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A first order phase transition with non-constant density

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ABSTRACT

We introduce a new model for first order phase transitions accounting for non-constant densities of the phases during the process. The resulting initial and boundary value problem for a PDE system is recovered by thermodynamical principles. The resulting system presents some singularities and strong nonlinearities accounting for internal constraints, ensuring in particular the positivity of the pressure and the temperature. Physical consistency for the order parameter comes from a maximum principle argument. Existence of a weak solution is proved by a regularization-passage to the limit procedure.

1. Introduction

This paper deals with a first order phase transition problem (ice–water phase change) in which the different phases are characterized by non-constant densities. In the literature many differential models have been investigated under the hypothesis of constant densities for the ice and water phases and of a constant transition temperature providing a vertical melting line (see [21] for some references on the subject). The novelty of this paper consists in removing these restrictions (see [7,13] for a fairly different model), taking into account the evolution of the density during the ice–water phase transition. This leads to a melting line with a negative slope, so to describe the experimental diagram in the picture. As a consequence the pressure of the phases plays a role in the thermo-mechanical behavior of the system and is included as a new variable. Note that, towards physical consistency, the positivity of the pressure p has to be guaranteed. In our framework this is obtained by the presence of an internal constraint included in the energy functional. The resulting PDE system is recovered by the classical conservation laws of the mass and momentum and a new law on the second order structure, motivated by the actions of micro-forces resulting from motions at the microscopic level. As far as thermal properties of the system, we describe the evolution of the entropy (see, e.g., [2,4]). The main advantage of this approach consists in the possibility to prove directly the positivity of the absolute temperature, avoiding any a posteriori maximum principle argument, and to get energy and dissipation estimates directly by the variational formulation of the problem. The evolution of the phase transition is described by an order parameter, as it is usually done in the phase-field theory (cf. among others [14,15,18]). In particular, we are referring to some fairly recent theories in which microscopic movements and microscopic forces responsible for the phase transition process are included in the momentum and energy balance [9,11,17]. Hence, which is new in this framework, the physical consistency of the phase transition is not ensured by internal constraint (as it is done for the pressure), but it comes directly by the evolution equation due to the choice of the energy potentials.

Here is the outline of the paper. In Section 2 we introduce the model. In Section 3 we make precise the variational formulation and the main existence result. In Section 4 a maximum principle argument shows that the phase parameter remains bounded during the evolution. In Section 5 we introduce the approximated problem and state the corresponding

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existence result, by use of a fixed point argument. In Section 6, after proving suitable a priori estimates on the approximated solution, we pass to the limit and get the existence of a solution for the original problem.

2. The model

In any phase transition, a transformation from a more ordered material structure to a less one or vice versa is shown. Moreover, the structure order for many materials increases if the temperature goes below a critical value. Actually, in the solid–fluid first order phase transition, the solid phase has a greater structure due to the crystal symmetry group. Landau [16] suggested that the symmetry or the structure of the material should be measured by a new unknown χ, which he called order parameter. Hence, as usually thermal effects are considered in phase transitions model, so that the state variables are the order parameter and the absolute temperature θ. The resulting model consists into two differential equations, the balance equation governing the evolution of the order parameter and the heat equation. They are written in some bounded domain Ω, where the system is assumed to be located. Actually, to describe the general ice–water phase transition the variation of the density of the material is taken into account. Indeed, as the density of the water and the ice are different, the phase transition cannot be described by a model in which the density is considered as a constant coefficient, as in the classical phase transition works. Thus, in addition to θ and χ we consider the density ρ as a new variable for which there holds the so-called continuity equation

$$\rho_t + \operatorname{div} \rho\mathbf{v} = 0, \tag{2.1}$$

$\mathbf{v}$ being the velocity. Assuming the phenomenon of the phase transition in a quasi-static regime, the motion equation reads

$$\operatorname{div}(\mathbf{T}_e - p\mathbf{1}) + \rho\mathbf{b} = \mathbf{0}, \tag{2.2}$$

where p is the pressure, $\mathbf{1}$ the identity matrix, $\mathbf{T}_e$ the *extra-stress*, and $\mathbf{b}$ the body forces. Considering the ice and the water as incompressible viscous materials, the extra-stress can be written as

$$\mathbf{T}_e = \nu(\chi)\varepsilon(\mathbf{v}), \tag{2.3}$$

where the viscosity $\nu(\chi)$ is a positive scalar function (we will assume to be constant in the sequel), and $\varepsilon(\cdot)$ is the symmetric tensor, defined by $\varepsilon_{ij}(\mathbf{v}) = \frac{1}{2}(\partial_{x_i} v_j + \partial_{x_j} v_i)$, $\mathbf{v} = (v_1, v_2, v_3)$ (we restrict ourselves to small perturbations assumption).

Since we have assumed the hypothesis of incompressibility both for ice and water, it is correct to suppose that $\operatorname{div}\mathbf{v}$ is different from 0 only during the phase transition.

Now, let us introduce the equation describing the evolution of the order parameter, which is obtained by the balance law on the order structure, resulting from the principle of virtual power taking into account motion at the microscopic level and dual force at the same microscopic level (see [6,8–11,17]), given by

$$c_1\rho\chi_t = c_2 \operatorname{div}(\rho\nabla\chi) - \rho F'(\chi) - \rho(\theta + \lambda p)G'(\chi), \tag{2.4}$$

where the constants c_i, $i = 1, 2$, are strictly positive, while the coefficient λ is a small constant. Note in particular that λ is connected to the slope of line splitting up ice and water. The functions F and G characterize the order and the feature of the phase transition. In particular, in the case of a first order transition, as ice–water, they are addressed as follows [7]

$$F(\chi) = \frac{\chi^4}{4} - \frac{\chi^3}{3}, \qquad G(\chi) = \frac{\chi^4}{4} - \frac{2\chi^3}{3} + \frac{\chi^2}{2}. \tag{2.5}$$

Introducing the first principle of thermodynamics we are able to deduce in particular an evolution equation for the temperature. Letting e the internal energy, h the specific internal heat power, defined also as the rate at which the heat is absorbed per unit mass, P^i_χ the internal structure order power density, and P^i_v the internal mechanical power, the first principle assumes the form

$$\rho e_t = \rho h + P^i_\chi + P^i_v. \tag{2.6}$$

Let us recall that the above quantity h may be specified, by the heat balance law as follows

$$\rho h = -\operatorname{div}\mathbf{q} + \rho r, \tag{2.7}$$

$\mathbf{q}$ being the heat flux vector, which is governed by the Fourier law

$$\mathbf{q} = -k_0\nabla\theta, \quad k_0 > 0, \tag{2.8}$$

while r is the external heat supply. Hence, the internal order structure power P^i_χ follows from (2.4)

$$P^i_\chi = \rho\left(\frac{c_1}{2}(\chi)^2_t + F_t(\chi) + (\theta + \lambda p)G'(\chi)\chi_t + c_2\nabla\chi\cdot\nabla\chi_t\right). \tag{2.9}$$

Finally, P_v^i is given by

$$P_v^i = p\frac{\rho_t}{\rho} + \nu(\chi)\nabla\mathbf{v}\cdot\nabla\mathbf{v}. \tag{2.10}$$

The second law of thermodynamics implies that there exists a state function η, called entropy function, satisfying

$$\rho\eta_t \geqslant \rho\frac{h}{\theta} + \frac{1}{\theta^2}\mathbf{q}\cdot\nabla\theta. \tag{2.11}$$

Then from (2.6)–(2.10), it follows that

$$\rho\big(e_t - F_t - \theta G_t - \gamma\chi_t^2\big) - p\left(\frac{\rho_t}{\rho} + \lambda\rho G_t\right) - \nu(\chi)\nabla\mathbf{v}\cdot\nabla\mathbf{v} - \left(\frac{c_2(\nabla\chi)^2}{2}\right)_t = \kappa_0\Delta\theta + \rho r. \tag{2.12}$$

Moreover, the second law (2.11) yields the inequality

$$\rho\theta\eta_t \geqslant \rho e_t - \mathcal{P}_\chi^i - \mathcal{P}_v^i + \frac{1}{\theta}\mathbf{q}\cdot\nabla\theta. \tag{2.13}$$

Then, we obtain from (2.13)

$$\begin{aligned}&\rho(\psi_\theta + \eta)\theta_t - \rho\gamma\chi_t^2 + \rho\big(\psi_\chi - F' - \theta G'\big)\chi_t - p\left(\frac{\rho_t}{\rho} + \lambda\rho\dot{G}(\chi)\right)\\ &\quad - \nu(\chi)\nabla\mathbf{v}\cdot\nabla\mathbf{v} + \rho(\psi_{\nabla\chi} - c_2\nabla\chi)\cdot\nabla\chi_t - \frac{\kappa_0}{\theta}|\nabla\theta|^2 \leqslant 0.\end{aligned} \tag{2.14}$$

First, we let

$$\eta = -\psi_\theta, \tag{2.15}$$

$$\frac{\rho_t}{\rho} + \lambda\rho G_t(\chi) = 0 \tag{2.16}$$

so that there exists a constant ρ_0, which represents the water density, such that

$$\rho = \frac{\rho_0}{1 + \lambda\rho_0 G(\chi)}. \tag{2.17}$$

The density ρ is given by two contributions ρ_0 and ρ_1, such that

$$\frac{1}{\rho} = \frac{1}{\rho_0} + \frac{1}{\rho_1} \tag{2.18}$$

where

$$\frac{1}{\rho_1} = \lambda G(\chi). \tag{2.19}$$

Moreover, in (2.14) we consider

$$F' + \theta G' = \psi_\chi, \tag{2.20}$$

and

$$\psi_{\nabla\chi} = c_2\nabla\chi. \tag{2.21}$$

At the end we get

$$\rho\big(e_\theta\theta_t - \theta G_t - \gamma\chi_t^2\big) - \nu(\chi)\nabla\mathbf{v}\cdot\nabla\mathbf{v} - \left(\frac{c_2(\nabla\chi)^2}{2}\right)_t = \kappa_0\Delta\theta + \rho r$$

from which we obtain the entropy equation

$$\rho\left(\frac{e_\theta}{\theta}\theta_t - G_t - \frac{\gamma}{\theta}\chi_t^2\right) - \frac{\nu}{\theta}(\chi)\nabla\mathbf{v}\cdot\nabla\mathbf{v} - \left(\frac{c_2(\nabla\chi)^2}{2\theta}\right)_t = \frac{\kappa_0}{\theta}\Delta\theta + \rho\frac{r}{\theta}. \tag{2.22}$$

The differential system which describes the ice–water phase transition is given by Eqs. (2.2), (2.4), (2.16), (2.22). In the small perturbations framework, the terms of second order are disregarded. Then we obtain the following differential system written in the domain Ω.

$$\mathbf{0} = \frac{1}{\rho_0}\operatorname{div}(-p\mathbf{1} + \mathbf{T}_e) + \mathbf{b}, \tag{2.23}$$

$$c_1\chi_t = c_2\operatorname{div}(\nabla\chi) - F'(\chi) - (\theta + \lambda p)G'(\chi), \tag{2.24}$$

$$\frac{e_\theta}{\theta}\theta_t - \dot{G}(\chi) = \frac{\kappa_0}{\rho_0}\Delta\log\theta + \frac{r}{\theta}. \tag{2.25}$$

Note that in particular, by (2.18), ρ and ρ_0 are related by the equation (see also (2.19))

$$\frac{1}{\rho} - \frac{1}{\rho_0} = \lambda G(\chi). \tag{2.26}$$

Finally, accounting for a constraint on p ensuring that $p \geqslant 0$, we rewrite the continuity equation as follows

$$-\operatorname{div}\mathbf{v} + \lambda\rho_0 G_t(\chi) + \partial I_{[0,+\infty)}(p) \ni 0 \tag{2.27}$$

where $\partial I_{[0,+\infty)}(p)$ is the empty set for p negative, $\partial I_{[0,+\infty)}(p) = 0$ if $p > 0$ and $\partial I_{[0,+\infty)}(0) = (-\infty, 0]$. This relationship proves that $\partial I_{[0,+\infty)}(p)$ is not empty, thus that pressure p is non-negative. When pressure p is null, relationship (2.27) gives a negative right-hand side for continuity equation (2.1). The quantity of mass present at a point decreases due to the apparition of voids. This is the cavitation phenomenon in fluids.

The resulting system (2.22)–(2.27) is complemented with suitable initial and boundary conditions. More precisely, it is assumed that a known traction $\mathbf{t}$ is applied on a part of the boundary, say Γ_2, while no heat flux appears through the boundary, i.e.

$$\mathbf{q}\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega \tag{2.28}$$

and

$$(\mathbf{T}_e - p\mathbf{1})\cdot\mathbf{n} = \mathbf{t} \quad \text{on } \Gamma_2. \tag{2.29}$$

Then, velocities are fixed to be zero on the remaining part of the boundary, say Γ_1,

$$\mathbf{v} = \mathbf{0} \quad \text{on } \Gamma_1. \tag{2.30}$$

Finally, for χ we assume, as usual, homogeneous Neumann boundary condition

$$\partial_n\chi = 0 \quad \text{on } \partial\Omega. \tag{2.31}$$

3. The variational formulation and main analytical results

In this section, we make precise the analytical problem we are dealing with and state our main results. We let Ω be a bounded and smooth domain in $\mathbb{R}^3$ and denote by Γ its boundary $\partial\Omega$. We assume that $\Gamma = \Gamma_1 \cup \Gamma_2$, the measure of Γ_1 being strictly positive. Hence, denoting by T a fixed final time, we investigate the evolution of the phenomenon we are considering in the cylinder $Q := \Omega \times (0, T)$. Throughout the paper, given a Banach space X, we denote by ${}_{X'}\langle\cdot,\cdot\rangle_X$ the duality pairing between X' and X, and by $\|\cdot\|_X$ both the norm in X and in any power of it. Henceforth, we introduce the Hilbert triplet

$$V \hookrightarrow H \hookrightarrow V', \quad V := H^1(\Omega), \qquad H := L^2(\Omega),$$

H being identified as usual with its dual space. Then, let us take

$$W = \{\mathbf{v} \in H^1(\Omega)^3,\ \mathbf{v} = \mathbf{0} \text{ in } \Gamma_1\}.$$

To write the abstract version of our problem, the following bounded operators are used

$$\mathcal{A} : W \to W', \quad {}_{W'}\langle\mathcal{A}\mathbf{u}, \mathbf{v}\rangle_W = \sum_{i,j=1}^{3} \varepsilon_{ij}(\mathbf{u})k_{ij}\varepsilon_{ij}(\mathbf{v}), \tag{3.1}$$

$$A : V \to V', \quad {}_{V'}\langle Au, v\rangle_V = \int_\Omega \nabla u\cdot\nabla v. \tag{3.2}$$

We recall that the viscosity matrix $K = (k_{ij})$ is symmetric and positive definite. Thus, the Poincaré inequality yields that there exists $C > 0$ such that

$$ {}_{W'}\langle\mathcal{A}\mathbf{u}, \mathbf{u}\rangle_W \geqslant C\|\mathbf{u}\|_W^2. \tag{3.3}$$

We introduce $F, G : \mathbb{R} \to \mathbb{R}$ such that

$$F, G \in C^2(\mathbb{R}), \qquad F \geqslant c_*, \qquad G \geqslant 0, \tag{3.4}$$

$$F', G' \leqslant 0 \quad \text{in } (-\infty, \chi_*), \qquad F', G' \geqslant 0 \quad \text{in } \big(\chi^*, +\infty\big) \tag{3.5}$$

where $c_*, \chi_*, \chi^* \in \mathbb{R}$, $\chi_* < \chi^*$. In particular, $[\chi_*, \chi^*]$ denotes the range of admissible values for the parameter χ.

Let us list assumptions on initial and boundary data. We have (here $0 < \theta_* < \theta^*$)

$$\theta_\Gamma \in L^2\big(0, T; H^{1/2}(\Gamma)\big) \cap H^1\big(0, T; H^{-1/2}(\Gamma)\big), \qquad \theta_* \leqslant \theta_\Gamma \leqslant \theta^*, \tag{3.6}$$

$$\theta_0 \in H, \qquad \ln\theta_0 \in H, \tag{3.7}$$

$$\chi_0 \in V, \qquad \chi_0 \in \big[\chi_*, \chi^*\big] \quad \text{a.e. in } \Omega. \tag{3.8}$$

Hence, we introduce the harmonic extension of θ_Γ, defined for a.e. t by

$$\theta_H(t) \in V, \qquad \theta_{H_{|\Gamma}} = \theta_\Gamma, \qquad \Delta\theta_H(t) = 0 \quad \text{in } \Omega. \tag{3.9}$$

The general theory for harmonic functions and (3.6) ensure that

$$\theta_H \in H^1(0, T; H) \cap L^2(0, T; V), \qquad \theta_* \leqslant \theta_H \leqslant \theta^*. \tag{3.10}$$

As a consequence, it is a standard matter to infer that

$$\ln\theta_H \in L^\infty(Q) \cap H^1(0, T; H) \cap L^2(0, T; V). \tag{3.11}$$

In addition, let

$$\begin{aligned} &\mathbf{f} \in L^2\big(0, T; W'\big), \\ &R \in L^2(0, T; H) \end{aligned} \tag{3.12}$$

with

$$_{W'}\langle \mathbf{f}, \mathbf{v}\rangle_W = \int_\Omega \mathbf{b} \cdot \mathbf{v} + \int_{\Gamma_2} \mathbf{t} \cdot \mathbf{v}.$$

Finally, we use the notation $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ for the subdifferential $\partial I_{[0,+\infty)}$. Actually, we could extend our results to more general graphs

$$\beta = \partial\widehat{\beta}, \quad \widehat{\beta} \text{ being a lower semicontinuous, convex, proper function,}$$

$$\operatorname{dom}\beta \subseteq [0, +\infty), \quad 0 \in \beta(0).$$

Remark 3.1. Note that, to ensure the physical constraint on the pressure $p \geqslant 0$, we have to require that the domain of the operator β, acting on p, is included in $[0, +\infty)$.

Here is the problem we are dealing with.

Problem. Find $(\theta, \chi, \mathbf{v}, p)$ fulfilling for a.e. $t \in [0, T]$ the following system

$$\chi_t + A\chi + F'(\chi) + G'(\chi)(\theta + p) = 0 \quad \text{in } V', \tag{3.13}$$

$$\mathcal{A}(\mathbf{v}) + \nabla p = \mathbf{f} \quad \text{in } W', \tag{3.14}$$

$$-G'(\chi)\chi_t + \operatorname{div}\mathbf{v} + \beta(p) \ni 0 \quad \text{a.e. in } \Omega, \tag{3.15}$$

$$\begin{aligned} &\theta_t - G'(\chi)\chi_t + A\log\theta = R \quad \text{in } V_0', \\ &\ln\theta = \ln\theta_\Gamma \quad \text{a.e. in } \Gamma \end{aligned} \tag{3.16}$$

combined with initial conditions

$$\chi(0) = \chi_0, \qquad \theta(0) = \theta_0. \tag{3.17}$$

In particular it is used the following notation

$$_{W'}\langle \nabla p, \mathbf{v}\rangle_W = -\int_\Omega p \operatorname{div}\mathbf{v}. \tag{3.18}$$

The following existence result holds.

Theorem 3.2. *Let* (3.4), (3.6)–(3.8)*, and* (3.12) *hold. Then, there exist*

$$\theta \in H^1(0,T;V') \cap L^\infty(0,T;H), \qquad \ln\theta \in L^2(0,T;V), \tag{3.19}$$

$$\chi \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)), \tag{3.20}$$

$$\mathbf{v} \in L^2(0,T;W), \tag{3.21}$$

$$p \in L^2(0,T;H) \tag{3.22}$$

solving for a.e. $t \in [0,T]$ (3.13)–(3.16) *combined with* (3.17)*. In particular,* (3.13) *also holds a.e. in* Ω*.*

Theorem 3.2 will be proved in the next sections. First, it is introduced an approximated system, regularizing nonlinearities and adding some dissipative terms. Then, by use of the Schauder fixed point theorem, it is proved the existence of a solution for the regularized system, at least locally in time. Hence, suitable a priori estimates allow us to extend the existence of a solution on the whole time interval $[0,T]$. Finally, we pass to the limit (w.r.t. to the approximating parameter), using a priori (uniform) estimates, compactness tools, and semicontinuity arguments.

Concerning the solutions to (3.13)–(3.16), exploiting a maximum principle argument, we can deduce that the χ-component of the solution is uniformly bounded and belongs (see (3.8)) to $[\chi_*, \chi^*]$ a.e. Let us point out that the physical constraint on χ is not a priori ensured as an internal constraint but it results from the evolving of the solution itself, whose existence is stated by Theorem 3.2.

Theorem 3.3. *Under the same assumptions of Theorem* 3.2*, let* $(\theta, \chi, p, \mathbf{v})$ *be a solution to* (3.13)–(3.16), (3.17) *given by Theorem* 3.2*. Then, the following bound holds a.e. in* Q

$$\chi_* \leqslant \chi \leqslant \chi^*. \tag{3.23}$$

4. A maximum principle (proof of Theorem 3.3)

The prove of Theorem 3.3 is based on a maximum principle, following the argument presented in [5] for a different model. However, for the sake of completeness we detail the proof, as some differences are introduced in our case.

A Lipschitz function $H : \mathbb{R} \to \mathbb{R}$ is fixed, sufficiently smooth (of class C^1) and such that

$$H(x) = 0 \quad \text{if } x \in [\chi_*, \chi^*], \qquad H'(x) > 0 \quad \text{otherwise}. \tag{4.1}$$

Then, let $(\theta, \mathbf{v}, \chi, p)$ be a solution to (3.13)–(3.16), (3.17) introduced by Theorem 3.2. As χ turns out to belong to $L^\infty(0,T;V)$ (cf. (3.20)) we are allowed to use as test function $H(\chi)$ in (3.13). After integrating over $(0,t)$, we get

$$\int_0^t \int_\Omega \chi_t H(\chi) + \int_0^t \int_\Omega \nabla\chi \cdot \nabla H(\chi) + \int_0^t \int_\Omega \big(F'(\chi) + G'(\chi)(\theta + p)\big) H(\chi) = 0. \tag{4.2}$$

Denoting by $\widehat{H}$ the primitive of H vanishing at χ_*, integrating by parts in time and using (3.8), it is straightforward to obtain

$$\int_0^t \int_\Omega \chi_t H(\chi) = \int_\Omega \widehat{H}(\chi(t)) - \int_0 \widehat{H}(\chi_0) = \int_\Omega \widehat{H}(\chi(t)) \geqslant 0. \tag{4.3}$$

Using the chain rule (cf. (4.1)) leads to

$$\int_0^t \int_\Omega \nabla\chi \cdot \nabla H(\chi) = \int_0^t \int_\Omega H'(\chi)|\nabla\chi|^2 \geqslant 0. \tag{4.4}$$

Analogously, the last integral is non-negative, i.e.

$$\int_0^t \int_\Omega \big(F'(\chi) + G'(\chi)(\theta + p)\big) H(\chi) \geqslant 0. \tag{4.5}$$

Indeed, F', G', and H have the same sign (cf. (3.4) and (4.1)) and $\theta > 0$, $p \geqslant 0$. Let us point out that the positivity of θ and p comes from the presence of the logarithm in (3.16) and the assumptions on the domain of the operator β. Therefore, all the integrals in (4.2) vanish identically. In particular, we can deduce that $\widehat{H}(\chi(t)) = 0$ a.e. in Ω, and for every t, so that (3.23) follows.

5. The approximated problem

In this section, we regularize system (3.13)–(3.16), approximating the nonlinearity β and adding some viscosity terms in the equations. Let us fix an approximating parameter $\varepsilon > 0$. Then, we introduce the Moreau–Yosida approximation of the operator β (cf. [1])

$$\beta_\varepsilon(w) := \frac{1}{\varepsilon}\big(w - \rho_\varepsilon(w)\big),$$

where $\rho_\varepsilon : \mathbb{R} \to \mathbb{R}$ is the ε-resolvent operator associated to β, defined for every $w \in \mathbb{R}$ as the unique solution of the inclusion

$$\rho_\varepsilon(w) - w + \beta\big(\rho_\varepsilon(w)\big) \ni 0.$$

In particular, β_ε is a Lipschitz continuous function. Then, we replace F and G by new functions, we still denote by F and G, such that in addition to (3.4) they satisfy

$$F', G' \in L^\infty(\mathbb{R}). \tag{5.1}$$

Let us point out that we are allowed to add the assumption (5.1) as, due to Theorem 3.3, a solution of the problem is such that $\chi \in [\chi_*, \chi^*]$.

To write the approximated system, we introduce in addition

$$\begin{aligned}
&\chi_{1\varepsilon} \in H, \qquad \mathbf{v}_{0\varepsilon} \in W \cap H^2(\Omega)^3,\\
&p_{0\varepsilon} \in V, \qquad p_{0\varepsilon} \in \operatorname{dom}\beta \quad \text{a.e.},
\end{aligned} \tag{5.2}$$

such that

$$\varepsilon^{1/2}\|\chi_{1\varepsilon}\|_H \to 0, \qquad \varepsilon^{1/2}\|\mathbf{v}_{0\varepsilon}\|_W \to 0, \qquad \varepsilon^{1/2}\|p_{0\varepsilon}\|_V \to 0, \tag{5.3}$$

as $\varepsilon \searrow 0$.

During this section, to simplify notation, we do not make precise the dependence on ε of the solutions $(\chi, \theta, \mathbf{v}, p)$. Here is the PDE system we are dealing with in $(0, T)$,

$$\varepsilon\chi_{tt} + \chi_t + \varepsilon A\chi_t + A\chi + F'(\chi) + G'(\chi)(\theta + p) = 0 \quad \text{in } V', \tag{5.4}$$

$$\varepsilon\mathcal{A}(\mathbf{v}_t) + \mathcal{A}(\mathbf{v}) + \nabla p = \mathbf{f} \quad \text{in } W', \tag{5.5}$$

$$\theta_t + \ln\theta - G'(\chi)\chi_t = R \quad \text{in } V_0', \qquad \ln\theta = \ln\theta_\Gamma \quad \text{a.e. in } Q, \tag{5.6}$$

$$\varepsilon(p_t + Ap) + \beta_\varepsilon(p) = G'(\chi)\chi_t - \operatorname{div}\mathbf{v} \quad \text{in } V'. \tag{5.7}$$

Eqs. (5.4), (5.6) are combined with initial conditions (3.17) and

$$\chi_t(0) = \chi_{1\varepsilon}, \qquad \mathbf{v}(0) = \mathbf{v}_{0\varepsilon}, \qquad p(0) = p_{0\varepsilon}. \tag{5.8}$$

The following theorem is proved.

Theorem 5.1. *Let $\varepsilon > 0$ and* (3.4), (3.6)–(3.8), (3.12), *and* (5.1), (5.2), (5.3) *hold. Then, there exists a solution* $(\theta, \chi, \mathbf{v}, p)$ *to* (5.4)–(5.7) *combined with* (3.17), (5.8).

Remark 5.2. Note that in the model proposed in [13] Eq. (5.7) is considered without the presence of the potential β_ε and the term $\operatorname{div}\mathbf{v}$. In our case, after proving existence of a solution for the approximated parabolic equation, we can pass to the limit as ε converging to 0.

5.1. A fixed point argument

To prove the existence of a solution for the approximated problem, we exploit a fixed point argument. Before proceeding let us point out that in this section the symbol c is used to denote possibly different positive constants depending on the data of the problem but also on ε. In particular, we use the notation $c(M)$ to put in evidence the dependence of a constant c on the constant M. Then, let us point out that we are proceeding with formal estimates. Our argument could be made rigorous following the analogous procedure exploited in [3].

We first consider, for $M > 0$ and for a final time $\widehat{T}$ to be fixed later,

$$\widehat{\chi} \in H^1(0, \widehat{T}; H), \tag{5.9}$$

$$\widehat{\mathbf{v}} \in L^2(0, \widehat{T}; W), \tag{5.10}$$

such that

$$\|\widehat{\chi}\|_{H^1(0,\widehat{T};H)} + \|\widehat{\mathbf{v}}\|_{L^2(0,\widehat{T};W)} \leqslant M. \tag{5.11}$$

First step. We first fix $\widehat{\chi}$ and $\widehat{\mathbf{v}}$ in (5.7) in place of χ and $\mathbf{v}$, and state Cauchy condition (5.8). Then, by monotonicity of β_ε and fairly standard results for parabolic equations (note that the right-hand side of (5.7) belongs to $L^2(0,\widehat{T};H)$) there exists a unique solution $p = \mathcal{T}_1(\widehat{\chi},\widehat{\mathbf{v}})$. Formally testing the resulting equation by p_t and integrating over $(0,t)$ lead to

$$\varepsilon\|p_t\|^2_{L^2(0,t;H)} + \frac{\varepsilon}{2}\big\|\nabla p(t)\big\|^2_H - \frac{\varepsilon}{2}\|\nabla p_{0\varepsilon}\|^2_H + \int_0^t\int_\Omega \beta_\varepsilon(p)p_t \leqslant \int_0^t \big\|G'\big\|_\infty\|\widehat{\chi}_t\|_H\|p\|_H + c\int_0^t \|\widehat{\mathbf{v}}\|_W\|p\|_H. \tag{5.12}$$

Then, by the chain rule

$$\int_0^t\int_\Omega \beta_\varepsilon(p)p_t = \int_\Omega \widehat{\beta}_\varepsilon\big(p(t)\big) - \int_\Omega \widehat{\beta}_\varepsilon(p_{0\varepsilon}), \tag{5.13}$$

where $\beta_\varepsilon = \partial\widehat{\beta}_\varepsilon$. Note that, by definition of the Yosida regularization of β (see [1]) there holds $0 \leqslant \widehat{\beta}_\varepsilon(x) \leqslant \widehat{\beta}(x)$ for any x belonging to the closure of $\mathrm{dom}(\beta)$. Thus, exploiting the Gronwall lemma and by virtue of (5.11), we are able to deduce that

$$\|p\|^2_{H^1(0,\widehat{T};H)\cap L^\infty(0,\widehat{T};V)} \leqslant c(M), \tag{5.14}$$

and, secondly, by a comparison in (5.7) and recalling that β_ε is Lipschitz continuous

$$\|p\|_{L^2(0,\widehat{T};H^2(\Omega))} \leqslant c. \tag{5.15}$$

Note that in order to apply the Gronwall lemma to (5.12) and to deduce (5.14), the following standard inequality is used

$$\big\|p(t)\big\|_{L^\infty(0,t;H)} \leqslant \int_0^t \|p_t\|_H.$$

Second step. We fix $\widehat{\chi}$ in (5.6) in place of χ. Using the results by [5], we can infer that there exists a unique solution $\theta = \mathcal{T}_2(\widehat{\chi})$ for the resulting equation, combined with (3.17). Note in particular that $G'(\widehat{\chi})\chi_t + R$ in (5.6) belongs to $L^2(0,\widehat{T};H)$ due to (3.12), (5.9), and (5.1). Then, we test by $(\theta-\theta_H) + (\ln\theta - \ln\theta_H)$ and integrate over $(0,t)$.

In the following, we will argue formally. To make our argument rigorous, we should proceed as in [5] and regularize the logarithm by a function $L_{n\varepsilon}: \mathbb{R}\to\mathbb{R}$ defined as follows

$$L_{n\varepsilon} r := \varepsilon r + \ln_\varepsilon r, \quad \ln_\varepsilon \text{ being the Moreau–Yosida regularization of } \ln. \tag{5.16}$$

Then, owing to the fact that a priori bounds can be proved on $L_{n\varepsilon}$ independently of ε one can pass to the limit (see [3]).

Integrating by parts in time, there holds

$$\begin{aligned}
&\frac{1}{2}\big\|\theta(t)-\theta_H(t)\big\|^2_H - \frac{1}{2}\big\|\theta_0-\theta_H(0)\big\|^2_H + \int_0^t\int_\Omega \nabla\ln\theta\cdot\nabla\theta + \int_0^t\int_\Omega \theta_t\ln\theta + \int_0^t\int_\Omega \big|\nabla(\ln\theta-\ln\theta_H)\big|^2 \\
&\quad\leqslant \int_0^t \big(\big\|G'\big\|_\infty\|\widehat{\chi}_t\|_H + \|R\|_H\big)\|\theta-\theta_H\|_H + \int_0^t \big(\big\|G'\big\|_\infty\|\widehat{\chi}_t\|_H + \|R\|_H\big)\|\ln\theta-\ln\theta_H\|_H - \int_0^t\int_\Omega \theta_{Ht}(\theta-\theta_H) \\
&\qquad + \int_0^t\int_\Omega \theta_t\ln\theta_H + \int_0^t\int_\Omega \big|\nabla\ln\theta_H\cdot\nabla(\ln\theta-\ln\theta_H)\big| + \int_0^t\int_\Omega |\nabla\ln\theta\cdot\nabla\theta_H|.
\end{aligned} \tag{5.17}$$

We first observe that, formally,

$$\int_0^t\int_\Omega \nabla\ln\theta\cdot\nabla\theta = \int_0^t\int_\Omega \frac{|\nabla\theta|^2}{\theta} = 4\int_0^t\int_\Omega \big|\nabla\theta^{1/2}\big|^2. \tag{5.18}$$

Then, using the chain rule, after introducing $\mathcal{L}_n(r) := \int_1^r \ln(s)\,ds$, we get

$$\int_0^t \int_\Omega \theta_t \ln\theta = \int_\Omega \mathcal{L}_n\big(\theta(t)\big) - \int_\Omega \mathcal{L}_n(\theta_0) \geqslant \int_\Omega \mathcal{L}_n\big(\theta(t)\big) - c, \tag{5.19}$$

where $\mathcal{L}_n$ turns out to be convex and bounded from below. Hence, let us handle the integrals on the right-hand side of (5.17). We first observe that $(\|G'\|_\infty \|\widehat{\chi}_t\|_H + \|R\|_H)$ belongs to $L^2(0,\widehat{T})$ (see (5.9), (5.1), and (3.12)). Then, the Young inequality is used to deal with

$$\begin{aligned}\int_0^t \big(\|G'\|_\infty \|\widehat{\chi}_t\|_H + \|R\|_H\big)\|\ln\theta - \ln\theta_H\|_H &\leqslant \frac{1}{8}\int_0^t \|\ln\theta - \ln\theta_H\|_H^2 + c\left(\int_0^t \|G'\|_\infty^2 \|\widehat{\chi}_t\|_H^2 + \|R\|_H^2\right)\\ &\leqslant \frac{1}{8}\int_0^t \|\ln\theta - \ln\theta_H\|_H^2 + c.\end{aligned} \tag{5.20}$$

The next integral is bounded as follows (cf. (3.10))

$$-\int_0^t \int_\Omega \theta_{Ht}(\theta - \theta_H) \leqslant c\|\theta_H\|_{H^1(0,T;H)}^2 + \int_0^t \|\theta - \theta_H\|_H^2. \tag{5.21}$$

Integrating by parts we infer that (cf. (3.10)–(3.11) and (3.7))

$$\begin{aligned}\int_0^t \int_\Omega \theta_t \ln\theta_H &= \int_\Omega (\theta - \theta_H)_t \ln\theta_H + \int_0^t \int_\Omega \theta_{Ht} \ln\theta_H\\ &= \int_\Omega (\theta - \theta_H)(t)\ln\theta_H(t) - \int_\Omega (\theta - \theta_H)(0)\ln\theta_H(0) - \int_0^t \int_\Omega (\theta - \theta_H)\partial_t \ln\theta_H + \int_0^t \int_\Omega \theta_{Ht}\ln\theta_H\\ &\leqslant \frac{1}{8}\int_\Omega \big|(\theta - \theta_H)(t)\big|^2 + c\int_0^t \int_\Omega |\theta - \theta_H|^2 + c\int_\Omega \big|\ln\theta_H(t)\big|^2\\ &\quad + c\int_0^t \int_\Omega \big(|\partial_t \ln\theta_H|^2 + |\ln\theta_H|^2 + \|\partial_t\theta_H\|_H^2\big) + c\\ &\leqslant \frac{1}{8}\int_\Omega \big|(\theta - \theta_H)(t)\big|^2 + c\int_0^t \int_\Omega |\theta - \theta_H|^2 + c.\end{aligned} \tag{5.22}$$

Analogously, using (3.10), (3.11), and the Young inequality we have

$$\begin{aligned}\int_0^t \int_\Omega |\nabla \ln\theta \cdot \nabla\theta_H| &\leqslant \int_0^t \int_\Omega \big|\nabla(\ln\theta - \ln\theta_H)\cdot\nabla\theta_H\big| + |\nabla\ln\theta_H \cdot \nabla\theta_H|\\ &\leqslant \frac{1}{8}\int_0^t \int_\Omega \big|\nabla(\ln\theta - \ln\theta_H)\big|^2 + c\int_0^t \int_\Omega |\nabla\theta_H|^2 + c\int_0^t \int_\Omega |\nabla\ln\theta_H|^2\\ &\leqslant \frac{1}{8}\int_0^t \int_\Omega \big|\nabla(\ln\theta - \ln\theta_H)\big|^2 + c.\end{aligned} \tag{5.23}$$

Moreover, it is now easy to verify that

$$\int_0^t \int_\Omega |\nabla\ln\theta_H|\big|\nabla(\ln\theta - \ln\theta_H)\big| \leqslant \frac{1}{8}\int_0^t \int_\Omega \big|\nabla(\ln\theta - \ln\theta_H)\big|^2 + c. \tag{5.24}$$

Combining in (5.17) the above estimates, we can apply the Gronwall lemma and the Poincaré inequality to yield

$$\left\|\nabla\theta^{1/2}\right\|_{L^2(0,\widehat{T};H)}+\|\theta-\theta_H\|_{L^\infty(0,\widehat{T};H)}+\|\ln\theta-\ln\theta_H\|_{L^2(0,\widehat{T};V)}\leqslant c(M) \tag{5.25}$$

so that by (3.10)–(3.11)

$$\|\theta\|_{L^\infty(0,\widehat{T};H)}+\|\ln\theta\|_{L^2(0,\widehat{T};V)}\leqslant c(M). \tag{5.26}$$

Hence, after observing that

$$\nabla\theta=2\theta^{1/2}\nabla\theta^{1/2},$$

from (5.25) and (5.26) we can deduce

$$\|\nabla\theta\|_{L^2(0,\widehat{T};L^{4/3}(\Omega))}\leqslant c(M).$$

Eventually (cf. (5.26)), also due to a comparison in (5.6), it follows

$$\|\theta\|_{H^1(0,\widehat{T};V_0')\cap L^\infty(0,\widehat{T};H)\cap L^2(0,T;W^{1,4/3}(\Omega))}+\|\ln\theta\|_{L^2(0,\widehat{T};V)}\leqslant c(M). \tag{5.27}$$

Third step. Fix $p=\mathcal{T}_1(\widehat{\chi},\widehat{\mathbf{v}})$ (see (5.14) and (5.15)) in (5.5) and find the unique $\mathbf{v}=\mathcal{T}_3(p)$ solving the resulting equation with (5.8). The existence of such a solution is a standard result in the theory of parabolic equation. Note that, testing by $\mathbf{v}_t$, and integrating over $(0,t)$ we are able to infer that

$$\begin{aligned}\frac{1}{2}\|\mathbf{v}_t\|^2_{L^2(0,t;W)}+\frac{1}{2}\left\|\mathbf{v}(t)\right\|^2_W&\leqslant\frac{1}{2}\|\mathbf{v}_{0\varepsilon}\|_W+c\big(\|\mathbf{f}\|^2_{L^2(0,t;W')}+\|p\|^2_{L^2(0,t;H)}\big)\\&\leqslant c\big(1+\|\widehat{\chi}_t\|^2_{L^2(0,\widehat{T};H)}+\|\widehat{\mathbf{v}}\|^2_{L^2(0,\widehat{T};W)}\big)\leqslant c(M).\end{aligned} \tag{5.28}$$

Note that in (5.28) we have in particular used the Young inequality. Hence, testing formally by $\mathcal{A}\mathbf{v}$ and integrating over $(0,t)$ it is now a standard matter to eventually deduce (cf. (5.2))

$$\|\mathbf{v}\|_{H^1(0,T;W)\cap L^\infty(0,T;H^2(\Omega)^3\cap W)}\leqslant c. \tag{5.29}$$

Fourth step. Finally, fixing θ (cf. (5.27)) and p (cf. (5.14)–(5.15)) in (5.4) there exists a unique corresponding solution $\chi=\mathcal{T}_4(\theta,p)$. Testing the resulting equation by χ_t and integrating over $(0,t)$ we can deduce

$$\begin{aligned}&\frac{\varepsilon}{2}\left\|\chi_t(t)\right\|^2_H-\frac{\varepsilon}{2}\|\chi_{1\varepsilon}\|^2_H+\varepsilon\|\chi_t\|^2_{L^2(0,t;V)}+\frac{1}{2}\left\|\nabla\chi(t)\right\|^2_H-\frac{1}{2}\|\nabla\chi_0\|^2_H\\&\quad\leqslant\int_0^t\int_\Omega\left|F'(\chi)\right||\chi_t|+\int_0^t\int_\Omega\left|G'(\chi)\right||\theta||\chi_t|+\int_0^t\int_\Omega\left|G'(\chi)\right||p||\chi_t|\\&\quad\leqslant c\left(\int_0^t\|\chi_t\|_H+\int_0^t\|\theta\|_H\|\chi_t\|_H+\int_0^t\|p\|_H\|\chi_t\|_H\right)\end{aligned} \tag{5.30}$$

(here c depends in particular on $\|F'\|_\infty$ and $\|G'\|_\infty$). Thus, the Gronwall lemma and (5.27) and (5.14) lead to

$$\|\chi_t\|_{L^\infty(0,\widehat{T};H)}+\|\chi_t\|_{L^2(0,\widehat{T};V)}\leqslant c(M), \tag{5.31}$$

and by a comparison in (5.4)

$$\|\chi\|_{H^2(0,\widehat{T};V')}\leqslant c. \tag{5.32}$$

5.2. The operator $\mathcal{T}$

We define the operator $\mathcal{T}$ as follows. Fixing $(\widehat{\chi},\widehat{\mathbf{v}})$ as in (5.9)–(5.10) and (5.11), we let

$$(\chi,\mathbf{v})=\mathcal{T}(\widehat{\chi},\widehat{\mathbf{v}})=\big(\mathcal{T}_4\big(\mathcal{T}_2(\widehat{\chi})\big),\mathcal{T}_3\big(\mathcal{T}_1(\widehat{\chi},\widehat{\mathbf{v}})\big)\big),$$

where $\mathcal{T}_i$ are defined by the above detailed constructing procedure. Note that a fixed point for $\mathcal{T}$ should provide a solution for our problem. Hence, (5.31) leads to

$$\|\chi\|_{H^1(0,\widehat{T};H)}\leqslant\widehat{T}^{1/2}\|\chi\|_{W^{1,\infty}(0,\widehat{T};H)}\leqslant\widehat{T}^{1/2}c(M), \tag{5.33}$$

and analogously (cf. (5.28))

$$\|\mathbf{v}\|_{L^2(0,T;W)}\leqslant\widehat{T}^{1/2}\|\mathbf{v}\|_{L^\infty(0,\widehat{T};W)}\leqslant\widehat{T}^{1/2}c(M). \tag{5.34}$$

In particular, by choosing $\widehat{T}$ sufficiently small it is ensured that $(\chi,\mathbf{v})=\mathcal{T}(\widehat{\chi},\widehat{\mathbf{v}})$ satisfies (5.11), i.e.

$$\|\chi\|_{H^1(0,\widehat{T};H)}+\|\mathbf{v}\|_{L^2(0,T;W)}\leqslant M. \tag{5.35}$$

Hence, to apply the Schauder theorem it remains to prove that $\mathcal{T}$ is compact and continuous with respect to the topology of $H^1(0,\widehat{T};H)\times L^2(0,\widehat{T};W)$. The fact that $\mathcal{T}$ is compact follows from (5.31)–(5.32) and (5.28)–(5.29). To prove that $\mathcal{T}$ is continuous, let us take $(\widehat{\chi}_n,\widehat{\mathbf{v}}_n)$ satisfying (5.9), (5.10), (5.11), such that

$$\widehat{\chi}_n\to\widehat{\chi},\qquad \widehat{\mathbf{v}}_n\to\widehat{\mathbf{v}},$$

strongly in the topology of $H^1(0,\widehat{T};H)\times L^2(0,\widehat{T};W)$, and show that the analogous convergence is ensured for the corresponding $\mathcal{T}(\widehat{\chi}_n,\widehat{\mathbf{v}}_n)=(\chi_n,\mathbf{v}_n)$. We denote by $p_n,\theta_n,\chi_n,\mathbf{v}_n$ the solution obtained in the procedure of the construction of the operator $\mathcal{T}$, once $(\widehat{\chi}_n,\widehat{\mathbf{v}}_n)$ is fixed. Proceeding as above (cf. (5.14)–(5.15), (5.27), (5.29), (5.31)–(5.32)) we can deduce that the following estimates hold independently of n

$$\begin{aligned}&\|p_n\|_{H^1(0,\widehat{T};H)\cap L^\infty(0,\widehat{T};V)\cap L^2(0,\widehat{T};H^2(\Omega))}+\|\theta_n\|_{H^1(0,\widehat{T};V_0')\cap L^\infty(0,\widehat{T};H)\cap L^2(0,\widehat{T};W^{1,4/3}(\Omega))}+\|\ln\theta_n\|_{L^2(0,\widehat{T};V)}\\&\quad+\|\mathbf{v}_n\|_{H^1(0,\widehat{T};W)\cap L^\infty(0,T;(W\cap H^2(\Omega))^3)}+\|\chi_n\|_{H^2(0,\widehat{T};V')\cap H^1(0,\widehat{T};V)\cap W^{1,\infty}(0,\widehat{T};H)}\leqslant c.\end{aligned} \tag{5.36}$$

Thus, using weak and weak star compactness results, we infer that, at least for some subsequences still denoted by the index n for the sake of simplicity,

$$p_n\to p\quad\text{weakly star in } H^1(0,\widehat{T};H)\cap L^\infty(0,\widehat{T};V)\cap L^2\big(0,\widehat{T};H^2(\Omega)\big), \tag{5.37}$$

$$\theta_n\to\theta\quad\text{weakly star in } H^1\big(0,\widehat{T};V_0'\big)\cap L^\infty(0,\widehat{T};H)\cap L^2\big(0,\widehat{T};W^{1,4/3}(\Omega)\big), \tag{5.38}$$

$$\ln\theta_n\to\eta\quad\text{weakly in } L^2(0,\widehat{T};V), \tag{5.39}$$

$$\mathbf{v}_n\to\mathbf{v}\quad\text{weakly star in } H^1(0,\widehat{T};W)\cap L^2\big(0,\widehat{T};\big(W\cap H^2(\Omega)\big)^3\big), \tag{5.40}$$

$$\chi_n\to\chi\quad\text{weakly star in } H^2\big(0,\widehat{T};V'\big)\cap H^1(0,\widehat{T};V)\cap W^{1,\infty}(0,\widehat{T};H). \tag{5.41}$$

Hence, strong compactness theorems ensure, in particular, that

$$p_n\to p\quad\text{strongly in } C^0\big([0,\widehat{T}];H\big), \tag{5.42}$$

$$\theta_n\to\theta\quad\text{strongly in } L^2(0,T;H)\cap C^0\big([0,\widehat{T}];V\big), \tag{5.43}$$

$$\chi_n\to\chi\quad\text{strongly in } H^1(0,\widehat{T};H)\cap W^{1,\infty}\big(0,\widehat{T};V'\big), \tag{5.44}$$

$$\mathbf{v}_n\to\mathbf{v}\quad\text{strongly in } L^2(0,T;W). \tag{5.45}$$

The above convergences allow us to pass to the limit in the equations (written for the index n) and prove that, at the limit $n\to+\infty$, $(\chi,\mathbf{v})=\mathcal{T}(\widehat{\chi},\widehat{\mathbf{v}})$, concluding the proof of the continuity of $\mathcal{T}$, as (5.44)–(5.45) hold. In particular, let us briefly comment about the identification of nonlinear terms in the passage to the limit procedure. Due to (5.39) and (5.43) we can identify $\ln\theta=\eta$ (as the logarithm is monotone). Hence, due to (5.44) we can prove also that $F'(\chi_n)$ and $G'(\chi_n)$ strongly converge to $F'(\chi)$ and $G'(\chi)$, respectively, e.g. in $L^2(0,\widehat{T};H)$ (see (3.4) and (5.1)). Finally, also for β_ε we can pass to the limit exploiting (5.42) and the regularity of β_ε.

6. The existence result

The existence result stated by Theorem 3.2 is proved passing to the limit as $\varepsilon\searrow 0$ in the approximated system (5.4)–(5.7). We first proceed by performing a priori estimates on the approximated solutions independently of ε and then passing to the limit by compactness and semicontinuity arguments. Analogous estimates actually allow us to extend the solution to the whole interval $(0,T)$, as they do not depend on $\widehat{T}$ but just on the final time. Thus, we will avoid to repeat the procedure and, formally, we directly write all the estimates and the passage to the limit procedure on the whole time interval $(0,T)$. Finally, let us point out that in the sequel we denote by c possibly different positive constants not depending on ε.

6.1. A priori estimates

First a priori estimate. We test (5.4) by χ_t, (5.5) by $\mathbf{v}$, (5.7) by p, and (5.6) by $(\theta-\theta_{\mathcal{H}})+\delta(\ln\theta-\ln\theta_H)$ ($\delta>0$ to be fixed later). Then, we integrate over $(0,t)$ and add the resulting equations. Observing that some terms cancel we get (cf. in particular (5.19))

$$
\begin{aligned}
&\frac{\varepsilon}{2}\|\chi_t(t)\|_H^2-\frac{\varepsilon}{2}\|\chi_t(0)\|_H^2+\varepsilon\int_0^t\|\nabla\chi_t\|_H^2+\int_0^t\|\chi_t\|_H^2+\frac{1}{2}\|\nabla\chi(t)\|_H^2-\frac{1}{2}\|\nabla\chi_0\|_H^2+\int_0^t\int_\Omega F'(\chi)\chi_t+\frac{\varepsilon}{2}\|\mathbf{v}(t)\|_W^2\\
&\quad-\frac{\varepsilon}{2}\|\mathbf{v}(0)\|_W^2+\int_0^t\|\mathbf{v}\|_W^2+\frac{\varepsilon}{2}\|p(t)\|_H^2-\frac{\varepsilon}{2}\|p_{0\varepsilon}\|_H^2+\varepsilon\int_0^t\|\nabla p\|_H^2+\frac{1}{2}\|(\theta-\theta_H)(t)\|_H^2-\frac{1}{2}\|(\theta-\theta_H)(0)\|_H^2\\
&\quad+\int_0^t\int_\Omega G'(\chi)\chi_t\theta_H+\int_0^t\int_\Omega\frac{|\nabla\theta|^2}{\theta}-\int_0^t\int_\Omega\nabla\ln\theta\cdot\nabla\theta_H+\int_0^t\int_\Omega\partial_t\theta_H(\theta-\theta_H)+\delta\int_\Omega\big(\mathcal{L}_n\big(\theta(t)\big)-\mathcal{L}_n\big(\theta(0)\big)\big)\\
&\quad+\delta\int_0^t\|\nabla(\ln\theta-\ln\theta_H)\|_H^2-\delta\int_0^t\int_\Omega G'(\chi)\chi_t(\ln\theta-\ln\theta_H)-\delta\int_0^t\int_\Omega\theta_t\ln\theta_H-\delta\int_0^t\int_\Omega\nabla\ln\theta_H\cdot\nabla(\ln\theta-\ln\theta_H)\\
&\leqslant c\left(\int_0^t\|R\|_H\big(\|\theta-\theta_H\|_H+\delta\|\ln\theta-\ln\theta_H\|_H\big)+\int_0^t\|\mathbf{f}\|_{W'}\|\mathbf{v}\|_W\right). \qquad (6.1)
\end{aligned}
$$

In particular, we have exploited the fact that

$$
\int_0^t\int_\Omega\beta_\varepsilon(p)p\geqslant 0.
$$

We first infer that (cf. (3.10))

$$
\begin{aligned}
\left|\int_0^t\int_\Omega G'(\chi)\chi_t\theta_H\right|&\leqslant\int_0^t\|G'\|_{L^\infty}\|\theta_H\|_H\|\chi_t\|_H\\
&\leqslant\frac{1}{8}\int_0^t\|\chi_t\|_H^2+c\|\theta_H\|_{L^2(0,T;H)}^2. \qquad (6.2)
\end{aligned}
$$

Then, analogously proceeding as in (5.23), we obtain (cf. (3.11))

$$
\left|\int_0^t\int_\Omega\nabla\ln\theta\cdot\nabla\theta_H\right|\leqslant\frac{\delta}{8}\int_0^t\|\nabla(\ln\theta-\ln\theta_H)\|_H^2+c\big(\|\theta_H\|_{L^2(0,T;V)}^2+\|\ln\theta_H\|_{L^2(0,T;V)}^2\big). \qquad (6.3)
$$

Hence, we proceed as follows

$$
\begin{aligned}
\left|\int_0^t\int_\Omega\partial_t\theta_H(\theta-\theta_H)\right|&\leqslant\int_0^t\|\partial_t\theta_H\|_H\|\theta-\theta_H\|_H\\
&\leqslant c\int_0^t\|\theta-\theta_H\|_H^2+c\|\theta_H\|_{H^1(0,T;H)}^2, \qquad (6.4)
\end{aligned}
$$

and (also exploiting Poincaré's inequality with constant $C(\Omega)$)

$$
\begin{aligned}
\delta\left|\int_0^t\int_\Omega G'(\chi)\chi_t(\ln\theta-\ln\theta_H)\right|&\leqslant\delta\int_0^t\|G'\|_{L^\infty}\|\chi_t\|_H\|\ln\theta-\ln\theta_H\|_H\\
&\leqslant\frac{\delta}{8}\|\nabla(\ln\theta-\ln\theta_H)\|_H^2+2\delta C(\Omega)^2\|G'\|_{L^\infty}^2\int_0^t\|\chi_t\|_H^2. \qquad (6.5)
\end{aligned}
$$

Following the argument used in (5.22) we get

$$\delta\left|\int_0^t\int_\Omega \theta_t \ln\theta_H\right| \leqslant \frac{1}{8}\left\|(\theta-\theta_H)(t)\right\|_H^2 + c\big(\|\theta-\theta_H\|^2_{L^2(0,t;H)} + \|\ln\theta_H\|^2_{H^1(0,T;H)} + \|\theta_H\|^2_{H^1(0,T;H)} + 1\big). \tag{6.6}$$

Finally, we handle with the last integral

$$\left|\delta\int_0^t\int_\Omega \nabla\ln\theta_H\cdot\nabla(\ln\theta-\ln\theta_H)\right| \leqslant \frac{\delta}{8}\int_0^t \left\|\nabla(\ln\theta-\ln\theta_H)\right\|_H^2 + c\|\ln\theta_H\|^2_{L^2(0,T;V)}. \tag{6.7}$$

After recalling that (3.10)–(3.11) and (3.12), (5.2), (5.3) hold, choosing δ sufficiently small, and applying the Young inequality, we can infer that

$$\begin{aligned}
&\|\chi_t\|^2_{L^2(0,t;H)} + \left\|\nabla\chi(t)\right\|_H^2 + \|\mathbf{v}\|^2_{L^2(0,t;W)} + \left\|(\theta-\theta_H)(t)\right\|_H^2 + \|\ln\theta-\ln\theta_H\|^2_{L^2(0,t;V_0)} + \left\|\nabla\theta^{1/2}\right\|^2_{L^2(0,t;H)}\\
&\quad + \varepsilon\big(\left\|\chi_t(t)\right\|_H^2 + \|\nabla\chi_t\|^2_{L^2(0,t;H)} + \left\|p(t)\right\|_H^2 + \|\nabla p\|^2_{L^2(0,t;V)} + \|\mathbf{v}_t\|^2_{L^2(0,t;W)}\big)\\
&\leqslant c\big(1+\|\theta-\theta_H\|^2_{L^2(0,t;H)}\big).
\end{aligned} \tag{6.8}$$

Eventually, the Gronwall lemma ensures that

$$\|\chi\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \leqslant c, \tag{6.9}$$

$$\|\mathbf{v}\|_{L^2(0,T;W)} \leqslant c, \tag{6.10}$$

$$\|\theta-\theta_{\mathcal{H}}\|_{L^\infty(0,T;H)} \leqslant c, \tag{6.11}$$

$$\|\ln\theta-\ln\theta_{\mathcal{H}}\|_{L^2(0,T;V_0)} \leqslant c, \tag{6.12}$$

$$\left\|\nabla\theta^{1/2}\right\|_{L^2(0,T;H)} \leqslant c, \tag{6.13}$$

$$\varepsilon^{1/2}\big(\|\chi_t\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|p\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|\mathbf{v}\|_{L^\infty(0,T;W)}\big) \leqslant c. \tag{6.14}$$

In addition, by a comparison we can deduce that

$$\|\theta_t\|_{L^2(0,T;V')} \leqslant c. \tag{6.15}$$

Now, (3.10), (3.11), (6.12), and (6.15) yield

$$\|\theta\|_{H^1(0,T;V')\cap L^\infty(0,T;H)} \leqslant c, \tag{6.16}$$

$$\|\ln\theta\|_{L^2(0,T;V)} \leqslant c. \tag{6.17}$$

Second a priori estimate. Recalling that $\nabla\theta = 2\theta^{1/2}\nabla\theta^{1/2}$, (6.16) and (6.13) entail

$$\|\theta\|_{L^2(0,T;W^{1,4/3}(\Omega))} \leqslant c. \tag{6.18}$$

Third a priori estimate. By a comparison in (5.5) we deduce that (cf. (6.10), (6.14))

$$\|\nabla p\|_{L^2(0,T;W')} \leqslant c. \tag{6.19}$$

It follows that for any $\mathbf{u}\in W$ there holds

$$\left|\int_0^t\int_\Omega p\,\mathrm{div}\,\mathbf{u}\right| \leqslant c. \tag{6.20}$$

Hence, we choose $\mathbf{v}_*\in W$ such that (cf. Remark 6.1)

$$\int_\Omega \mathrm{div}\,\mathbf{v}_* = \int_\Gamma \mathbf{v}_*\cdot\mathbf{n} \neq 0, \tag{6.21}$$

and introduce a seminorm in H defined by

$$m(v) = \int_\Omega v\,\mathrm{div}\,\mathbf{v}_*. \tag{6.22}$$

In particular, let us point out that m is a norm for constant functions. Moreover, there holds

$$m(p) \leqslant c, \tag{6.23}$$

due to (6.20). Now, we are in the position of applying the result by [20] leading to

$$\|\phi\|_H \leqslant c(\Omega)\big(m(\phi) + \|\nabla\phi\|_{W'}\big),$$

for any $\phi \in H$ with $\nabla\phi \in W'$. In particular, owing to (6.22) and (6.23), the following bound for the pressure p follows

$$\|p\|_{L^2(0,T;H)} \leqslant c. \tag{6.24}$$

Remark 6.1. Let us now make precise how to construct $\mathbf{v}^*$ as in (6.21). We follow the argument introduced in [12]. Assuming that Ω is regular (e.g. Lipschitz) one can take for $x \in \Gamma_2$ the ball $B_\varepsilon(x)$ centered in x with radius ε, such that $B_\varepsilon(x) \cap \Gamma_1$ is empty. Then, considering the Lipschitz parametrization $(x_1, x_2) \to (x_1, x_2, \phi(x_1, x_2))$. The associated normal vector is

$$\mathbf{n} = \frac{(\partial_{x_1}\phi, -\partial_{x_2}\phi, 1)}{(1 + |\nabla\phi|^2)^{1/2}}.$$

Then, take

$$\mathbf{v}^* = (0, 0, \eta)$$

without $\eta(y) = \exp(-\frac{1}{1-\frac{|x-y|^2}{\varepsilon^2}})$ if $|x - y| \leqslant \varepsilon$, and 0 otherwise. It results that $\mathbf{v}^* \in W$. In addition, it is proved that

$$\frac{1}{(1 + |\nabla\phi|^2)^{1/2}} \geqslant \frac{1}{(1 + L^2)^{1/2}},$$

where L is the Lipschitz constant associated to ϕ. It follows

$$\int_\Omega \operatorname{div}\mathbf{v}^* = \int_{\Gamma_2} \mathbf{v}^* \cdot \mathbf{n} = \int_{\Gamma_2 \cap B_\varepsilon(x)} \frac{\eta}{(1 + |\nabla\phi|^2)^{1/2}} \neq 0.$$

Remark 6.2. Note that in the case of a model assuming Dirichlet boundary condition for the velocity $\mathbf{v}$ on the whole boundary Γ (taking e.g. $\mathbf{v} = \mathbf{0}$ on Γ), from (6.19) we could deduce just that p is bounded in a quotient space, i.e. in L^2 modulus constant functions.

Fourth estimate. We test (5.7) by $\xi = \beta_\varepsilon(\chi)$ and integrate over $(0, t)$. By monotonicity of β_ε and its definition (recalling that $\beta = \partial I_{[0,+\infty)}$), also integrating by parts, we get

$$\varepsilon \int_0^t \langle Ap, \xi\rangle + \varepsilon \int_0^t \int_\Omega p_t \xi \geqslant 0. \tag{6.25}$$

Hence, exploiting the Young inequality and (6.9), (6.10) we get

$$\|\xi\|_{L^2(0,T;H)} \leqslant c. \tag{6.26}$$

6.2. Passage to the limit

This section is devoted to the passage to the limit in (5.4)–(5.7) as $\varepsilon \searrow 0$. Note that here we explicitely write the dependence on ε of the solutions $(\theta_\varepsilon, \chi_\varepsilon, p_\varepsilon, \mathbf{v}_\varepsilon)$. Owing to (6.9), (6.10), (6.19), (6.16), (6.17), and (6.18) we can apply well-known weak and weak star compactness results to deduce the following convergences (at least holding for some subsequences still denoted by the index ε)

$$\theta_\varepsilon \to \theta \quad \text{weakly star in } H^1\big(0, T; V_0'\big) \cap L^\infty(0, T; H) \cap L^2\big(0, T; W^{1,4/3}(\Omega)\big), \tag{6.27}$$

$$\ln\theta_\varepsilon \to \eta \quad \text{weakly in } L^2(0, T; V), \tag{6.28}$$

$$p_\varepsilon \to p \quad \text{weakly in } L^2(0, T; H), \tag{6.29}$$

$$\mathbf{v}_\varepsilon \to \mathbf{v} \quad \text{weakly in } L^2(0, T; W), \tag{6.30}$$

$$\chi_\varepsilon \to \chi \quad \text{weakly star in } H^1(0, T; H) \cap L^\infty(0, T; V). \tag{6.31}$$

Hence, by virtue of (6.14) it is proved

$$\varepsilon\chi_{\varepsilon t} \to 0 \quad \text{strongly in } L^2(0,T;V) \cap L^\infty(0,T;H), \tag{6.32}$$

$$\varepsilon\mathbf{v}_\varepsilon \to 0 \quad \text{strongly in } L^\infty(0,T;W), \tag{6.33}$$

$$\varepsilon p_\varepsilon \to 0 \quad \text{strongly in } L^2(0,T;V) \cap L^\infty(0,T;H). \tag{6.34}$$

Note that by a comparison in the equations there holds

$$\varepsilon\big(\|\chi_{\varepsilon tt}\|_{L^2(0,T;V')} + \|\mathcal{A}\mathbf{v}_{\varepsilon t}\|_{L^2(0,T;W')} + \|p_t\|_{L^2(0,T;V')}\big) \leqslant c, \tag{6.35}$$

independently of ε. Thus, using (6.32)–(6.34), we deduce

$$\varepsilon\chi_{\varepsilon tt} \to 0 \quad \text{weakly in } L^2\big(0,T;V'\big), \tag{6.36}$$

$$\varepsilon\mathcal{A}\mathbf{v}_{\varepsilon t} \to 0 \quad \text{weakly in } L^2\big(0,T;W'\big), \tag{6.37}$$

$$\varepsilon p_{\varepsilon t} \to 0 \quad \text{weakly in } L^2\big(0,T;V'\big). \tag{6.38}$$

Actually, note that also the following strong convergences hold due to (6.32) and (6.34)

$$\varepsilon A\chi_{\varepsilon t} \to 0 \quad \text{strongly in } L^2\big(0,T;V'\big), \tag{6.39}$$

$$\varepsilon A p_\varepsilon \to 0 \quad \text{strongly in } L^2\big(0,T;V'\big). \tag{6.40}$$

Moreover, owing to (6.26) there holds (recall that $\xi_\varepsilon = \beta_\varepsilon(p_\varepsilon)$)

$$\xi_\varepsilon \to \xi \quad \text{weakly in } L^2(0,T;H). \tag{6.41}$$

In addition, by strong compactness (cf. [19]) (6.27) and (6.31) lead to

$$\theta_\varepsilon \to \theta \quad \text{strongly in } L^2(0,T;H) \cap C^0\big([0,T];V_0'\big), \tag{6.42}$$

$$\chi_\varepsilon \to \chi \quad \text{strongly in } C^0\big([0,T];H\big). \tag{6.43}$$

Thus, combining (6.13), (6.16) with (6.42) (implying in particular that, at least for some subsequence, θ_ε a.e. converges to θ, and thus $\theta_\varepsilon^{1/2}$), we also deduce

$$\theta_\varepsilon^{1/2} \to \theta^{1/2} \quad \text{weakly in } L^2(0,T;V). \tag{6.44}$$

Now, exploiting the above convergences we can pass to the limit (weakly) in (5.4)–(5.7).

We first deal with (5.4) observing that (6.43) and (5.1), (3.4) imply (at least for some subsequence)

$$F'(\chi_\varepsilon) \to F'(\chi), \qquad G'(\chi_\varepsilon) \to G'(\chi) \quad \text{strongly in } L^p\big(\Omega \times (0,T)\big),\ p < +\infty. \tag{6.45}$$

Thus, we are able to pass to the limit in (5.4) by (6.31), (6.36), (6.39), (6.42), and (6.45) getting at the limit (3.13). Hence, to show that (3.20) holds and (3.13) is solved a.e. we formally proceed by testing (3.13) by $A\chi$. After integrating over $(0,t)$, it is now a standard matter to get $A\chi$ bounded in $L^2(0,T;H)$, from which the result easily follows.

Hence, (6.33), (6.37), and (6.29) entail that we can pass to the limit weakly in (5.5) in W' to get (3.14).

Eq. (3.16) is obtained passing to the weak limit in V_0' in (5.6) making use of (6.27), (6.28), (6.45), (6.31), and (6.30). Then, to identify η in (6.28) with $\ln\theta$, we exploit a semicontinuity argument (cf. [1]), as the logarithm is monotone and (6.42) and (6.28) hold.

Finally, we deal with (5.7) to get at the limit (3.15). Due to (6.30), (6.31), (6.45), (6.38), (6.41), and (6.40) and passing to the limit in (5.7) we eventually write

$$\xi = G'(\chi)\chi_t - \operatorname{div}\mathbf{v}, \tag{6.46}$$

in which it remains to identify $\xi \in \beta(p)$. Exploiting the monotonicity of β and (6.41), (6.29), we proceed by semicontinuity showing that

$$\limsup_{\varepsilon\searrow 0} \int_0^t \int_\Omega \xi_\varepsilon p_\varepsilon \leqslant \int_0^t \int_\Omega \xi p. \tag{6.47}$$

Testing (5.7) by p_ε and integrating over $(0,t)$ yields

$$\limsup_{\varepsilon\searrow 0}\int_0^t\int_\Omega \xi_\varepsilon p_\varepsilon = \limsup_{\varepsilon\searrow 0} -\frac{\varepsilon}{2}\left\|p_\varepsilon(t)\right\|_H^2 + \frac{\varepsilon}{2}\left\|p_\varepsilon(0)\right\|_H^2 - \varepsilon\int_0^t\int_\Omega |\nabla p_\varepsilon|^2 + \int_0^t\int_\Omega G'(\chi_\varepsilon)\chi_{\varepsilon t} p_\varepsilon - \int_0^t\int_\Omega \operatorname{div}\mathbf{v}_\varepsilon p_\varepsilon. \tag{6.48}$$

We first point out that (5.3) and (6.14) imply that

$$\limsup_{\varepsilon\searrow 0} -\frac{\varepsilon}{2}\left\|p_\varepsilon(t)\right\|_H^2 + \frac{\varepsilon}{2}\left\|p_\varepsilon(0)\right\|_H^2 - \varepsilon\int_0^t\int_\Omega |\nabla p_\varepsilon|^2 \leqslant 0. \tag{6.49}$$

Now, we have to deal with $\limsup_{\varepsilon\searrow 0}\int_0^t\int_\Omega G'(\chi_\varepsilon)\chi_{\varepsilon t} - \int_0^t\int_\Omega \operatorname{div}\mathbf{v}_\varepsilon p_\varepsilon$. To this aim we test (5.4) by $\chi_{\varepsilon t}$, (5.5) by $\mathbf{v}_\varepsilon$, (5.6) by θ_ε, and then combine the resulting equations integrating in time. Thus, we can rewrite

$$\begin{aligned}
&\limsup_{\varepsilon\searrow 0}\int_0^t\int_\Omega G'(\chi_\varepsilon)\chi_{\varepsilon t} p_\varepsilon - \int_0^t\int_\Omega \operatorname{div}\mathbf{v}_\varepsilon p_\varepsilon \\
&= \limsup_{\varepsilon\searrow 0} -\varepsilon\int_0^t\int_\Omega \chi_{\varepsilon tt}\chi_{\varepsilon t} - \varepsilon\int_0^t\int_\Omega |\nabla\chi_{\varepsilon t}|^2 - \int_0^t\int_\Omega |\chi_{\varepsilon t}|^2 - \frac{1}{2}\left\|\nabla\chi_\varepsilon(t)\right\|_H^2 + \frac{1}{2}\left\|\nabla\chi_\varepsilon(0)\right\|_H^2 - \int_0^t\int_\Omega F'(\chi_\varepsilon)\chi_{\varepsilon t} \\
&\quad - \frac{1}{2}\left\|\theta_\varepsilon(t)\right\|_H^2 + \frac{1}{2}\left\|\theta(0)\right\|_H^2 - \int_0^t\int_\Omega \left|\nabla\theta_\varepsilon^{1/2}\right|^2 + \int_0^t\int_\Omega R\theta_\varepsilon + \int_0^t {}_{W'}\langle\mathbf{f},\mathbf{v}_\varepsilon\rangle_W - \int_0^t\int_\Omega |\nabla\mathbf{v}_\varepsilon|^2 \\
&\quad - \frac{\varepsilon}{2}\left\|\nabla\mathbf{v}_\varepsilon(t)\right\|_H^2 + \frac{\varepsilon}{2}\left\|\nabla\mathbf{v}_\varepsilon(0)\right\|_H^2.
\end{aligned} \tag{6.50}$$

Let us deal with the right-hand side of (6.48). We first observe that due to (5.3) and integrating by parts in time, we have

$$\limsup_{\varepsilon\searrow 0} -\varepsilon\int_0^t\int_\Omega \chi_{\varepsilon tt}\chi_{\varepsilon t} = \limsup_{\varepsilon\searrow 0} -\frac{\varepsilon}{2}\left(\left\|\chi_{\varepsilon t}(t)\right\|_H^2 - \|\chi_{1\varepsilon}\|_H^2\right) \leqslant 0. \tag{6.51}$$

Then, we have

$$\limsup_{\varepsilon\searrow 0} -\varepsilon\int_0^t\int_\Omega |\nabla\chi_{\varepsilon t}|^2 \leqslant 0, \tag{6.52}$$

and analogously (cf. also (5.3))

$$\limsup_{\varepsilon\searrow 0} -\frac{\varepsilon}{2}\int_\Omega \left|\nabla\mathbf{v}_\varepsilon(t)\right|^2 + \frac{\varepsilon}{2}\int_\Omega \left|\nabla\mathbf{v}(0)\right|^2 \leqslant 0. \tag{6.53}$$

By weak convergence and lower semicontinuity results (see (6.31)) it follows

$$\begin{aligned}
\limsup_{\varepsilon\searrow 0} -\int_0^t\int_\Omega |\chi_{\varepsilon t}|^2 &= -\liminf_{\varepsilon\searrow 0}\int_0^t\int_\Omega |\chi_{\varepsilon t}|^2 \\
&\leqslant -\int_0^t\int_\Omega |\chi_t|^2,
\end{aligned} \tag{6.54}$$

and

$$\limsup_{\varepsilon\searrow 0} -\frac{1}{2}\int_\Omega \left|\nabla\chi_\varepsilon(t)\right|^2 = -\frac{1}{2}\liminf_{\varepsilon\searrow 0}\int_\Omega \left|\nabla\chi_\varepsilon(t)\right|^2 \leqslant -\frac{1}{2}\int_\Omega \left|\nabla\chi(t)\right|^2. \tag{6.55}$$

Combining (6.31) and (6.45) yields

$$\lim_{\varepsilon\searrow 0} -\int_0^t\int_\Omega F'(\chi_\varepsilon)\chi_{\varepsilon t} = -\int_0^t\int_\Omega F'(\chi)\chi_t. \tag{6.56}$$

Hence, by lower semicontinuity (6.27) and (6.44) lead to

$$\limsup_{\varepsilon\searrow 0} -\frac{1}{2}\int_\Omega \left|\theta_\varepsilon(t)\right|^2 \leqslant -\frac{1}{2}\int_\Omega \left|\theta(t)\right|^2, \tag{6.57}$$

and

$$\limsup_{\varepsilon\searrow 0} -\int_0^t\int_\Omega \left|\nabla\theta_\varepsilon^{1/2}\right|^2 \leqslant -\int_0^t\int_\Omega \left|\nabla\theta^{1/2}\right|^2. \tag{6.58}$$

Analogously, (6.30) implies

$$\limsup_{\varepsilon\searrow 0} -\int_0^t\int_\Omega |\nabla\mathbf{v}_\varepsilon|^2 \leqslant -\int_0^t\int_\Omega |\nabla\mathbf{v}|^2. \tag{6.59}$$

Finally, (6.27) and (6.30) give

$$\lim_{\varepsilon\searrow 0}\int_0^t\int_\Omega R\theta_\varepsilon +_{W'}\langle\mathbf{f},\mathbf{v}_\varepsilon\rangle_W = \int_0^t\int_\Omega R\theta +_{W'}\langle\mathbf{f},\mathbf{v}\rangle_W. \tag{6.60}$$

Eventually, combining (6.48)–(6.60), (6.47) follows from which we can identify $\xi \in \beta(p)$, concluding our passage to the limit procedure and the proof of Theorem 3.2.

Remark 6.3. As far as uniqueness, this is not expected as the pressure is not uniquely determined during the evolution. Note that in the case of a phase transition in which the pressure is known the reader may refer to the uniqueness result proved in [5] for a PDE system coupling (3.13), without the term $G'(\chi)p$, and (3.16).

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