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Interpolation by Convex Algebraic Hypersurfaces

Eugenii Shustin

School of Mathematical Sciences, Tel Aviv University, Ramat Aviv, Tel Aviv 69978, Israel

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Let Σ be the set of vertices of a convex non-degenerate polyhedron in $\mathbf{R}^n$, $n \geqslant 2$. We suggest an algorithm to construct smooth convex algebraic hypersurfaces of degree as small as possible, going through Σ.

INTRODUCTION

The standard algebraic interpolation problem in $\mathbf{R}^n$ is to find an algebraic hypersurface of degree as small as possible, passing through a given finite set Σ. The convex algebraic interpolation problem is: Given a finite set $\Sigma \subset \mathbf{R}^n$ in convex position, we look for a smooth convex algebraic hypersurface of degree as small as possible, going through Σ. A Hermite-type convex algebraic interpolation problem is to find a smooth convex algebraic hypersurface, going through Σ and tangent at Σ to given hyperplanes. A modification of this problem—interpolation by convex piecewise algebraic curves and surfaces—was treated in [2–6, 11]. In [8] a solution to the convex algebraic interpolation problem in $\mathbf{R}^2$ is presented: namely, a family of convex curves of degree $[(m+1)/2]$, going through all vertices of a convex m-gon, is constructed. This algorithm gives a solution to the Hermite-type convex algebraic interpolation problem in $\mathbf{R}^2$ as well, but in general does not work for higher dimensions.

In the present paper we suggest an approach to both ordinary and Hermite-type convex algebraic interpolation problems in any dimension $n \geqslant 2$ using hyperbolic hypersurfaces. The advantage is that, given a real polynomial of degree >2, it is hard to check whether the polynomial defines a convex hypersurface or a hypersurface with a convex connected component (see Section 2 below) and the hyperbolicity is a well-controllable property. An explicit equation of convex hypersurfaces that we construct below can be found from some systems of non-linear equations which have a simple linear approximation.

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Throughout the article we number statements and equations separately.

1. FORMULATION OF RESULTS

To simplify the notation in the following we denote an algebraic hypersurface and a polynomial, defining it by the same symbol.

A finite set $\Sigma \subset \mathbf{R}^n$ is called *convexly located* if no point of Σ belongs to the convex hull of the other points. The convex hull $\mathscr{C}(\Sigma)$ of Σ is a convex polyhedron.

A connected component of a smooth algebraic hypersurface in $\mathbf{R}^n$ is called *convex* if all its finite subsets are convexly located. An algebraic hypersurface H is said to be *convex interpolatory* for a convex set Σ if H has a smooth convex connected component containing Σ. If such a component of H is bounded then the hypersurface H will be called *bounded convex interpolatory* for Σ.

Let $\Sigma \subset \mathbf{R}^n$ be a convexly located set of m points. We denote by $d(\Sigma)$ the minimal number of disjoint proper faces of $\mathscr{C}(\Sigma)$ such that they all are simplices (not necessary equidimensional) and their union contains Σ. Such a set of faces we will call a covering set. It is easy to see that for $n=2$

$$d(\Sigma) \leqslant \frac{m+1}{2}. \tag{1}$$

If $n=3$ then from [1]

$$d(\Sigma) < \tfrac{2}{3} m, \tag{2}$$

and the constant 2/3 is tight. For any $n \geqslant 4$,

$$d(\Sigma) < m, \tag{3}$$

and this estimate is asymptitically tight; i.e., there are examples with

$$d(\Sigma) = m - O(m^{1/[n/2]}).$$

These examples, shown to me by Professor N. Alon, are presented in the proof of Theorem 5 below.

Our main result is the following

THEOREM 1. *Let $\Sigma \subset \mathbf{R}^n$ be a set of $m > n$ convexly located points, and* $\dim \mathscr{C}(\Sigma) = n$. *Then there exists a hypersurface in $\mathbf{R}^n$ of degree $d(\Sigma)$, convex* *interpolatory for* Σ, *and there exists a hypersurface in* $\mathbf{R}^n$ *of degree* $\leqslant d(\Sigma)+1$, *bounded convex interpolatory for* Σ.

THEOREM 2. *Let* $\Sigma=\{z_1, ..., z_m\} \subset \mathbf{R}^n$, $m \geqslant n$, *be a convexly located set, and let* $L_1, ..., L_m$ *be a set of hyperplanes in* $\mathbf{R}^n$ *such that*

$$L_i \cap \mathscr{C}(\Sigma)=\{z_i\}, \qquad i=1, ..., m.$$

There exist a hypersurface $F \subset \mathbf{R}^n$ *of degree* m, *convex interpolatory for* Σ, *and a hypersurface* $G \subset \mathbf{R}^n$ *of degree* $\leqslant m+1$, *bounded convex interpolatory for* Σ, *such that* F *and* G *are tangent to* L_i *at* z_i *for all* $i=1, ..., m$.

The same approach allows us to solve a problem of the mixed type as well. For a proper nonempty subset $\Sigma' \subset \Sigma$ let us denote by $d(\Sigma', \Sigma)$ the minimal number of proper faces of $\mathscr{C}(\Sigma)$ such that they all are simplices and their union contains $\Sigma \backslash \Sigma'$.

THEOREM 3. *Let* $\Sigma'=\{z_1, ..., z_p\}$ *be a subset of a convexly located set* $\Sigma \subset \mathbf{R}^n$ *of* $m \geqslant n$ *points. Let* $L_1, ..., L_p$ *be a set of hyperplanes in* $\mathbf{R}^n$ *such that*

$$L_i \cap \mathscr{C}(\Sigma)=\{z_i\}, \qquad i=1, ..., p.$$

There exist a hypersurface $F \subset \mathbf{R}^n$ *of degree* $p+d(\Sigma', \Sigma)$, *convex interpolatory for* Σ, *and a hypersurface* $G \subset \mathbf{R}^n$ *of degree* $\leqslant p+d(\Sigma', \Sigma)+1$, *bounded convex interpolatory for* Σ, *such that* F *and* G *are tangent to* L_i *at* z_i *for all* $i=1, .., p$.

In the case $n=2$ Theorem 1 gives a convex interpolatory curve for Σ of degree $[(m+1)/2]$, whereas, actually, in [8] a bounded convex interpolatory curve of the same degree was constructed. The following statement specifies this result.

THEOREM 4. *For any convexly located set* $\Sigma \subset \mathbf{R}^2$ *of* $m \geqslant 4$ *points there exist a convex interpolatory curve of degree* $[m/2]$ *and a bounded convex interpolatory curve of degree* $[(m+1)/2]$.

The following statement together with estimates (1), (2), (3) shows how optimal are the results of Theorems 1 and 4 with respect to the degree of interpolatory hypersurfaces.

THEOREM 5. (1) *For any* $m \geqslant 4$ *there exists a convexly located set* $\Sigma_m \subset \mathbf{R}^2$ *of* m *points such that there is no convex smooth curve through* Σ_m *of degree* $<[m/2]$.

(2) *For any* $m \geqslant 10$ *there exists a convexly located set* $\Sigma_m \subset \mathbf{R}^3$ *of* m *points such that there is no convex smooth surface through* Σ_m *of degree* $<[(2m-4)/3]$.

(3) *For any* $n \geqslant 4$ *and* $m \geqslant n+1$ *there exists a convexly located set* $\Sigma_m \subset \mathbf{R}^n$ *of* m *points such that there is no convex smooth hypersurface through* Σ_m *of degree less than* $m - s_0$, *where*

$$s_0 = \min \left\{ s \geqslant n+1 \mid s + \binom{s-[(n+1)/2]}{s-n} + \binom{s-[n/2]-1}{s-n} \geqslant m \right\}.$$

Remarks. (1) In the case of odd m there may not exist a bounded convex interpolatory curve of degree $(m-1)/2$. For example, five points on a hyperbola determine this hyperbola as the unique conic curve through the chosen points.

(2) In the third statement of Theorem 5

$$m - s_0 \geqslant \binom{s_0 - 1 - [(n+1)/2]}{s_0 - 1 - n} + \binom{s_0 - 2 - [n/2]}{s_0 - 1 - n} - 1,$$

where the right-hand side is a polynomial in s_0 of degree $[n/2]$. Hence the statements of Theorems 1 and 5 mean that there are convexly located sets $\Sigma \subset \mathbf{R}^n$, $n \geqslant 4$, of m points with $d(\Sigma) = m - O(m^{1/[n/2]})$.

(3) The convex interpolatory curves and hypersurfaces, which we construct in the proofs of Theorems 1–4, are not unique. For instance, by Theorem 4, given a convexly located set $\Sigma \subset \mathbf{R}^2$ of $m = 2k+1$ points, there exists a convex interpolatory curve C_k of degree k through Σ. For $k \geqslant 3$ the space of curves of degree k (being the space of polynomials in two variables of degree k, taken up to a constant factor) has dimension $k(k+3)/2 > 2k+1$; hence the set of curves of degree k, going through Σ, is a projective space of a positive dimension. and hereby any such curve, sufficiently close to C_k, is a convex interpolatory for Σ as well.

2. HYPERBOLIC POLYNOMIALS

Let q be a point in the real projective space $\mathbf{R}P^n$. A real homogeneous polynomial $F(x_0, ..., x_n)$ of degree d is called *q-hyperbolic* (*strict q-hyperbolic*) if the hypersurface $F=0$ in $\mathbf{R}P^n$ intersects any straight line through q at d real points counting multiplicities (resp., at d distinct real points).

LEMMA 1. *Any q-hyperbolic polynomial is the limit of strict q-hyperbolic polynomials of the same degree.*

Proof. In fact, this statement is due to Nuij [9]. We will present here the required family of strict q-hyperbolic polynomials in a slightly modified form, suggested in [10]. Let $F(x_1, ..., x_n)$ be a q-hyperbolic polynomial of degree d, and let $q=(1, 0, ..., 0)$. Then according to [9] the polynomials

$$\mathcal{T}_\xi F(x_0, ..., x_n) = \left(\mathrm{Id} + \xi x_1 \frac{\partial}{\partial x_1}\right)^d \circ \cdots \circ \left(\mathrm{Id} + \xi x_n \frac{\partial}{\partial x_n}\right)^d F(x_0, ..., x_n), \quad (4)$$

where Id is the identity operator, are strict q-hyperbolic for all constants $\xi \in \mathbf{R} \backslash \{0\}$, while $\mathcal{T}_0 F = F$. ∎

LEMMA 2. *Let F be a strict q-hyperbolic polynomial of degree d if $d=2k$ then the hypersurface F in $\mathbf{R}P^n$ consists of k smooth connected components homeomorphic to the $(n-1)$-sphere S^{n-1}. Each connected component bounds in $\mathbf{R}P^n$ a domain homeomorphic to the n-dimensional ball D^n and containing q, and all these balls form an ascending sequence. If $d=2k+1$ then the hypersurface F consists of $k+1$ smooth connected components, k of them are homeomorphic to S^{n-1} and situated as described above; one more component is homeomorphic to $\mathbf{R}P^{n-1}$ and does not bound any part of $\mathbf{R}P^n$.*

Proof. The fact is well known in real algebraic geometry. We will explain it shortly. First, it is known (see, for example [13]) that a smooth real hypersurface consists of orientable components homologous to zero in $\mathbf{R}P^n$, and, in the case of odd degree, contains one more component realizing a non-zero homology class. At last, note that the natural projection of a strict q-hyperbolic hypersurface onto the space $\mathbf{R}P^{n-1}$ of lines going through q is a d-sheeted covering, which completes the proof because $\mathbf{R}P^{n-1}$ can be covered either by $\mathbf{R}P^{n-1}$ or by S^{n-1}, and, for $n>2$, a component $\mathbf{R}P^{n-1}$ cannot bound anything in $\mathbf{R}P^n$, because it intersects any (projective) straight line L through q at one point and hence does not divide L into two or more connected components. ∎

In particular, a strict q-hyperbolic hypersurface of degree >1 has a unique component homeomorphic to S^{n-1} which bounds a component of $\mathbf{R}P^n \backslash F$ homeomorphic to a ball. We will call it the *inner component*.

LEMMA 3. *Given a strict q-hyperbolic hypersurface F, let S be its inner component. For any hyperplane $H \subset \mathbf{R}P^n$, the set $S \backslash H \subset \mathbf{R}^n = \mathbf{R}P^n \backslash H$ consists either of one or two convex components of the affine hypersurface $F \backslash H$.*

Proof. Note that F is q'-hyperbolic with respect to any point q' belonging to the ball bounded by the inner component. Therefore any straight line in $\mathbf{R}P^n$ meets the inner component at at most two points, and we are done. ∎

3. PROOF OF THEOREM 1

3.1. *Construction of a Convex Interpolatory Hypersurface*

Let $\sigma_1, ..., \sigma_d$, $d = d(\Sigma)$, be a covering set of faces of $\mathscr{C}(\Sigma)$. Through these faces one can draw hyperplanes $H_1, ..., H_d$ such that

$$H_i \cap \mathscr{C}(\Sigma) = \sigma_i, \qquad i = 1, ..., d.$$

Let $\tilde{H}_1, ..., \tilde{H}_d \subset \mathbf{R}P^n$ be the projective closures of $H_1, ..., H_d$, respectively.

It is clear that the hypersurface

$$F = \tilde{H}_1 \cdots \tilde{H}_d$$

is q-hyperbolic with respect to any point q in the interior of $\mathscr{C}(\Sigma)$. In order to get a strict q-hyperbolic polynomial we apply the deformation (4). The problem is how to keep these hyperbolic hypersurfaces from passing through Σ. So we will slightly modify the deformation (4).

Let $\Sigma = \{z_1, ..., z_m\}$. In a neighborhood of each point z_i, $i = 1, ..., m$, on the straight line $\langle qz_i \rangle$ we introduce a local coordinate s_i such that the coordinate of z_i is 0.

LEMMA 4. *There exist $\varepsilon > 0$ and smooth families $\tilde{H}_{1,\bar{s}}, ..., \tilde{H}_{d,\bar{s}}$ of hyperplanes, depending on parameters*

$$\bar{s} = (s_1, ..., s_m), \qquad |s_i| < \varepsilon, \qquad i = 1, ..., m,$$

such that $\tilde{H}_{j,0} = \tilde{H}_j$, $j = 1, ..., d$, and, for any pair $z_i \in \tilde{H}_j$, the hyperplane $\tilde{H}_{j,\bar{s}}$ meets the line $\langle qz_i \rangle$ at the point with coordinate s_i.

Proof. This follows immediately from the fact that the points of Σ belonging to $\tilde{H}_j$ are verices of a simplex, hence are linearly independent. ∎

Let us consider the family of hypersurfaces $\mathscr{T}_\xi F_{\bar{s}}$, where the operator $\mathscr{T}_\xi$ is defined by (4), q is assumed to be $(1. 0, ..., 0)$, and

$$F_{\bar{s}} = \tilde{H}_{1,\bar{s}} \cdots \tilde{H}_{d,\bar{s}},$$

$$\bar{s} = (s_1, ..., s_m), \qquad |\xi| < \delta, \quad |s_1|, ..., |s_m| < \varepsilon,$$

with some fixed positive δ, ε. These are strict q-hyperbolic hypersurfaces of degree d for all $\xi \neq 0$. Now we seek $s_1, ...,. s_m$ as functions of ξ such that $s_i(0) = 0$, $i = 1, ..., m$, and, for any $\xi \in (-\varepsilon, \varepsilon)$, the hypersurface $\mathscr{T}_\xi F_{\bar{s}}$ contains Σ.

Homogeneous polynomials in $n+1$ variables of degree d, close to F_0, can be parametrized by the collection of their coefficients

$$\mathscr{A} = \{A_{i_0, ..., i_n}, i_0 + \cdots + i_n = d, i_0 < d\},$$

assuming $A_{d,0,\ldots,0} = \text{const.} \neq 0$. By construction, the coefficients $\mathscr{A}$ of the polynomials $\mathscr{T}_\xi F_{\bar{s}}$ are smooth functions of $\xi, \bar{s}$ in a neighborhood of zero. On the other hand, since any straight line $\langle qz_i \rangle$ meets F_0 transversally at distinct points (by choice of q in a generic position), this straight line meets transversaly each hypersurface $\mathscr{T}_\xi F_{\bar{s}}$ at d distinct points. Therefore, the coordinate S_i of the intersection point of the line $\langle qz_i \rangle$ and $\mathscr{T}_\xi F_{\bar{s}}$, which is close to z_i, is a smooth function of $\mathscr{A}$. Thereby, our problem can be reformulated as to find a solution $s_1(\xi), \ldots, s_m(\xi)$ of the system

$$S_i(\mathscr{A}(\xi, s_1, \ldots, s_m)) = 0, \qquad i = 1, \ldots, m,$$

in a neighborhood of zero. Note that

$$S_i(\mathscr{A}(0, s_1, \ldots, s_m)) = s_i, \qquad i = 1, \ldots, m,$$

for all $s_1, \ldots, s_m$ close to zero. Hence

$$\det \left(\frac{\partial S_i}{\partial s_j} \right)_{1 \leqslant i, j \leqslant m} \Bigg|_{\xi = 0, \bar{s} = 0} = 1,$$

therefore by the implicit function theorem there exists a solution $s_1(\xi), \ldots, s_m(\xi)$, defined on some interval $\xi \in (-\delta, \delta)$ and satisfying

$$s_i(0) = 0, \quad |s_i(\xi)| < \varepsilon, \qquad \xi \in (-\delta, \delta), \quad i = 1, \ldots, m.$$

That completes the construction of a convex interpolatory hypersurface.

3.2. *Construction of a Bounded Convex Interpolatory Hypersurface*

If the hyperplanes $H_1, \ldots, H_d$, introduced above, bound a compact polyhedron which contains Σ, then the previous procedure gives a bounded inner component of the hyperbolic hypersurface constructed.

Assume that the component of the complement to $H_1 \cup \cdots \cup H_d$ in $\mathbf{R}^n$, containing q, is unbounded. Since the points of Σ lying in H_1 are vertices of a simplex, there exists an $(n-2)$-sphere S^{n-1} in H_1 going through these points. Obviously, there exists an $(n-1)$-sphere S^{n-1} in $\mathbf{R}^n$ going through S^{n-2} and bounding a ball which contains Σ. Thus, substituting H_1 for S^{n-1} and performing the procedure from Section 3.1, we get a bounded convex interpolatory hypersurface of degree $d(\Sigma) + 1$.

4. PROOF OF THEOREMS 2 AND 3

We will perform, actually, the above procedure. Let $\tilde{H}_1, \ldots, \tilde{H}_m \subset \mathbf{R}P^n$ be the projective closures of the affine hyperplanes $L_1, \ldots, L_m$. Define the normal vector of a hyperplane $a_0x_0+a_1x_1+\cdots+a_nx_n=0$ in $\mathbf{R}P^n$ with $a_0\neq 0$ as

$$v=\left(\frac{a_1}{a_0}, ..., \frac{a_n}{a_0}\right)\in\partial\mathbf{R}^n.$$

Denote by $\bar{v}_1, ..., \bar{v}_m$ the normal vectors of $\tilde{H}_1, ..., \tilde{H}_m$, and introduce families

$$\tilde{H}_1(s_1, \bar{w}_1), ..., \tilde{H}_m(s_m, \bar{w}_m)$$

of hyperplanes depending on parameters $s_1, ..., s_m\in\mathbf{R}$ close to zero, and vectors

$$\bar{w}_i=(w_{i1}, ..., w_{in})\in\mathbf{R}^n, \qquad i=1, ..., m,$$

close to $\bar{v}_1, ..., \bar{v}_m$, respectively, such that the hyperplane $\tilde{H}_i(s_i, \bar{w}_i)$ meets the line $\langle qz_i\rangle$ at the point with coordinate s_i and has the normal vector $\bar{w}_i$, $i=1, ..., m$.

As in Section 3.1, we look for the required convex interpolatory hypersuface F in the family $\mathscr{T}_\xi F(\bar{s}, \bar{w}_1, ..., \bar{w}_m)$, where

$$F(\bar{s}, \bar{w}_1, ..., \bar{w}_m)=\prod_{i=1}^{m}\tilde{H}_i(s_i, \bar{w}_i), \qquad \bar{s}=(s_1, ..., s_m).$$

Clearly, the coefficients of F

$$\mathscr{A}=\{A_{i_0, ..., i_n}, i_0+\cdots+i_n=m, i_0\neq m\}$$

are smooth functions of $\bar{s}, \bar{w}_1, ..., \bar{w}_m$, if $A_{m,0,...,0}=\text{const}\neq 0$. On the other hand, the coordinates $S_1, ..., S_m$ of the intersection points of F with the lines $\langle qz_1\rangle, ..., \langle qz_m\rangle$ in neighborhoods of points $z_1, ..., z_m$, respectively, and the normal vectors $V_1, ..., V_m$ of the tangent hyperplanes to F at these intersection points depend smoothly on $\mathscr{A}$. Thus, our problem is reduced to solution of the system

$$S_i(\mathscr{A}(\xi, \bar{s}, \bar{w}_1, ..., \bar{w}_m))=0, \qquad V_i(\mathscr{A}(\xi, \bar{s}_1, \bar{w}_1, ..., \bar{w}_m))=\bar{v}_i, \qquad i=1, ..., m, \tag{5}$$

with respect to $\bar{s}, \bar{w}_1, ..., \bar{w}_m$ as functions of ξ. Since

$$S_i(\mathscr{A}(0, \bar{s}, \bar{w}_1, ..., \bar{w}_m))=s_i,$$

$$V_i(\mathscr{A}(0, \bar{s}, \bar{w}_1, ..., \bar{w}_m))=\bar{w}_i, \qquad i=1, ..., m,$$

the Jacobian of the left-hand sides of (5) with respect to $s_1, \ldots, s_m$, $w_{11}, \ldots, w_{mn}$ is non-degenerate which, by the implicit function theorem, provides the existence of the required solution to (5) which completes the construction of F.

If the hyperplanes $L_1, \ldots, L_m$ bound a compact polyhedron in $\mathbf{R}^n$ containing Σ, then we put $G = F$. Otherwise, we substitute the hyperplanek L_1 for a $(n-1)$-sphere, tangent to L_1 at z_1 and embracing Σ, in the construction described above, and get a bounded convex interpolatory hypersurface G of degree $m+1$, which completes the proof of Theorem 2.

The proof of Theorem 3 is a simple combination of the proofs of Theorems 1 and 2.

5. PROOF OF THEOREM 4

5.1. *Existence of a Bounded Convex Interpolatory Curve*

In [8] a convex interpolatory curve of degree $[(m+1)/2]$ was constructed. We will show that this algorithm gives also a bounded convex interpolatory curve. Assume that $m = 2k$ $(k \geqslant 2)$. Let us number successively the edges of the m-gon $\mathscr{C}(\Sigma)$ and put

$$F_\lambda = \lambda \prod_{i=1}^{k} H_{2i} + (1-\lambda) \prod_{i=1}^{k} H_{2i-1}, \qquad \lambda = \text{const} \in (0, 1),$$

where H_i is the straight line through the ith edge, $i = 1, \ldots, m$. The fact that all the curves F_λ, $\lambda \in (0, 1)$, are convex interpolatory for Σ is proved in [8, Theorem 1]. It was shown in the proof of Theorem 1 in [8] that the convex component of F_λ lies in the closure of the set $(\Pi_1 \cup \Pi_2) \backslash (\Pi_1 \cap \Pi_2)$, where

$$\Pi_1 = \bigcap_{i=1}^{k} \pi_{2i-1}, \qquad \Pi_2 = \bigcap_{i=1}^{k} \pi_{2i},$$

and $\pi_i \subset \mathbf{R}^2$, $1 \leqslant i \leqslant m$, denotes the closed half plane, bounded by H_i and containing $\mathscr{C}(\Sigma)$. It is not difficult to see that there exists a straight line $H \subset \mathbf{R}^2$, which bounds a half plane π such that $\pi \supset \mathscr{C}(\Sigma)$, and $\pi \cap (\Pi_1 \cup \Pi_2)$ is bounded. Now we shift H in a parallel way, keeping the above property, until $H \cap \mathscr{C}(\Sigma) = \{z_i\}$, $z_i \in \Sigma$. Note that the tangent to F_λ at $z_i = H_i \cap H_{i+1}$ runs over that interval (H_i, H_{i+1}) of the line pencil through z_i, which contains H, as λ varies in the interval $(0, 1)$. Choosing $\lambda \in (0, 1)$ so that F_λ is tangent to H at z_i, we get a convex curve through Σ lying in the bounded set $(\Pi_1 \cup \Pi_2) \cap \pi$.

The case of odd m can be considered analogously.

5.2. *Existence of a Convex Interpolatory Curve*

We only have to construct a convex interpolatory curve of degree k for a set Σ, consisting of $m=2k+1$ $(k \geqslant 2)$ points. We will use a slightly modified procedure from [8]. Let $\Sigma=\{z_1, ..., z_{2k+1}\} \subset \mathbf{R}^2$ be the set of successively numbered vertices of a convex $(2k+1)$-gon. Introduce the straight lines

$$H_1=\langle z_1, z_2\rangle, \qquad H_2=\langle z_2, z_3\rangle, ..., H_{2k-1}=\langle z_{2K-1}, z_{2k}\rangle,$$
$$H_{2k}=\langle z_{2k}, z_1\rangle.$$

As in the previous subsection, we define the family F_λ of curves of degree k and the sets Π_1, Π_2, assuming that π_i, $i=1, ..., 2k$, is the closed half plane, bounded by H_i and containing the points $z_1, ..., z_{2k}$. Clearly, the point z_{2k+1} belongs to $(\Pi_1 \cup \Pi_2) \backslash (\Pi_1 \cap \Pi_2)$. Since the curves F_λ cover the interior of the latter set as λ runs through the interval (0, 1), there exists $\mu \in (0, 1)$ such that the curve F_μ (convex by [8, Theorem 1]) does through z_{2k+1}, which completes the construction.

6. PROOF OF THEOREM 5

(1) Let us consider a convexly located set $\Sigma_m \subset \mathbf{R}^2$, consisting of $m-1$ points on a convex conic curve C and of one more point outside the disk bounded by C. Then any curve F of degree

$$d \leqslant \left[\frac{m}{2}\right]-1=\left[\frac{m-2}{2}\right],$$

going through Σ_m, meets C at least at

$$m-1>2 \cdot \frac{m-2}{2} \geqslant 2d$$

points, hence, by Bezout's theorem [12], F must contain C as component and cannot be interpolatory for Σ.

(2) Let $m=3s_0-1-r \geqslant 10$, where $r=3$, 4, or 5, and s_0 is an integer. Clearly, there exists a convex polyhedron Δ in $\mathbf{R}^3$ with s_0 vertices z_i, $i=1, ..., s_0$, such that one of its facets (faces of codimension 1) is an r-angle and the other are triangles. Denote by s_1, s_2 the numbers of edges and facets of Δ, respectively. From

$$3s_2+r-3=2s_1, \qquad s_0-s_1+s_2=2,$$

one derives $s_2 = 2s_0 - r - 1$. Denote by w_i, $i = 1, ..., s_2$, the baricenters of the facets of Δ, and by $\bar{v}_i$, $i = 1, ..., s_2$, the normal vectors of the corresponding facets, oriented in the exterior of Δ. For a given $\varepsilon > 0$ denote by $w_i(\varepsilon)$ the point $w_i + \varepsilon \bar{v}_i$, $1, ..., s_2$. For a sufficiently small $\varepsilon_1 < 0$ the sets

$$\Sigma_m(\varepsilon) = \{z_1, ..., z_{s_0}, w_1(\varepsilon), ..., w_{s_2}(\varepsilon)\}, \qquad 0 < \varepsilon < \varepsilon_0,$$

of m points are convexly located in $\mathbf{R}^2$. Put $d(\varepsilon)$ to be the minimal degree a convex interpolary surface through $\Sigma_m(\varepsilon)$. This integral-valued function defines a semi-algebraic subdivision of the interval $(0, \varepsilon_0)$. Hence there is $\varepsilon_1 \in (0, \varepsilon_0)$ such that

$$d(\varepsilon) = d^* = \text{const}, \qquad \varepsilon \in (0, \varepsilon_1).$$

Since the (projective) space of real surfaces of degree d is compact, there exists a sequence $H_1, H_2, H_3, ...$ of convex interpolatory surfaces of degree d^* for the sets $\Sigma_m(\varepsilon_1), \Sigma_m(\varepsilon_2), \Sigma_m(\varepsilon_3), ..$, respectively, such that $\lim \varepsilon_k = 0$ and there exists $\lim H_k = H \neq 0$. Since $w_i(\varepsilon_k) \to w_i$ as $k \to \infty$, the convexity condition implies that the limit shape of the convex component H_k must be Δ, hence H contains the planes through all the s_2 facets of Δ as components. Thereby,

$$d^* \geqslant s_2 = 2s_0 - r - 1 = \left[\frac{2m-4}{3}\right],$$

which completes the proof of the second part.

(3) Let us fix $n \geqslant 4$. The convex hull $\Delta(s)$ of $s > n$ generic points $z_1, ..., z_s$ on the curve

$$\{(t, t^2, ..., t^n) \in \mathbf{R}^n \mid t \in \mathbf{R}\}$$

is a so-called cyclic polyhedron with q vertices and

$$\mu(s, n) = \binom{s - [(n+1)/2]}{s-n} + \binom{s - [n/2] - 1}{s-n}$$

facets [7]. For a given $m \geqslant n+1$, let

$$s_0 = \min\{s \geqslant n \mid s + \mu(s, n) \geqslant m\}.$$

Clearly, $m \geqslant s_0 \geqslant n+1$, since $\mu(n, n) = 2$. Put $r = m - s_0$. As above, we introduce the normal vectors $\bar{v}_i$, $i = 1, ..., r$, of r distinct facets $\sigma_1, ..., \sigma_r$ of

$\Delta(s_0)$, oriented in the exterior of $\Delta(s_0)$. Also, we fix one point w_i inside each facet σ_i, $i = 1, ..., r$. For a sufficiently small $\varepsilon > 0$ the set $\Sigma_m(\varepsilon)$ of m points

$$z_i, \quad i = 1, ..., s_0, \qquad w_i(\varepsilon) = w_i + \varepsilon \bar{v}_i, \quad i = 1, ..., r,$$

is convexly located in $\mathbf{R}^n$. As above one shows that the minimal degree of a convex interpolatory hypersurface for $\Sigma_m(\varepsilon)$ is a constant d^* as $\varepsilon \in (0, \varepsilon_1)$, and $d^* \geqslant r$, which implies the required statement.

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