF of the PD $\{p_i(m)\}$ of the number (m) of customers in Q_i at an arbitrary epoch,

$P_i^-(x)$:= GF of the PD $\{p_i^-(m)\}$ of the number of customers in Q_i just before an arrival epoch,

$\Pi_i(x)$:= GF of the PD $\{\pi_i(m)\}$ of the number of customers in Q_i just after a departure epoch of a customer from the system,

$\Pi_i^-(x)$:= GF of the PD $\{\pi_i^-(m)\}$ of the number of customers in Q_i just before a departure epoch of a customer from the system.

LEMMA 1.

$$\Pi_i(x) = P_i(x) R_i(x), \qquad R_i(x) := \frac{1 - G_i(x)}{g_i(1 - x)}. \tag{5}$$

PROOF. Let a set $S_m := \{m+1, m+2, \dots\}$ and $S_m^C := \{0, 1, \dots, m\}$ for $m \geq 0$ with respect to the number of customers in Q_i. Then using the exit rate (r_{out}) from the set S_m and the entry rate (r_{in}) into S_m, that is, from the discrete-state level-crossing analysis regarding the state $\{m+1\}$, we get

$$\begin{aligned} r_{\text{out}} &:= \lambda_{\text{down}} \pi_i^-(m+1) = \lambda g_i \pi_i(m), \\ r_{\text{in}} &:= \lambda_{\text{up}} \sum_{j=0}^{m} p_i^-(j) \sum_{k=m-j+1}^{\infty} \Pr\{G_i = k\} \\ &= \lambda \sum_{j=0}^{m} p_i(j) \left[1 - \sum_{k=0}^{m-j} \Pr\{G_i = k\} \right], \end{aligned}$$

where λ_{down} (λ_{up}) is the long run rate of downward (upward) jumps, e.g., see [11] and [12], and we have used the PASTA (Poisson arrivals see time averages) property for the second equation with λ_{up}. Equating the exit and entry rates, we obtain

$$g_i \pi_i(m) = \sum_{j=0}^{m} p_i(j) \left[1 - \sum_{k=0}^{m-j} \Pr\{G_i = k\} \right]. \tag{6}$$

Finally, taking the generating function of (6) we get (5). ∎

The GF $R_i(x)$ represents the GF for the forward (also backward) recurrence time in the discrete-time renewal process, where the interval between two successive renewal points is given by G_i. A probabilistic interpretation for (5) is given as follows: first of all, we assume the FIFO discipline for Q_i. Then, $P_i(x)R_i(x)$ represents the GF for the number of all customers placed before an arbitrary tagged customer chosen randomly from an arriving batch in Q_i when the tagged customer has arrived at Q_i because of the PASTA property, while $\Pi_i(x)$ is the GF for the number of customers being behind the tagged customer in Q_i just after the tagged customer's departure epoch. Both GFs should be equal in steady state. Note here that (5) holds for any service mechanism with nonbatch and nonpreemptive services, since the number of customers in Q_i is independent of service disciplines as FIFO, LIFO, and so on. Equation (5) for the time-limited service polling system is a generalized result including one for the $M^X/G/1$ single-queue without vacation times in [13], and also follows from the result derived by Takine and Takahashi [14] as a special case of a batch Markovian arrival process (BMAP).

4. QUEUEING ANALYSIS

We will derive Theorem 1 for the time-limited service system using the GFs formulated on both service-beginning and customer-departure epochs. In what follows, an epoch is a polling instant, a service completion or a service beginning for a customer in Q_i. We consider a sequence of pairs of random variables (Y_n, J_n), $n = 1, 2, \dots$ defined as follows: Y_n denotes the number of customers at the n^{th} epoch, while $J_n = 0$ if the epoch marks a polling instant of Q_i, $J_n = 1$ if the epoch marks a service completion of a customer in Q_i, and $J_n = 2$ if the epoch marks a service beginning for each customer.

THEOREM 1. PSEUDO-CONSERVATION LAW.

For a stable $M^X/G/1$ cyclic-service system with an exponential time-limited service specified by a vector $(\bar{T}_1, \bar{T}_2, \dots, \bar{T}_N)$, the following relationship among the mean waiting times holds:

$$\sum_{i=1}^{N} \rho_i \left[1 - \frac{\lambda_i \bar{D}}{1-\hat{\rho}} (1 - H_i^*(\alpha_i))\right] \boldsymbol{E}(W_i)$$
$$= \frac{\lambda}{2(1-\hat{\rho})} \sum_{i=1}^{N} \left(\hat{\rho} g_i h_i^{(2)} + g_i^{(2)} h_i^2\right) + \frac{\lambda}{1-\hat{\rho}} \sum_{i=2}^{N} h_i \sum_{j=1}^{i-1} g_{i,j} h_j + \frac{\hat{\rho}\sigma_D^2}{2\bar{D}} + \frac{\bar{D}}{2(1-\hat{\rho})} \left[\hat{\rho} - \sum_{i=1}^{N} \rho_i^2\right]$$
$$+ \frac{\lambda \bar{D}}{2(1-\hat{\rho})} \sum_{i=1}^{N} h_i \left[2\rho_i g_i + 2\lambda_i g_i \frac{d}{d\alpha_i} H_i^*(\alpha_i) + g_i^{(2)} (1 - H_i^*(\alpha_i))\right]. \tag{7}$$

PROOF. First of all, we define the following GFs:

$$\Phi_i(x) := \lim_{n\to\infty} \boldsymbol{E}\left[x^{Y_n} \mid J_n = 0\right],$$
$$\Pi_i(x) = \lim_{n\to\infty} \boldsymbol{E}\left[x^{Y_n} \mid J_n = 1\right],$$
$$\Pi_i^+(x) := \lim_{n\to\infty} \boldsymbol{E}\left[x^{Y_n} \mid J_n = 2\right], \qquad \left(= \sum_{k=1}^{\infty} \pi_i^+(k) x^k\right),$$

where $\Pi_i^+(0) \equiv 0$. Then we obtain a functional relationship between $\Pi_i(x)$ and $\Pi_i^+(x)$,

$$\Pi_i(x) = \Pi_i^+(x) \left[Q_{f_i}(x) + Q_{\bar{f}_i}(x)\right] \frac{1}{x} = \Pi_i^+(x) Q_{H_i}(x) \frac{1}{x} \tag{8}$$

and a functional relationship with $\Pi_i^+(x)$,

$$\Pi_i^+(x) = \kappa \left[\Phi_i(x) - \Phi_i(0)\right] + \Pi_i^+(x) Q_{f_i}(x) \frac{1}{x} - \pi_i^+(1) x Q_{f_i}(0) \frac{1}{x}, \tag{9}$$

where

$$\kappa := \lim_{n\to\infty} \frac{\Pr\{J_n = 0\}}{\Pr\{J_n = 2\}} = \frac{(1-\hat{\rho})}{\lambda_i \bar{D}}. \tag{10}$$

Note here that $1/\kappa$ is nothing but the mean number of customers served in one service-period at Q_i, i.e., $1/\kappa = E(N_i)$. Combining (8), (9), and Lemma 1, we obtain finally

$$(x - Q_{f_i}(x))R_i(x)P_i(x) = \frac{(1-\hat{\rho})\,Q_{H_i}(x)}{\lambda_i \bar{D}} \left[\Phi_i(x) + \frac{\lambda_i \bar{p}_i \bar{D}}{1-\hat{\rho}} - 1\right]. \tag{11}$$

Therefore, taking the first derivative of both sides of (11) with respect to x and applying Little's formula to $P_i'(1)$ lead to

$$\Phi_i'(1) = \frac{\lambda_i \bar{D}}{1-\hat{\rho}} \left[1 + \lambda_i \frac{d}{d\alpha_i} H_i^*(\alpha_i) + (1 - H_i^*(\alpha_i)) \left\{\frac{g_i^{(2)}}{2g_i} + \lambda_i \boldsymbol{E}(W_i)\right\}\right]. \tag{12}$$

Furthermore, it follows from the decomposition theorem for vacation systems that

$$\begin{aligned}\sum_{i=1}^{N} \rho_i \boldsymbol{E}(W_i) = {} & \frac{\lambda b^{(2)}}{2(1-\hat{\rho})} - \sum_{i=1}^{N} \frac{\rho_i h_i^{(2)}}{2h_i} + \frac{\hat{\rho}\sigma_D^2}{2\bar{D}} + \frac{\bar{D}}{2(1-\hat{\rho})} \left[\hat{\rho} - \sum_{i=1}^{N} \rho_i^2\right] \\ & + \sum_{i=1}^{N} h_i \boldsymbol{E}(L_i),\end{aligned} \tag{13}$$

$$\boldsymbol{E}(L_i) = \Phi_i'(1) - \frac{\lambda_i(1-\rho_i)\bar{D}}{1-\hat{\rho}}, \tag{14}$$

where we have used (3), (4), and $\boldsymbol{E}(K_i) = \Phi_i'(1)$ for derivation of (14), see [15] and [8]. Hence, arranging (12)–(14), we reach (7). ∎

Next we give an exact mean waiting time formula for the symmetric polling system

$$\begin{aligned}\boldsymbol{E}(W) = {} & \frac{1}{2\left[1 - \hat{\rho} - \lambda g \bar{D}\{1 - H^*(\alpha)\}\right]} \left[\frac{\hat{\rho} g h^{(2)} + g^{(2)} h^2}{gh} + (N-1)\frac{g^{[2]} h}{g}\right. \\ & \left. + (1-\hat{\rho})\frac{\sigma_D^2}{\bar{D}} + \bar{D}\left\{1 + \lambda g h + 2\lambda g \frac{d}{d\alpha} H^*(\alpha) + \frac{g^{(2)}}{g}(1 - H^*(\alpha))\right\}\right],\end{aligned} \tag{15}$$

where $g := g_i$, $g^{(2)} := g_i^{(2)}$, $g^{[2]} := g_{i,j}$, $i, j = 1, 2, \ldots, N$ $(i \neq j)$.

REMARK 4.1. Relationship (7) reduces to equation (9) in Theorem 1 in [8] setting $g_i = 1$, $g_i^{(2)} = g_{i,j} = 0$, and $\lambda_i = \lambda g_i$. Similarly, formula (15) reduces to equation (18) in Theorem 2 in [8] setting $g = 1$, $g^{(2)} = g^{[2]} = 0$, and $\hat{\rho} = N\rho$. If all the switchover times are zero, (7) reduces to the conservation law for $M^X/G/1$ queueing systems with multiple customer classes.

REMARK 4.2. A necessary and sufficient condition for stability of the time-limited service polling system is given by

$$\hat{\rho} < 1 \quad \text{and} \quad \lambda_i < \frac{1 - \hat{\rho} + \rho_i}{\bar{p}_i \bar{D} + h_i}, \qquad \text{for } i \in \{1, 2, \ldots, N\}. \tag{16}$$

An intuitive proof of (16) is given as follows: Under a situation that the queue length of Q_i is infinite at a polling instant at Q_i, the probability denoted by $p(n)$ that n customers in Q_i are served during one service-period is given by $p(n) = p_i^{n-1}\bar{p}_i$, $n = 1, 2, 3, \ldots$. Then the average maximum number of customers served in one service-period equals $\sum_{n=1}^{\infty} np(n) = 1/\bar{p}_i$. Recall here that Georgiadis and Szpankowski [17] have proved the following necessary and sufficient condition for the polling system with k_i-limited service at Q_i, in which the number of customers served at Q_i per visit of the server is exhaustively limited by k_i:

$$\hat{\rho} < 1 \quad \text{and} \quad \lambda_i \boldsymbol{E}(C) < k_i, \qquad \text{for } i \in \{1, 2, \dots, N\}. \tag{17}$$

Thus, replacing k_i in (17) by $1/\bar{p}_i$, we get

$$\hat{\rho} < 1 \quad \text{and} \quad \lambda_i \boldsymbol{E}(C) < \frac{1}{\bar{p}_i}, \qquad \text{for } i \in \{1, 2, \dots, N\}, \tag{18}$$

which leads to (16) using $\boldsymbol{E}(C) = \bar{D}/(1-\hat{\rho})$, where $\lambda_i \boldsymbol{E}(C)$ represents the average number of customers arrived at Q_i during one cycle-time. In a stable system, $\lambda_i \boldsymbol{E}(C)$ should be less than the average maximum number of customers served in one service-period. The second relationship in (16) is identical with the stability condition (necessary condition) for the polling system with Bernoulli schedules as given in [16].

5. CONCLUDING REMARKS

Under the gated time-limited service, the server serves at most those customers that are found at the polling instant of a queue. For the polling system with gated time-limited services, it remains as an open problem to derive the pseudoconservation law, that is, the similar relationship to (7) has not been obtained even for the $M/G/1$ polling system with gated time-limited service as discussed in [8].

Lastly, combining Theorem 1 and the well-known results for exhaustive, gated, limited, and decrementing service disciplines, we give the following general pseudoconservation law for the $M^X/G/1$ polling system with combined basic service disciplines:

$$\begin{aligned} &\sum_{i \in T} \rho_i \left\{ 1 - \frac{\lambda_i \bar{D}}{1-\hat{\rho}} \left(1 - H_i^*(\alpha_i)\right) \right\} \boldsymbol{E}(W_i) + Z_W = C + Z_C \\ &+ \frac{\lambda \bar{D}}{2(1-\hat{\rho})} \sum_{i \in T} h_i \left[2\rho_i g_i + 2\lambda_i g_i \frac{d}{d\alpha_i} H_i^*(\alpha_i) + g_i^{(2)} \left(1 - H_i^*(\alpha_i)\right) \right], \end{aligned} \tag{19}$$

$$\begin{aligned} C &:= \frac{\lambda}{2(1-\hat{\rho})} \sum_{i=1}^{N} \left(\hat{\rho} g_i h_i^{(2)} + g_i^{(2)} h_i^2 \right) + \frac{\lambda}{1-\hat{\rho}} \sum_{i=2}^{N} h_i \sum_{j=1}^{i-1} g_{i,j} h_j + \frac{\hat{\rho} \sigma_D^2}{2\bar{D}} \\ &\quad + \frac{\bar{D}}{2(1-\hat{\rho})} \left[\hat{\rho} - \sum_{i=1}^{N} \rho_i^2 \right], \\ Z_C &:= \frac{\bar{D} \sum_{i \in G,L} \rho_i^2}{1-\rho} + \frac{\lambda \bar{D} \sum_{i \in L} g_i^{(2)} h_i}{2(1-\rho)} + \frac{\lambda \bar{D} \sum_{i \in D} \left[(1-2\rho_i) g_i^{(2)} h_i - (\lambda g_i)^2 g_i h_i h_i^{(2)} \right]}{2(1-\rho)}, \\ Z_W &:= \sum_{i \in E,G} \rho_i \boldsymbol{E}(W_i) + \sum_{i \in L} \rho_i \left\{ 1 - \frac{\lambda_i \bar{D}}{1-\hat{\rho}} \right\} \boldsymbol{E}(W_i) + \sum_{i \in D} \rho_i \left\{ 1 - \frac{\lambda_i (1-\rho_i) \bar{D}}{1-\hat{\rho}} \right\} \boldsymbol{E}(W_i), \end{aligned} \tag{20}$$

where E, G, L, D, and T stand for the index sets of queues with exhaustive, gated, one-limited, one-decrementing, and time-limited service disciplines, respectively. The relationship $Z_W = C + Z_C$ is given in [4].

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