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ORIGINAL ARTICLE

Existence, uniqueness and stability of random impulsive neutral partial differential equations

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Abstract In this paper, the existence, uniqueness and stability via continuous dependence of mild solution of neutral partial differential equations with random impulses are studied under sufficient condition via fixed point theory.

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1. Introduction

Neutral differential equations arise in many area of science and engineering have received much attention in the last decades. The ordinary neutral differential equation is very extensive to study the theory of aeroelasticity and the lossless transmission lines [1] and the references therein. Neutral partial differential equations with delays are motivated from stabilization of lumped control systems, theory of heat conduction in materials [2,3] and the references therein. Hernández and O'Regan [4], studied some neutral partial differential equations by assuming some temporal and spatial regularity type condition.

Recently impulsive differential equations are well to model problems see [5,6]. There is much notice in the field of fixed impulsive type equations [2,7] and the references therein. When the impulses are exist at random, the solutions of the equation behave as a stochastic process. It is quite different from deterministic impulsive differential equations and stochastic differential equations (SDEs). Iwankievicz and Nielsen [8], investigated dynamic response of non-linear systems to poisson distributed random impulses. Wu and Meng [9], first gave the general random impulsive ordinary differential equations and investigated boundedness of solutions to these models by Liapunov's direct method. Wu et al. [10–12], have studied some qualitative properties of differential equations with random impulses. In [13], the author studied the existence and exponential stability for random impulsive semilinear functional differential equations through the fixed point technique under non-uniqueness. The existence, uniqueness and stability results were discussed in [14] through Banach fixed point method for the

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system of differential equations with random impulsive effect. The author [15], studied the existence results for the random impulsive neutral functional differential equations with delays. In [16], the author studied existence results of random impulsive neutral non-autonomous differential inclusions with delays via Dhage's fixed point theorem. In [17], random impulsive semilinear functional differential inclusions were studied using the Martelli fixed point theorem and the fixed point theorem due to Covitz and Nadler. In [18], the authors generalized the distribution of random impulses with the Erlang distribution. Further we refer [19–22]. Motivated by the above mentioned works, the main purpose of this paper is to study the random impulsive neutral partial differential equations (RINDEs) to fill the gap in the above works. Aeroelasticity and the lossless transmission lines can also be modeled in the form of RINDEs.

This paper is organized as follows. In Section 2, we recall briefly the notations, definitions, lemmas and preliminaries which are used throughout this paper. In Section 3, we study the existence and uniqueness of the RINDEs by relaxing the linear growth conditions. In Section 4, we study the stability through continuous dependence on the initial values of the RINDEs. Finally in Section 5, an example is presented to illustrate our results.

2. Preliminaries

Let X be a real separable Hilbert space and Ω a nonempty set. Assume that τ_k is a random variable defined from Ω to $D_k \overset{def.}{=} (0, d_k)$ for $k = 1, 2, \ldots$, where $0 < d_k < +\infty$. Furthermore, assume that τ_k follow Erlang distribution, where $k = 1, 2, \ldots$ and let τ_i and τ_j are independent with each other as $i \neq j$ for $i, j = 1, 2, \ldots$. For the sake of simplicity, we denote $\mathfrak{R}^+ = [0, +\infty)$.

We consider neutral partial differential equations with random impulses of the form

$$\frac{d}{dt}[x(t) + g(t, x_t)] = Ax(t) + f(t, x_t), \quad t \neq \xi_k, \; t \geqslant 0, \tag{2.1}$$

$$x(\xi_k) = b_k(\tau_k)x(\xi_k^-), \quad k = 1, 2, \ldots, \tag{2.2}$$

$$x_{t_0} = \varphi, \tag{2.3}$$

where A is the infinitesimal generator of an analytic semigroup of bounded linear operators $\{S(t); t \geqslant 0\}$ with $D(A) \subset X$. If $S(t)$ is uniformly bounded analytic semigroup such that $0 \in \rho(A)$, then it is possible to define the fractional power A^η, for $0 < \eta \leqslant 1$, as a closed linear operator with dense domain $D(A^\eta)$ in X. If X_η represents the space $D(A^\eta)$ endowed with norm $\|\cdot\|$, then we have

Lemma 2.1 [21]. *Assume that the following conditions hold:*

(*i*) *For* $0 < \eta \leqslant 1$, X_η *is a Banach space.*
(*ii*) *For* $0 < \eta \leqslant \beta \leqslant 1$, *the embedding* $X_\beta \hookrightarrow X_\eta$ *is continuous.*
(*iii*) *There exists a constant* $C_\eta > 0$ *depending on* $0 < \eta \leqslant 1$ *such that*

$$\|A^\eta S(t)\|^2 \leqslant \frac{C_\eta}{t^{2\eta}}, \; t > 0.$$

Now we make the system (2.1), (2.2) and (2.3) precious: The functional $g: \mathfrak{R}^+ \times \widehat{C} \to X; f: \mathfrak{R}^+ \times \widehat{C} \to X$, $\widehat{C} = \widehat{C}((-\infty, 0], X_\eta)$ is the set of piecewise continuous functions with left-hand limit φ from $(-\infty, 0]$ into X_η. The phase space $\widehat{C}((-\infty, 0], X_\eta)$ is assumed to be equipped with the norm $\|\varphi\|_t = \sup_{-\infty < \theta \leqslant 0} |\varphi(\theta)|$. x_t is a function defined by $x_t(s) = x(t+s)$ for all $s \in (-\infty, 0]$ and fixed $t; \xi_0 = t_0$ and $\xi_k = \xi_{k-1} + \tau_k$ for $k = 1, 2, \ldots$, here $t_0 \in \mathfrak{R}^+$ is arbitrary given real number. Obviously, $t_0 = \xi_0 < \xi_1 < \xi_2 < \cdots < \lim_{k\to\infty} \xi_k = \infty$; $b_k: D_k \to X$ for each $k = 1, 2, \ldots; x(\xi_k^-) = \lim_{t \uparrow \xi_k} x(t)$ according to their paths with the norm $\|x\|_t = \sup_{-\infty < s \leqslant t} |x(s)|$ for each t satisfying $t \geqslant 0$ and $T \in \mathfrak{R}^+$ is a given number, $\|\cdot\|$ is any given norm in X_η.

Denote $\{B_t, t \geqslant 0\}$ the simple counting process generated by $\{\xi_n\}$, that is, $\{B_t \geqslant n\} = \{\xi_n \leqslant t\}$, and denote $\mathcal{F}_t$ the σ-algebra generated by $\{B_t, t \geqslant 0\}$. Then $(\Omega, P, \{\mathcal{F}_t\})$ is a probability space. Let $L_2 = L_2(\Omega, \mathcal{F}_t, X)$ denote the Hilbert space of all $\mathcal{F}_t$-measurable square integrable random variables with values in X.

Let $\mathcal{B}_\mathcal{T}$ denote the Banach space $\mathcal{B}_\mathcal{T}([t_0, T], L_2)$, the family of all $\mathcal{F}_t$-measurable, $\widehat{C}$-valued random variables ψ with the norm

$$\|\psi\|^2_{\mathcal{B}_\mathcal{T}} = \sup_{t \in [t_0, T]} E\|\psi\|^2_t.$$

Let $L_2^0(\Omega, \mathcal{B}_\mathcal{T})$ denote the family of all $\mathcal{F}_0$-measurable, $\mathcal{B}_\mathcal{T}$-valued random variable φ.

Definition 2.1. A semigroup $\{S(t), t \geqslant t_0\}$ is said to be uniformly bounded if $\|S(t)\| \leqslant M$ for all $t \geqslant t_0$, where $M \geqslant 1$ is some constant. If $M = 1$, then the semigroup is said to be contraction semigroup.

Definition 2.2. For a given $T \in (t_0, +\infty)$, a stochastic process $\{x(t) \in \mathcal{B}_\mathcal{T}, -\infty < t \leqslant T\}$ is called a mild solution to system (2.1), (2.2) and (2.3) in $(\Omega, P, \{\mathcal{F}_t\})$, if

(i) $x(t) \in \mathcal{B}_\mathcal{T}$ is $\mathcal{F}_t$-adapted;
(ii) $x(t_0 + s) = \varphi(s)$ when $s \in (-\infty, 0]$, and

$$\begin{aligned} x(t) = \sum_{k=0}^{+\infty} \Bigg[& \prod_{i=1}^{k} b_i(\tau_i) S(t - t_0)[\varphi(0) + g(0, \varphi)] - \prod_{i=1}^{k} b_i(\tau_i) g(t, x_t) \\ & - \left[\sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} AS(t-s) g(s, x_s) ds + \int_{\xi_k}^{t} AS(t-s) g(s, x_s) ds \right] \\ & + \left[\sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} S(t-s) f(s, x_s) ds + \int_{\xi_k}^{t} S(t-s) f(s, x_s) ds \right] \Bigg] I_{[\xi_k, \xi_{k+1})}(t), \quad t \in [t_0, T], \end{aligned}$$

where $\prod_{j=m}^{n}(\cdot) = 1$ as $m > n$, $\prod_{j=i}^{k} b_j(\tau_j) = b_k(\tau_k)b_{k-1}(\tau_{k-1})\cdots b_i(\tau_i)$, and $I_A(\cdot)$ is the index function, i.e.,

$$I_A(t) = \begin{cases} 1, & \text{if } t \in A, \\ 0, & \text{if } t \notin A. \end{cases}$$

Now we introduce the following hypotheses used in our discussion:

Hypotheses

(H_1): The function $f : [t_0, T] \times \widehat{C} \to X$ satisfies the Lipschitz condition, that is there exists a constant $L_f = L_f(T) > 0$ such that

$$E\|f(t,x_t) - f(t,y_t)\|^2 \leqslant L_f E\|x-y\|_t^2 \text{ for } x,y \in \widehat{C}, t \in [t_0, T].$$

(H_2): The mapping $g : [t_0, T] \times \widehat{C} \to X$ satisfies that there exists a number $\eta \in [0,1]$ such that for any $x, y \in \widehat{C}$, $t \in [t_0, T]$ and $g(t,x_t) \in D(A^\eta)$ and

$$E\|A^\eta g(t,x_t) - A^\eta g(t,y_t)\|^2 \leqslant L_g E\|x-y\|_t^2, \; L_g > 0.$$

$$(\Phi x)(t) = \sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k} b_i(\tau_i) S(t-t_0)[\varphi(0) + g(0,\varphi)] - \prod_{i=1}^{k} b_i(\tau_i) g(t,x_t) - \Bigg[\sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i} AS(t-s)g(s,x_s)ds + \int_{\xi_k}^{t} AS(t-s)g(s,x_s)ds\Bigg] + \Bigg[\sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i} S(t-s)f(s,x_s)ds + \int_{\xi_k}^{t} S(t-s)f(s,x_s)ds\Bigg]\Bigg] I_{[\xi_k,\xi_{k+1})}(t).$$

(H_3): For all $t \in [t_0, T]$, it follows that $f(t,0), A^\eta g(t,0) \in L^1$ such that

$$E\|f(t,0)\|^2 \leqslant \kappa_f,$$
$$E\|A^\eta g(t,0)\|^2 \leqslant \kappa_g, \text{ where } \kappa_f, \kappa_g > 0 \text{ are constants.}$$

$$\begin{aligned}
\|(\Phi x)(t)\|^2 \leqslant\; & 4\Bigg[\sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k}\|b_i(\tau_i)\|^2\|S(t-t_0)\|^2\|\varphi(0)+g(0,\varphi)\|^2 I_{[\xi_k,\xi_{k+1})}(t)\Bigg] + \sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k}\|b_i(\tau_i)\|^2\|A^{-\eta}\|^2\|A^\eta g(t,x_t)\|^2 I_{[\xi_k,\xi_{k+1})}(t)\Bigg] \\
& + \Bigg[\max_{i,k}\Bigg\{1, \prod_{j=i}^{k}\|b_j(\tau_j)\|\Bigg\}\Bigg]^2\Bigg(\int_{t_0}^{t}\|A^{1-\eta}S(t-s)A^\eta g(s,x_s)\|ds I_{[\xi_k,\xi_{k+1})}(t)\Bigg)^2 \\
& + \Bigg[\max_{i,k}\Bigg\{1, \prod_{j=i}^{k}\|b_j(\tau_j)\|\Bigg\}\Bigg]^2\Bigg(\int_{t_0}^{t}\|S(t-s)\|\|f(s,x_s)\|ds I_{[\xi_k,\xi_{k+1})}(t)\Bigg)^2\Bigg] \\
\leqslant\; & 8M^2\Bigg[\max_{k}\Bigg\{\prod_{i=1}^{k}\|b_i(\tau_i)\|^2\Bigg\}\Bigg][\|\varphi(0)\|^2 \\
& + \|g(0,\varphi)\|^2] + 8\Bigg[\max_{k}\Bigg\{\prod_{i=1}^{k}\|b_i(\tau_i)\|^2\Bigg\}\Bigg]\|A^{-\eta}\|^2[\|A^\eta g(t,x_t) - A^\eta g(t,0)\|^2 + \|A^\eta g(t,0)\|^2] \\
& + 4\Bigg[\max_{i,k}\Bigg\{1, \prod_{j=i}^{k}\|b_j(\tau_j)\|^2\Bigg\}\Bigg](t-t_0)\int_{t_0}^{t}\|A^{1-\eta}S(t-s)A^\eta g(s,x_s)\|^2 ds + 4M^2\Bigg[\max_{i,k}\Bigg\{1, \prod_{j=i}^{k}\|b_j(\tau_j)\|^2\Bigg\}\Bigg](t-t_0)\int_{t_0}^{t}\|f(s,x_s)\|^2 ds.
\end{aligned}$$

(H_4) : The condition $E\left\{\max_{i,k}\left\{\prod_{j=i}^{k}\|b_j(\tau_j)\|\right\}\right\}$ is uniformly bounded. That is, there is a constant $C > 0$ such that

$$E\left\{\max_{i,k}\left\{\prod_{j=i}^{k}\|b_j(\tau_j)\|\right\}\right\} \leqslant C \quad \text{for all } \tau_j \in D_j, \quad j = 1,2,\ldots.$$

3. Existence and uniqueness

In this section, we discuss the existence and uniqueness of the mild solution of the system (2.1), (2.2) and (2.3).

Theorem 3.1. *Let the hypotheses (H_1)–(H_4) be hold. Then there exists a unique (local) continuous mild solution to (2.1), (2.2) and (2.3) for any initial value (t_0, φ) with $t_0 \geqslant 0$ and $\varphi \in \mathcal{B}_T$.*

Proof 1. Let T be an arbitrary number $t_0 < T < +\infty$. In order to apply the contraction principle, we define the non-linear operator $\Phi : \mathcal{B}_T \to \mathcal{B}_T$ as follows

$$(\Phi x)(t) = \varphi(t - t_0), \quad \text{for } t \in (-\infty, t_0],$$

and for $t \in [t_0, T]$

It is easy to prove the continuity of Φ. Now, we have to show that Φ maps $\mathcal{B}_T$ into itself

By Lemma 2.1, (H_2) and (H_3) the following relation holds:

$$E\|AS(t-s)g(s,x_s)\|^2 = E\|A^{1-\eta}S(t-s)A^\eta g(s,x_s)\|^2 \leqslant 2\|A^{1-\eta}S(t-s)\|^2[E\|A^\eta g(s,x_s) - A^\eta g(s,0)\|^2 + E\|A^\eta g(s,0)\|^2] \leqslant \frac{2C_{1-\eta}}{(t-s)^{2(1-\eta)}}\left[L_g E\|x\|_s^2 + k_g\right]. \quad (3.1)$$

Then,

$$E\|(\Phi x)\|_t^2 \leqslant 8M^2C^2E[\|\varphi(0)\|^2 + \|g(0,\varphi)\|^2] + 8C^2\|A^{-\eta}\|^2\left[L_gE\|x\|_t^2 + \kappa_g\right] + 8\max\{1,C^2\}T\int_{t_0}^t \frac{C_{1-\eta}}{(t-s)^{2(1-\eta)}}\left[L_gE\|x\|_s^2 + k_g\right]ds + 8M^2\max\{1,C^2\}T\int_{t_0}^t \left[L_fE\|x\|_s^2 + \kappa_f\right]ds.$$

Taking supremum over t, we get

$$\|\Phi x\|_{\mathcal{B}_T}^2 \leqslant c_1 + c_2\|x\|_{\mathcal{B}_T}^2,$$

where $c_i, i = 1,2,$ are constants. Hence Φ is bounded.

Now, we have to show Φ is a contraction mapping. For any $x, y \in \mathcal{B}_T$, we have

$$\|(\Phi x)(t) - (\Phi y)(t)\|^2 \leqslant 3\left[\max_k\left\{\prod_{i=1}^k \|b_i(\tau_i)\|^2\right\}\|A^{-\eta}\|^2\|A^\eta g(t,x_t) - A^\eta g(t,y_t)\|^2 I_{[\xi_k,\xi_{k+1})}(t)\right] + 3\left[\max_{i,k}\left\{1, \prod_{j=i}^k\|b_j(\tau_j)\|\right\}\right]^2 \times\left(\int_{t_0}^t \|A^{1-\eta}S(t-s)[A^\eta g(t,x_s) - A^\eta g(t,y_s))]\|dsI_{[\xi_k,\xi_{k+1})}(t)\right)^2 + 3\left[\max_{i,k}\left\{1, \prod_{j=i}^k\|b_j(\tau_j)\|\right\}\right]^2\left(\int_{t_0}^t \|S(t-s)\|\|f(s,x_s) - f(s,y_s)\|dsI_{[\xi_k,\xi_{k+1})}(t)\right)^2$$

$$E\|(\Phi x) - (\Phi y)\|_t^2 \leqslant 3C^2\|A^{-\eta}\|^2E\|A^\eta g(t,x_t) - A^\eta g(t,y_t)\|^2 + 3\max\{1,C^2\}(t-t_0)\int_{t_0}^t \frac{C_{1-\eta}}{(t-s)^{2(1-\eta)}}E\|A^\eta g(t,x_s) - A^\eta g(t,y_s)\|^2ds + 3\max\{1,C^2\}(t-t_0)M^2\int_{t_0}^t E\|f(s,x_s) - f(s,y_s)\|^2ds.$$

Thus,

$$E\|(\Phi x) - (\Phi y)\|_t^2 \leqslant \left[\left(3C^2\|A^{-\eta}\|^2 + 3\max\{1,C^2\}\frac{C_{1-\eta}T^{2\eta}}{2\eta-1}\right)L_g + 3\max\{1,C^2\}M^2T^2L_f\right]E\|x-y\|_t^2.$$

Hence,

$$\|(\Phi x) - (\Phi y)\|_{\mathcal{B}_T}^2 \leqslant \Lambda(T)\|x-y\|_{\mathcal{B}_T}^2,$$

with

$$\Lambda(T) = \left[\left(3C^2\|A^{-\eta}\|^2 + 3\max\{1,C^2\}\tfrac{C_{1-\eta}T^{2\eta}}{2\eta-1}\right)L_g + 3\max\{1,C^2\}M^2T^2L_f\right].$$

Then we can take a suitable $0 < T_1 < T$ sufficient small such that $\Lambda(T_1) < 1$, and hence Φ is a contraction on $\mathcal{B}_{T_1}$ ($\mathcal{B}_{T_1}$ denotes $\mathcal{B}_T$ with T substituted by T_1). Thus, by the well-known Banach fixed point theorem we obtain a unique fixed point $x \in \mathcal{B}_{T_1}$ for operator Φ, and hence $\Phi x = x$ is a mild solution of (2.1), (2.2) and (2.3). This procedure can be repeated to extend the solution to the entire interval $(-\infty, T]$ in finitely many similar steps, thereby completing the proof for the existence and uniqueness of mild solutions on the whole interval $(-\infty, T]$. □

Theorem 3.2. *Let $g : \Re^+ \times \widehat{C} \to X$ and $f : \Re^+ \times \widehat{C} \to X$ satisfy the assumptions $(H_1) - (H_4)$. Then there exists a unique, global, continuous solution x to (2.1), (2.2) and (2.3) for any initial value (t_0, φ) with $t_0 \geqslant 0$ and $\varphi \in \mathcal{B}_T$.*

Proof 2. Since T is arbitrary in the proof of the previous theorem, this assertion follows immediately. □

4. Stability

In this section, we study the stability through continuous dependence of solutions on initial condition.

Definition 4.1. A mild solution $x(t)$ of the system (2.1) and (2.2) with initial value ϕ satisfies (2.3) is said to be stable in the mean square if for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$E\|x(t) - \hat{x}(t)\|^2 \leqslant \epsilon \text{ whenever } E\|\phi - \hat{\phi}\|^2 < \delta, \quad \text{for all } t \in [t_0, T]. \quad (3.2)$$

where $\hat{x}(t)$ is another mild solution of the system (2.1) and (2.2) with initial value $\hat{\phi}$ defined in (2.3).

Theorem 4.1. *Let $x(t)$ and $y(t)$ be mild solutions of the system (2.1), (2.2) and (2.3) with initial values φ_1 and φ_2 respectively. If the assumptions of Theorem 3.2 are satisfied, then the mild solution of the system (2.1), (2.2) and (2.3) is stable in the mean square.*

Proof 3. By the assumptions, $x(t)$ and $y(t)$ are two mild solutions of Eqs. (2.1), (2.2) and (2.3) with initial values φ_1 and φ_2 respectively, then

$$x(t)-y(t)=\sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k}b_i(\tau_i)S(t-t_0)[[\varphi_1-\varphi_2]+[g(0,\varphi_1)-g(0,\varphi_2)]]-\prod_{i=1}^{k}b_i(\tau_i)[g(t,x_t)-g(t,y_t)]-\Bigg[\sum_{i=1}^{k}\prod_{j=i}^{k}b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}AS(t-s)[g(s,x_s)-g(s,y_s)]ds+\int_{\xi_k}^{t}AS(t-s)[g(s,x_s)-g(s,y_s)]ds\Bigg]+\Bigg[\sum_{i=1}^{k}\prod_{j=i}^{k}b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}S(t-s)[f(s,x_s)-f(s,y_s)]ds+\int_{\xi_k}^{t}S(t-s)[f(s,x_s)-f(s,y_s)]ds\Bigg]\Bigg]I_{[\xi_k,\xi_{k+1})}(t).$$

Then,

$$E\|x(t)-y(t)\|^2\leqslant 8M^2\left[\max_k\left\{\prod_{i=1}^{k}\|b_i(\tau_i)\|^2\right\}\right]E[\|\varphi_1-\varphi_2\|^2+\|A^{-\eta}\|^2E\|A^\eta g(0,\varphi_1)-A^\eta g(0,\varphi_2)\|^2]+4\left[\max_k\left\{\prod_{i=1}^{k}\|b_i(\tau_i)\|^2\right\}\right]\|A^{-\eta}\|^2E[\|A^\eta g(t,x_t)-A^\eta g(t,y_t)\|^2]+4\left[\max_{i,k}\left\{1,\prod_{j=i}^{k}\|b_j(\tau_j)\|\right\}\right]^2\times(t-t_0)\int_{t_0}^{t}E\|A^{1-\eta}S(t-s)\times[A^\eta g(s,x_s)-A^\eta g(t,y_s)]\|^2ds+4M^2\left[\max_{i,k}\left\{1,\prod_{j=i}^{k}\|b_j(\tau_j)\|\right\}\right]^2(t-t_0)\int_{t_0}^{t}E\|f(s,x_s)-f(s,y_s)\|^2ds$$

$$\sup_{t\in[t_0,T]}E\|x-y\|_t^2\leqslant 8M^2C^2[E\|\varphi_1-\varphi_2\|^2+\|A^{-\eta}\|^2L_gE\|\varphi_1-\varphi_2\|^2]+4C^2\|A^{-\eta}\|^2L_g\sup_{t\in[t_0,T]}E\|x-y\|_t^2+4\max\{1,C^2\}\times T\int_{t_0}^{t}\frac{C_{1-\eta}}{(t-s)^{2(1-\eta)}}L_g\sup_{s\in[t_0,t]}E\|x-y\|_s^2ds+4M^2\max\{1,C^2\}T\int_{t_0}^{t}L_f\sup_{s\in[t_0,t]}E\|x-y\|_s^2ds.$$

Therefore,

$$\sup_{t\in[t_0,T]}E\|x-y\|_t^2\leqslant\widetilde{\Upsilon}E\|\varphi_1-\varphi_2\|^2,$$

where

$$\widetilde{\Upsilon}=\frac{8M^2C^2[1+\|A^{-\eta}\|^2L_g]}{1-\left[\left[4C^2\|A^{-\eta}\|^2+4\max\{1,C^2\}\frac{C_{1-\eta}T^{2\eta}}{2\eta-1}\right]L_g+4M^2\max\{1,C^2\}T^2L_f\right]}.$$

Now given $\epsilon>0$, choose $\delta=\frac{\epsilon}{\Upsilon}$ such that $E\|\varphi_1-\varphi_2\|^2<\delta$. Then

$$\|x-y\|_{\mathcal{B}_T}^2\leqslant\epsilon.$$

This completes the proof. □

5. An example

We conclude this work with an example of the form

$$\begin{aligned}\frac{\partial}{\partial t}\left[u(t,x)+\int_0^\pi b(y,x)u(tsint,y)dy\right]&=\frac{\partial^2}{\partial x^2}u(t,x)+H(t,u(tsint,x)),\quad t\neq\xi_k,\\ u(x,\xi_k)&=q(k)\tau_ku\big(x,\xi_k^-\big),\quad t=\xi_k,\\ u(t,0)&=u(t,\pi)=0\\ u(x,t)&=\Phi(x,t)0\leqslant x\leqslant\pi,\quad-\infty<t\leqslant0,\quad t\geqslant0.\end{aligned}\tag{5.1}$$

Let $X=L^2([0,\pi])$, and τ_k be a random variable defined on $D_k\equiv(0,d_k)$for $k=1,2,\ldots,$ where $0<d_k<+\infty$. Furthermore, assume that τ_k follow Erlang distribution, where $k=1,2,\ldots$ and τ_i and τ_j are independent with each other as $i\neq j$ for $i,j=1,2,\ldots;$ q is a function of $k;\xi_0=t_0;\xi_k=\xi_{k-1}+\tau_k$ for $k=1,2,\ldots$ and $t_0\in\Re^+$ is an arbitrary given real number.

Define A an operator on X by $Au=\frac{\partial^2u}{\partial x^2}$ with the domain

$$D(A)=\left\{u\in X\middle|u\text{ and }\frac{\partial u}{\partial x}\text{ are absolutely continuous, }\frac{\partial^2u}{\partial x^2}\in X,\ u(0)=u(\pi)=0.\right\}$$

It is well known that A generates a strongly continuous semigroup $S(t)$ which is compact, analytic and self-adjoint. Moreover, the operator A can be expressed as

$$Au=\sum_{n=1}^{\infty}n^2\langle u,u_n\rangle u_n,\quad u\in D(A),$$

where $u_n(\zeta)=(\frac{2}{\pi})^{\frac{1}{2}}\sin(n\zeta)$, $n=1,2,\ldots,$ is the orthonormal set of eigenvectors of A. Then the operator $(A^{\frac{1}{2}})$ is given by

$$A^{\frac{1}{2}}u=\sum_{n=1}^{\infty}n\langle u,u_n\rangle u_n\text{ on the space }D\left[A^{\frac{1}{2}}\right]=\left\{u\in X;\sum_{n=1}^{\infty}n\langle u,u_n\rangle u_n\in X\right\}$$

This satisfies $\|S(t)\|\leqslant\exp(-\pi^2(t-t_0))$, $t\geqslant t_0$. Hence $S(t)$ is a contraction semigroup. In particular,

$$\|A^{-\frac{1}{2}}\|=\frac{1}{\Gamma\frac{1}{2}}\int_0^\infty t^{\frac{1}{2}-1}\|S(t)\|dt<\frac{1}{\pi}.$$

We assume that the following condition hold:

(*i*) The function b is measurable and

$$\int_0^\pi \int_0^\pi b^2(y,x)dydx < \infty.$$

(*ii*) The function $\frac{\partial}{\partial t} b(y,x)$ is measurable $b(y,0) = b(y,\pi) = 0$ and let

$$L_g = \left[\int_0^\pi \int_0^\pi \left(\frac{\partial}{\partial t} b(y,x)\right)^2 dydx\right]^{\frac{1}{2}} < \infty.$$

(*iii*) For the function $H : [0,\infty) \times \mathfrak{R} \to \mathfrak{R}$ the following three conditions are satisfied.

(1) For each $t \in [0,\infty)$, $H(t,\cdot)$ is continuous.

(2) For each $u \in X$, $H(\cdot,u)$ is measurable.

(3) There are positive functions $h_1, h_2 : [0,\infty) \to \mathfrak{R}^+$ such that

$$|H(t,u)| \leqslant h_1(t) + h_2|u|, \quad (t,u) \in [0,\infty) \times X;$$

(*iv*)

$$E\left[\max_{i,k}\left\{\prod_{j=i}^{k} \|q(j)(\tau_j)\|^2\right\}\right] < \infty.$$

Assuming that conditions (*i*) and (*iv*) are verified. Under these conditions, we can define the f, g and b_k by

$$f(t,x_t) = H(t,u(tsint,x)), \quad g(t,x_t) = \int_0^\pi b(y,x)u(tsint,y)dy$$

$$b_k(\tau_k) = q(k)\tau_k.$$

Then the problem (5.1) can be modeled as the abstract random impulsive neutral functional differential equation of the form (2.1), (2.2) and (2.3).The next results are consequence of Theorems 3.2 and 4.1 respectively.

Proposition 5.1. *Let the hypotheses* (H_1)–(H_4) *be hold. Then there exists a unique global mild solution u of the system (5.1).*

Proposition 5.2. *Let the conditions of Proposition* 5.1 *be hold. Then the mild solution u of the system (5.1) is stable in the mean square.*

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