Perturbation analysis of the maximal solution of the matrix equation $X + A^* X^{-1} A = P$. II

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Abstract

Consider the nonlinear matrix equation

$$X + A^* X^{-1} A = P,$$

where $A, P$ are $n \times n$ complex matrices with $P$ Hermitian positive definite, and $A^*$ denotes the conjugate transpose of a matrix $A$. In this paper a sharper perturbation bound for the maximal solution to the matrix equation is derived, explicit expressions of the condition number for the maximal solution are obtained, and the backward error of an approximate solution to the maximal solution is evaluated by using the techniques developed in [Linear Algebra Appl. 259 (1997) 183; Linear Algebra Appl. 350 (2002) 237]. The results are illustrated by using numerical examples.

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1. Introduction

This paper is a continuation of the paper [9], i.e., we continue to do the perturbation analysis of the maximal solution $X_L$ to the nonlinear matrix equation (NMEQ)

$$X + A^*X^{-1}A = P,$$  \hspace{1cm} (1.1)

where $A, P \in \mathbb{C}^{n \times n}$ with $P$ Hermitian positive definite. Here $\mathbb{C}^{n \times n}$ denotes the set of all $n \times n$ complex matrices, $A^*$ the conjugate transpose of a matrix $A$. Our main purpose of this paper is threefold. To begin with, we derive a sharper perturbation bound for the maximal solution $X_L$. Secondly, we apply the theory of condition developed by Rice [6] to define a condition number of $X_L$, and moreover, we use the techniques developed in [8] to derive its explicit expressions. Finally, we use the techniques developed in [7] to evaluate the backward error of an approximate solution to the maximal solution $X_L$.

Throughout this paper we always assume that the NMEQ (1.1) has the maximal solution $X_L$. By Theorem 3.4 of [1] it is easy to derive that

$$\rho(X_L^{-1}A) \leq 1,$$  \hspace{1cm} (1.2)

where $\rho(X_L^{-1}A)$ denotes the spectral radius of $X_L^{-1}A$. In the case where $\rho(X_L^{-1}A) = 1$ the maximal solution $X_L$ is, generally speaking, very sensitive to perturbations in the coefficient matrices $A$ and $P$. For example, let

$$A = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \hspace{1cm} (1.3)$$

It is easy to verify that the NMEQ (1.1) with this given data has the maximal solution $X_L = \frac{1}{2}P$ and the eigenvalues of the matrix $X_L^{-1}A$ are 1 and $-1$, which implies $\rho(X_L^{-1}A) = 1$. Now assume that $A$ is slightly perturbed to

$$A_\varepsilon = \begin{bmatrix} \varepsilon & \frac{1}{2} \\ \frac{1}{2} & \varepsilon \end{bmatrix}, \quad 0 < \varepsilon \ll 1,$$

and $P$ remains unchanged. Then we have

$$\|A_\varepsilon\|_2 = \frac{1}{2} + \varepsilon \gg \frac{1}{2},$$

and hence, it follows from Theorem 5.1 of [1] that the perturbed NMEQ (1.1) has no maximal solution, however small the positive number $\varepsilon$ is. On the other hand, if $A$ is slightly perturbed to

$$A_\varepsilon = \begin{bmatrix} \varepsilon & \frac{1}{2} - \varepsilon \\ \frac{1}{2} + \varepsilon & \varepsilon \end{bmatrix}, \quad 0 < \varepsilon \ll 1,$$

and $P$ still remains unchanged, then it is easy to verify that in this case the NMEQ (1.1) has the maximal solution.
\[
X_L(\varepsilon) = \frac{1}{2} P + \frac{1}{2} \sqrt{2\varepsilon(1-2\varepsilon)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},
\]

and
\[
\|X_L(\varepsilon) - X_L\|_2 = \sqrt{2\varepsilon(1-2\varepsilon)} \approx \sqrt{2\varepsilon^{1/2}},
\]

which means that $O(\varepsilon)$ order perturbations in the coefficient matrices can cause $O(\varepsilon^{1/2})$ order changes in the maximal solution. This shows that the maximal solution is very sensitive to perturbations in the coefficient matrices $A$ and $P$. By the way, this phenomenon shown in the above example has been explained from point of view of the underlying invariant Lagrangian subspaces in [5].

Consequently, in this paper we only consider the case where $\rho(X_L^{-1}A) < 1$. It is well known that in such a case there exists a compatible matrix norm $\| \cdot \|$ such that
\[
\|X_L^{-1}A\| < 1. \tag{1.4}
\]

In order to make the following discussion simpler and more concrete, without loss of generality, we may assume that (1.4) holds for some unitary invariant norm. Unless otherwise stated, we always make this assumption throughout this paper.

It is worthwhile to point out that in the case $\rho(X_L^{-1}A) < 1$ it follows the considerations in [1,3,4] that the maximal solution $X_L$ is Lipschitz stable. However, from this we cannot get explicit perturbation bounds, which are important. In view of this, the main results of this paper can be viewed as just finding explicit expressions for the Lipschitz constant.

Throughout this paper we use $\mathcal{C}^{n \times n}$ (or $\mathcal{H}^{n \times n}$) to denote the set of complex (or real) $n \times n$ matrices, $\mathcal{H}^{n \times n}$ the set of $n \times n$ Hermitian matrices. $A^*$ denotes the conjugate transpose of a matrix $A$, $A^\dagger$ the Moore–Penrose inverse of $A$, and $A^T$ the transpose of $A$. $I$ stands for the identity matrix, and $0$ the null matrix. The symbols $\| \cdot \|$, $\| \cdot \|_2$, and $\| \cdot \|_F$ denote a unitary invariant norm, the spectral norm, and the Frobenius norm, respectively. For $A = (a_1, \ldots, a_n) = (a_{ij}) \in \mathcal{C}^{n \times n}$ and a matrix $B$, $A \otimes B = (a_{ij}B)$ is a Kronecker product, and vec $A$ is a vector defined by vec $A = (a_1^T, \ldots, a_n^T)^T$.

**2. Perturbation bound**

Let $A$, $P$, and $X_L$ be slightly perturbed to
\[
\tilde{A} = A + \Delta A, \quad \tilde{P} = P + \Delta P, \quad \tilde{X}_L = X_L + \Delta X,
\]
respectively, where $\Delta A \in \mathcal{C}^{n \times n}$, $\Delta P \in \mathcal{H}^{n \times n}$. Then the NMEQ (1.1) is perturbed to
\[
\tilde{X}_L + \tilde{A}^* \tilde{X}_L^{-1} \tilde{A} = \tilde{P}. \tag{2.1}
\]

Noting that $X_L$ is the maximal solution to the NMEQ (1.1), we have
\[
X_L + A^* X_L^{-1} A = P. \tag{2.2}
\]
Subtracting (2.2) from (2.1) gives
\[ \Delta X - B^* \Delta XB = E + h(\Delta X), \] (2.3)
where
\[ B = X_L^{-1} A, \]
\[ E = \Delta P - (B^* \Delta A + \Delta A^* B) - \Delta A^* X_L^{-1} \Delta A, \]
\[ h(\Delta X) = \bar{A} X_L^{-1} \Delta X (I + X_L^{-1} \Delta X)^{-1} X_L^{-1} \Delta A \
+ \Delta A^* X_L^{-1} \Delta X (I + X_L^{-1} \Delta X)^{-1} B \
- B^* \Delta X X_L^{-1} \Delta X (I + X_L^{-1} \Delta X)^{-1} B. \] (2.4)

First define the linear operator
\[ L : \mathcal{H}^{n \times n} \rightarrow \mathcal{H}^{n \times n} \]
by
\[ LW = W - B^* W B, \quad W \in \mathcal{H}^{n \times n}. \] (2.5)
Since \( \rho(B) = \rho(X_L^{-1} A) < 1 \), the operator \( L \) is invertible. Further, define the operator
\[ Q : \mathcal{C}^{n \times n} \rightarrow \mathcal{H}^{n \times n} \]
by
\[ QZ = L^{-1} (B^* Z + Z^* B), \quad Z \in \mathcal{C}^{n \times n}. \] (2.6)
Note that the operator \( Q \) is not linear, but it satisfies that \( Q(\alpha Z) = \alpha Q Z \) for any positive number \( \alpha \). Thus, in terms of those symbols we can rewrite (2.3) as
\[ \Delta X = L^{-1} \Delta P - Q \Delta A - L^{-1} (\Delta A^* X_L^{-1} \Delta A) + L^{-1} (h(\Delta X)). \] (2.7)

Define
\[ \|L^{-1}\| = \max_{\|W\| = 1} \|L^{-1} W\|, \quad \|Q\| = \max_{\|Z\| = 1} \|Q Z\|. \]
Then it follows that
\[ \|L^{-1} W\| \leq \|L^{-1}\| \|W\|, \quad W \in \mathcal{H}^{n \times n}, \] (2.8)
\[ \|Q Z\| \leq \|Q\| \|Z\|, \quad Z \in \mathcal{C}^{n \times n}. \] (2.9)

Now let
\[ \alpha = \|A\|_2, \quad \beta = \|B\|_2, \quad \zeta = \|X_L^{-1}\|_2, \]
\[ q = \|Q\|, \quad l = \|L^{-1}\|^{-1}, \]
\[ \epsilon = \frac{1}{l} \|\Delta P\| + q \|\Delta A\| + \frac{\zeta}{l} \|\Delta A\|^2, \]
\[ \delta = \frac{\zeta}{l} \left[ (\alpha + \|\Delta A\|) \zeta + \beta \right] \|\Delta A\|. \] (2.10)

Then we can state the main result of this section as follows.

**Theorem 2.1.** If
\[ \delta < \min \left\{ 1, \frac{(1 - \beta)(\alpha \zeta + \beta)}{l} \right\} \] (2.11)
\[ \varepsilon < \min \left\{ \frac{l(1 - \delta)^2}{\zeta \left[ l^2 + 2\beta^2 + l\delta + 2\sqrt{l^2 + \beta^2}l(l + \beta^2) \right]}, \frac{(1 - \delta) \left[ (1 - \beta)(\alpha\zeta + \beta) - l\delta \right]}{\zeta \left[ (1 + \beta)(\alpha\zeta + \beta) + l\delta \right]} \right\}, \tag{2.12} \]

then the perturbed matrix equation (2.1) has the maximal solution \( \tilde{X}_L \), and moreover,
\[ \| \tilde{X}_L - X_L \| \leq \frac{2\varepsilon}{l(1 + \zeta\varepsilon - \delta) + \sqrt{l^2(1 + \zeta\varepsilon - \delta)^2 - 4\zeta l\varepsilon(l + \beta^2)}} \equiv \xi^*. \tag{2.13} \]

Remark 2.1. From Theorem 2.1 we get the first order absolute perturbation bound for the maximal solution \( X_L \) as follows:
\[ \| \tilde{X}_L - X_L \| \leq \frac{1}{l} \| \Delta P \| + q \| \Delta A \| + O(\| (\Delta A, \Delta P) \|^2), \]
\[ (\Delta A, \Delta P) \rightarrow 0. \tag{2.14} \]

Combining this with (2.7) gives
\[ \Delta X = L^{-1}\Delta P - Q\Delta A + O(\| (\Delta A, \Delta P) \|^2), \quad (\Delta A, \Delta P) \rightarrow 0. \tag{2.15} \]

Remark 2.2. Compared with Theorem 3.1 of [9], Theorem 2.1 is much sharper. In fact, the conditions (2.11) and (2.12) and the conditions (3.2) and (3.3) of Theorem 3.1 of [9] require that perturbations in all the coefficient matrices \( A \) and \( P \) are small. This is not a essential restriction for the sensitivity analysis of the maximal solution since we only concern how the maximal solution \( X_L \) changes when the coefficient matrices \( A \) and \( P \) are subject to a small perturbation. Consequently, the essential requirement of Theorem 3.1 of [9] is \( \| A \|_2 \| P^{-1} \|_2 < \frac{1}{2} \), while here the essential one is \( \| X_L^{-1}A \|_2 < 1 \) if we take the unitary invariant norm in Theorem 2.1 as the spectral norm. It is easy to verify that the former implies the latter. Conversely, it is not essentially true. For instance, let
\[ A = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \]

It is easy to derive that in this case the maximal solution of the NMEQ (1.1) exists if and only if \( |a| < 1 \), and the maximal solution is given by
\[ X_L = \begin{bmatrix} 1 & 0 \\ 0 & 1 - |a|^2 \end{bmatrix}. \]

However, if \( \frac{1}{2} < |a| < 1 \), it follows that \( \| X_L^{-1}A \|_2 = |a| < 1 \), but \( \| A \|_2 \| P^{-1} \|_2 = |a| > \frac{1}{2} \).
On the other hand, it follows from (2.14) that
\[
\frac{\|\tilde{X}_L - X_l\|_2}{\|X_L\|_2} \leq \frac{1}{I} \|\Delta P\|_2 \|\Delta A\|_2 \|X_L\|_2 + o(\|\Delta A, \Delta P\|_2^2),
\]
\[\Delta A, \Delta P \rightarrow 0 \]
if we take the unitary invariant norm as the spectral norm. When \(\|A\|_2 \|P^{-1}\|_2 < \frac{1}{2}\), from the proof of Theorem 3.1 of [9] we see that
\[
l \geq 1 - 2 \|A\|_2 \|P^{-1}\|_2, \quad q \leq \frac{2}{l}, \quad \|P\|_2 \|X_L\|_2 \leq 2, \quad \|A\|_2 \|X_L\|_2 \leq 1,
\]
and hence, we have
\[
\frac{1}{l} \|\Delta P\|_2 \|\Delta A\|_2 \|X_L\|_2 \leq \frac{1}{l} \frac{\|\Delta P\|_2 \|\Delta A\|_2}{2 - \|A\|_2 \|P^{-1}\|_2} \left( \frac{\|\Delta P\|_2 + \|\Delta A\|_2}{\|P\|_2 + \|A\|_2} \right),
\]
the right hand of which is just the perturbation bound given in Theorem 3.1 of [9]. This shows that the perturbation bound given here is sharper than that one of [9].

**Proof of Theorem 2.1.** Let
\[
f(\Delta X) = L^{-1} \Delta P - QA - L^{-1}(\Delta A^* X_L^{-1} \Delta A) + L^{-1} (h(\Delta X)).
\]
Obviously, \(f(\Delta X)\) can be regarded as a continuous mapping from \(\mathcal{H}^{n \times n}\) to \(\mathcal{H}^{n \times n}\).

Note that the conditions (2.11) and (2.12) imply that
\[
\delta < 1 \quad \text{and} \quad \epsilon < \frac{l(1 - \delta)^2}{\xi(l + 2\beta^2 + l\delta + 2\sqrt{(l\delta + \beta^2)(l + \beta^2)})},
\]
which ensure that the quadratical equation
\[
\xi(\beta^2 + l)\xi^2 - l(1 + \xi \epsilon - \delta)\xi + l\epsilon = 0
\]
has two positive real roots, and the smaller one is given by
\[
\xi_* = \frac{2l\epsilon}{l(1 + \xi \epsilon - \delta) + \sqrt{l^2(1 + \xi \epsilon - \delta)^2 - 4l\epsilon l\xi(l + \beta^2)}}.
\]

Now define
\[
\mathcal{F}_{\xi_*} = \left\{ \Delta X \in \mathcal{H}^{n \times n} : \|\Delta X\| \leq \xi_* \right\}.
\]
Then for any \(\Delta X \in \mathcal{F}_{\xi_*}\), we have
\[
\|X_L^{-1}\Delta X\| \leq \|X_L^{-1}\|_2 \|\Delta X\| \leq \xi \xi_* \leq \frac{2l\epsilon}{l(1 + \xi \epsilon - \delta) + 1 + \xi \epsilon - \delta}.\]
Note that

\[ \zeta \varepsilon + \delta - 1 \leq \zeta \cdot \frac{I(1 - \delta)^2}{\zeta [l + 2\beta^2 + l\delta + 2\sqrt{l\delta + \beta^2}(l + \beta^2)]} + \delta - 1 \]

\[ \leq \frac{I(1 - \delta)^2}{2\beta^2 + l + l\delta} + (\delta - 1) \]

\[ = -\frac{2(1 - \delta)(\beta^2 + l\delta)}{2\beta^2 + l + l\delta} < 0, \]

and hence we have

\[ \|X^{-1}_L \Delta X\| \leq \|X^{-1}_L\|_2 \|\Delta X\| \leq \zeta \xi_\ast < 1, \quad \Delta X \in \mathcal{A}_{\xi_\ast}, \tag{2.18} \]

which implies that the matrix \(I - X^{-1}_L \Delta X\) is nonsingular and

\[ \left\| \left( I - X^{-1}_L \Delta X \right)^{-1} \right\| \leq \frac{1}{1 - \|X^{-1}_L \Delta X\|} \leq \frac{1}{1 - \zeta \|\Delta X\|}. \tag{2.19} \]

Using (2.8)–(2.10) and (2.19), we have

\[
\|f(\Delta X)\| \leq \frac{1}{I} \|\Delta P\| + q \|\Delta A\| + \frac{\zeta}{I} \|\Delta A\|^2 \\
+ \frac{1}{I} \left( (\alpha + \|\Delta A\|) \xi \|\Delta X\| \cdot \frac{\zeta \|\Delta A\|}{1 - \zeta \|\Delta X\|} \\
+ \|\Delta A\| \xi \|\Delta X\| \cdot \frac{\beta}{1 - \zeta \|\Delta X\|} + \frac{\beta^2 \zeta \|\Delta X\|^2}{1 - \zeta \|\Delta X\|} \right) \\
= \xi + \frac{\delta \|\Delta X\|}{1 - \zeta \|\Delta X\|} + \frac{\beta^2 \zeta \|\Delta X\|^2}{l(1 - \zeta \|\Delta X\|)} \\
\leq \xi + \frac{\delta \xi_\ast}{1 - \zeta \xi_\ast} + \frac{\beta^2 \zeta \xi_\ast^2}{l(1 - \zeta \xi_\ast)} = \xi_\ast
\]

for \(\Delta X \in \mathcal{A}_{\xi_\ast}\), in which the last equality is due to the fact that \(\xi_\ast\) is a solution to the quadratic equation (2.16). Thus we have proved that \(f(\mathcal{A}_{\xi_\ast}) \subset \mathcal{A}_{\xi_\ast}\). By the Schauder fixed-point theorem (see, e.g., [2, Section 6.3]), there exists a \(\Delta X_\ast \in \mathcal{A}_{\xi_\ast}\) such that \(f(\Delta X_\ast) = \Delta X_\ast\), i.e., there exists a solution \(\Delta X_\ast\) to the perturbed equation (2.3) such that

\[ \|\Delta X_\ast\| \leq \xi_\ast. \tag{2.20} \]

Let \(\tilde{X}_L = X_L + \Delta X_\ast\). Then \(\tilde{X}_L\) is a Hermitian solution of the perturbed matrix equation (2.1). Noting that

\[ \|X^{-1}_L\|_2 \|\Delta X_\ast\|_2 \leq \|X^{-1}_L\|_2 \|\Delta X_\ast\| < 1 \]
for any unitary invariant norm $\|\cdot\|$ and that $X_L$ is a positive definite matrix, we know that $\tilde{X}_L$ is also positive definite. That is, we have found a positive definite solution to the perturbed matrix equation (2.1). Next we prove that $\tilde{X}_L$ is just its maximal solution.

It follows from the definition of $\delta$ that

$$\|\Delta A\| = \frac{2l\delta}{\zeta(\alpha\zeta + \beta + \sqrt{(\alpha\zeta + \beta)^2 + 4l\delta})}. \quad (2.21)$$

Using (2.20), (2.21), and the definition of $\xi_\ast$, we have

$$\|(X_L - \lambda A)^{-1}(\Delta X_\ast - \lambda \Delta A)\| \leq \frac{\xi}{1 - \beta} (\|\Delta X_\ast\| + \|\Delta A\|) \leq \frac{\xi}{1 - \beta} \left( \frac{\xi_\ast + \frac{2l\delta}{\zeta(\alpha\zeta + \beta + \sqrt{(\alpha\zeta + \beta)^2 + 4l\delta})}}{1 - \delta + \frac{l\delta}{\zeta(\alpha\zeta + \beta)}} \right) \leq \frac{\xi}{1 - \beta} \left( \frac{2\varepsilon}{1 + \zeta\varepsilon - \delta} + \frac{l\delta}{\zeta(\alpha\zeta + \beta)} \right)$$

for any $|\lambda| < 1$. Note that it follows from (2.11) and (2.12) that

$$\delta < \frac{(1 - \beta)(\alpha\zeta + \beta)}{l} \quad \text{and} \quad \varepsilon < \frac{(1 - \delta)(1 - \beta)(\alpha\zeta + \beta) - l\delta}{\zeta(1 + \beta)(\alpha\zeta + \beta) + l\delta},$$

which guarantee that

$$\frac{\xi}{1 - \beta} \left( \frac{2\varepsilon}{1 + \zeta\varepsilon - \delta} + \frac{l\delta}{\zeta(\alpha\zeta + \beta)} \right) < 1,$$

and so we have

$$\|(X_L - \lambda A)^{-1}(\Delta X_\ast - \lambda \Delta A)\| < 1$$

for $|\lambda| < 1$. Consequently, it follows that the matrix

$$\tilde{X}_L - \lambda \tilde{A} = (X_L - \lambda A)[I + (X_L - \lambda A)^{-1}(\Delta X_\ast - \lambda \Delta A)]$$

is nonsingular for $|\lambda| < 1$, and hence, by Theorem 3.4 of [1], $\tilde{X}_L$ is the maximal solution to the perturbed matrix equation (2.1). Thus, the inequality (2.20) implies that the inequality (2.13) holds. The proof is completed. □

3. Condition numbers

We now apply the theory of condition developed by Rice [6] to study condition numbers of the maximal solution $X_L$ to the NMEQ (1.1).
3.1. The complex case

Suppose that the coefficient matrices $A$ and $P$ are slightly perturbed to $\tilde{A} \in \mathbb{C}^{n \times n}$ and $\tilde{P} \in \mathbb{H}^{n \times n}$, respectively, and let

\[ \Delta A = \tilde{A} - A, \quad \Delta P = \tilde{P} - P. \]

From Theorem 2.1 and Remark 2.1 we see that if $\| (\Delta A, \Delta P) \|_F$ is sufficiently small, then the maximal solution $\tilde{X}_L$ to the perturbed matrix equation (2.1) exists, and

\[ u \Delta y \equiv \tilde{X}_L - X_L = L^{-1} (\Delta P \Delta A + O((\Delta A, \Delta P))_F), \quad (3.1) \]

as $(\Delta A, \Delta P) \rightarrow 0$.

By the theory of condition developed by Rice [6] we define the condition number of the maximal solution $X_L$ by

\[ c(X_L) = \lim_{\delta \to 0} \sup_{\| (\Delta A, \Delta P) \|_F \leq \delta} \frac{\| \Delta X \|_F}{\xi \delta}, \quad (3.2) \]

where $\xi, \alpha, \rho$ are positive parameters. Taking $\xi = \alpha = \rho = 1$ in (3.2) gives the absolute condition number $c_{abs}(X_L)$, and taking $\xi = \| X_L \|_F$, $\alpha = \| A \|_F$, $\rho = \| P \|_F$ in (3.2) gives the relative condition number $c_{rel}(X_L)$.

Substituting (3.1) into (3.2), we get

\[ c(X_L) = \frac{1}{\xi} \max_{\Delta A \in \mathbb{C}^{n \times n}, \Delta P \in \mathbb{H}^{n \times n}} \frac{\| L^{-1} (\Delta P \Delta A - \Delta A^* B) \|_F}{\| (\Delta A, \Delta P) \|_F}, \quad (3.3) \]

Let $L$ be the matrix representation of the linear operator $L$. Then it is easy to see that

\[ L = I - B^T \otimes B^* = I - (X_L^{-1} A)^T \otimes (X_L^{-1} A)^*. \quad (3.4) \]

Let

\[ L^{-1} = S + i \Sigma, \]

\[ L^{-1} (I \otimes B^*) = L^{-1} (I \otimes (X_L^{-1} A)^*) = U_1 + i \Omega_1, \]

\[ L^{-1} (B^T \otimes I) \Pi = L^{-1} ((X_L^{-1} A)^T \otimes I) \Pi = U_2 + i \Omega_2, \quad (3.5) \]

where $\Pi$ is the vec-permutation matrix, i.e.,

\[ \text{vec } A^T = \Pi \text{ vec } A. \]

Moreover, let

\[ S_c = \begin{bmatrix} S & -\Sigma \\ \Sigma & S \end{bmatrix}, \quad U_c = \begin{bmatrix} U_1 + U_2 & \Omega_2 - \Omega_1 \\ \Omega_1 + \Omega_2 & U_1 - U_2 \end{bmatrix}. \quad (3.6) \]
Then we have the following theorem.

**Theorem 3.1.** The condition number $c(X_L)$ defined by (3.2) has the explicit expression

$$c(X_L) = \frac{1}{\xi} \| (\rho S_c, \alpha U_c) \|_2,$$

where the matrices $S_c, U_c$ are defined by (3.4)–(3.6).

The proof of this result is quite similar to the proof of Theorem 2.1 of [8] and so is omitted. See [8] for the details.

**Remark 3.1.** From (3.7) we have the relative condition number

$$c_{rel}(X_L) = \frac{\| (\|P\|_F S_c, \|A\|_F U_c) \|_2}{\|X_L\|_F}.$$

### 3.2. The real case

In this subsection we consider the real case, i.e., all the coefficient matrices $A, P$ of the NMEQ (1.1) are real. In such a case the corresponding maximal solution $X_L$ is also real. Completely similar arguments as in [8] gives

**Theorem 3.2.** Let $A, P$ be real, and let $c(X_L)$ be the condition number defined by (3.2). Then $c(X_L)$ has the explicit expression

$$c(X_L) = \frac{1}{\xi} \| (\rho S_r, \alpha U_r) \|_2,$$

where

$$S_r = (I - (A^TX_L^{-1}) \otimes (A^TX_L^{-1}))^{-1},$$

$$U_r = -S_r [I \otimes (A^TX_L^{-1}) + ((A^TX_L^{-1}) \otimes I) P].$$

**Remark 3.2.** In the real case the relative condition number is given by

$$c_{rel}(X_L) = \frac{\| (\|P\|_F S_r, \|A\|_F U_r) \|_2}{\|X_L\|_F}.$$

### 4. Backward error

Let $\tilde{X} \in \mathcal{M}^{n \times n}$ be an approximation to the maximal solution $X_L$ to the NMEQ (1.1), and let $\Delta A$ and $\Delta P$ be the corresponding perturbations of the coefficient matrices $A$ and $P$ in the NMEQ (1.1). A backward error of the approximate solution $\tilde{X}$ can be defined by
\[ \eta(\tilde{X}) = \min \left\{ \left\| \begin{pmatrix} \Delta A & \Delta P \end{pmatrix} \right\|_F : \Delta A \in \mathbb{C}^{n \times n}, \, \Delta P \in \mathbb{H}^{n \times n}, \, \tilde{X} + (A + \Delta A)^+ \tilde{X}^{-1}(A + \Delta A) = P + \Delta P \right\}, \tag{4.1} \]

where \( \alpha \) and \( \rho \) are positive parameters. Taking \( \alpha = \|A\|_F \) and \( \rho = \|P\|_F \) in (4.1) gives the relative backward error \( \eta_{rel}(\tilde{X}) \), and taking \( \alpha = \rho = 1 \) in (4.1) gives the absolute backward error \( \eta_{abs}(\tilde{X}) \).

Let
\[ R = P - \tilde{X} - A^+ \tilde{X}^{-1}A. \tag{4.2} \]

Then from
\[ \tilde{X} + (A + \Delta A)^+ \tilde{X}^{-1}(A + \Delta A) = P + \Delta P, \]

we get
\[ \Delta A^+ \tilde{X}^{-1}A + A^+ \tilde{X}^{-1} \Delta A - \Delta P = R - \Delta A^+ \tilde{X}^{-1} \Delta A, \tag{4.3} \]

which shows that the problem of finding an explicit expression of the backward error \( \eta(\tilde{X}) \) defined by (4.1) is an optimal problem subject to a nonlinear constraint. It seems to be difficult to derive an explicit expression for the backward error \( \eta(\tilde{X}) \). In this section we only give some estimates for \( \eta(\tilde{X}) \).

4.1. The real case

In this subsection we assume that all the matrices \( A, P, \tilde{X}, \Delta A, \Delta P \) are real. In this case (4.3) can be written as
\[ T \begin{bmatrix} \text{vec} \Delta A \\ \text{vec} \Delta P \end{bmatrix} = \text{vec} R - \text{vec}(\Delta A^T \tilde{X}^{-1} \Delta A), \tag{4.4} \]

where
\[ T = \left( \alpha \left( \{A^T \tilde{X}^{-1}\} \otimes I \right) \| I \otimes (A^T \tilde{X}^{-1}) \right), -\rho I_n^2, \tag{4.5} \]

in which \( H \) is the vec-permutation. Since \( \rho > 0 \), the \( n^2 \times 2n^2 \) matrix \( T \) is full row rank, and hence, \( TT^\dagger = I_n^2 \), which implies that every solution to the equation
\[ T^\dagger \begin{bmatrix} \text{vec} \Delta A \\ \text{vec} \Delta P \end{bmatrix} = \begin{bmatrix} \text{vec} R - \text{vec}(\Delta A^T \tilde{X}^{-1} \Delta A) \end{bmatrix} \tag{4.6} \]

must be a solution to the equation (4.4). Consequently, for any solution
\[ \begin{bmatrix} \text{vec} \Delta A \\ \text{vec} \Delta P \end{bmatrix} \]
to Eq. (4.6) we have

\[
\eta(\tilde{X}) \leq \left\| \begin{bmatrix} \text{vec} \Delta A \\ \text{vec} \Delta P \end{bmatrix} \right\|_2.
\]  

(4.7)

Let

\[
\gamma = \|T^\dagger \text{vec } R\|_2, \quad \tau = \|T^\dagger\|^{-1}_2, \quad \mu = \|\tilde{X}^{-1}\|_2,
\]

and define

\[
\mathcal{L} \left( \frac{\Delta A}{\alpha}, \frac{\Delta P}{\rho} \right) = T^\dagger \left[ \text{vec } (\Delta A^T \tilde{X}^{-1} \Delta A) - \text{vec } (\Delta A^T \tilde{X}^{-1} \Delta A) \right].
\]

(4.9)

Then we have

\[
\left\| \mathcal{L} \left( \frac{\Delta A}{\alpha}, \frac{\Delta P}{\rho} \right) \right\|_2 \leq \gamma + \frac{\mu}{\tau} \|\Delta A\|_F^2 \leq \gamma + \frac{\alpha^2 \mu}{\tau} \left\| \left( \frac{\Delta A}{\alpha}, \frac{\Delta P}{\rho} \right) \right\|_F^2.
\]

(4.10)

Consider the equation

\[
\xi = \gamma + \frac{\alpha^2 \mu}{\tau} \xi^2.
\]

(4.11)

It is easy to verify that if

\[
\gamma \leq \frac{\tau}{4\alpha^2 \mu},
\]

(4.12)

then Eq. (4.11) has the positive number

\[
\xi_1 = \frac{2\gamma \tau}{\tau + \sqrt{\tau^2 - 4\gamma \tau \mu \alpha^2}}
\]

(4.13)

as its smallest positive real root. Thus, it follows from (4.10) that

\[
\left\| \left( \frac{\Delta A}{\alpha}, \frac{\Delta P}{\rho} \right) \right\|_F \leq \xi_1 \implies \left\| \mathcal{L} \left( \frac{\Delta A}{\alpha}, \frac{\Delta P}{\rho} \right) \right\|_2 \leq \xi_1.
\]

Therefore, by the Schauder fixed-point theorem, there exists a \((\Delta A_s/\alpha, \Delta P_s/\rho)\) satisfying

\[
\left\| \left( \frac{\Delta A_s}{\alpha}, \frac{\Delta P_s}{\rho} \right) \right\|_F \leq \xi_1
\]

such that

\[
\mathcal{L} \left( \frac{\Delta A_s}{\alpha}, \frac{\Delta P_s}{\rho} \right) = \begin{bmatrix} \text{vec } \Delta A_s \\ \text{vec } \Delta P_s \end{bmatrix}.
\]
which means that
\[
\begin{bmatrix}
\text{vec} \Delta A \\
\text{vec} \Delta P
\end{bmatrix}
\] is a solution to Eq. (4.6), and hence it follows from (4.7) that
\[
\eta(\tilde{X}) \leq \left\| \begin{bmatrix}
\text{vec} \Delta A \\
\text{vec} \Delta P
\end{bmatrix} \right\|_2 = \left\| \begin{bmatrix}
\Delta A \\
\Delta P
\end{bmatrix} \right\|_F \leq \xi_1.
\] (4.14)
i.e., \(\xi_1\) is an upper bound for \(\eta(\tilde{X})\).

Next we derive a lower bound for \(\eta(\tilde{X})\). Suppose that \((\Delta A_{\min}/\alpha, \Delta P_{\min}/\rho)\) satisfies
\[
\eta(\tilde{X}) = \left\| \begin{bmatrix}
\Delta A_{\min} \\
\Delta P_{\min}
\end{bmatrix} \right\|_F.
\] (4.15)
Then we have
\[
T \begin{bmatrix}
\text{vec} \Delta A_{\min} \\
\text{vec} \Delta P_{\min}
\end{bmatrix} = \text{vec} R - \text{vec}(\Delta A_{\min}^T \tilde{X}^{-1} \Delta A_{\min}).
\] (4.16)
Let a singular-value decomposition of \(T\) be
\[
T = W(\Omega, 0) Z^T,
\]
where \(W\) and \(Z\) are orthogonal matrices, \(\Omega = \text{diag}(\omega_1, \ldots, \omega_n)\) with \(\omega_1 \geq \cdots \geq \omega_n > 0\). Substituting this decomposition into (4.16), and letting
\[
Z^T \begin{bmatrix}
\text{vec} \Delta A_{\min} \\
\text{vec} \Delta P_{\min}
\end{bmatrix} = \begin{bmatrix} v^T \\
0
\end{bmatrix}, \quad v \in \mathbb{R}^n,
\]
we get
\[
v = \Omega^{-1} W^T [\text{vec} R - \text{vec}(\Delta A_{\min}^T \tilde{X}^{-1} \Delta A_{\min})],
\]
and so we have
\[
\eta(\tilde{X}) \geq \left\| \begin{bmatrix}
\text{vec} \Delta A_{\min} \\
\text{vec} \Delta P_{\min}
\end{bmatrix}
\right\|_2 = \left\| \begin{bmatrix} v \end{bmatrix} \right\|_2 \geq \|v\|_2
\]
\[
\geq \|\Omega^{-1} W^T \text{vec} R\|_2 - \|\Omega^{-1} W^T \text{vec}(\Delta A_{\min}^T \tilde{X}^{-1} \Delta A_{\min})\|_2
\]
\[
\geq \|T^\dagger \text{vec} R\|_2 - \|T^\dagger \|_2 \|\text{vec}(\Delta A_{\min}^T \tilde{X}^{-1} \Delta A_{\min})\|_2
\]
\[
\geq \gamma - \frac{\mu}{\tau} \|\Delta A_{\min}\|_F
\]
\[
\geq \gamma - \frac{\mu \alpha^2}{\tau} \|\begin{bmatrix}
\Delta A_{\min} \\
\Delta P_{\min}
\end{bmatrix} \|_F^2
\]
\[
\geq \gamma - \frac{\mu \alpha^2}{\tau} \xi_1^2,
\] (4.17)
in which the last inequality follows from the fact that
\[ \| \left( \frac{\Delta A_{\min}}{\alpha}, \frac{\Delta P_{\min}}{\rho} \right) \|_F = \eta(\tilde{X}) \leq \xi_1. \]

Let now
\[ l(\gamma) = \gamma - \frac{\mu \alpha^2}{\tau} \xi_1^2 = \gamma - \frac{\mu \alpha^2}{\tau} \left( \frac{2\gamma \tau}{\tau^2 - 4\gamma \tau \mu \alpha^2} \right)^2. \]

If we can prove that \( l(\gamma) > 0 \), then (4.17) just gives a useful lower bound for \( \eta(\tilde{X}) \). Therefore, we now devote to prove that \( l(\gamma) > 0 \). Since \( \xi_1 \) is a solution to Eq. (4.11), we have
\[ \xi_1 = \gamma + \frac{\mu \alpha^2}{\tau} \xi_1^2, \]
and hence we have
\[ l(\gamma) = \gamma - \frac{\mu \alpha^2}{\tau} \xi_1^2 = 2\gamma - \xi_1 = \frac{2\gamma \sqrt{\tau^2 - 4\gamma \mu \alpha^2}}{\tau^2 - 4\gamma \tau \mu \alpha^2} > 0. \]

In summary, we have proved the following theorem.

**Theorem 4.1.** Let \( A, P, \tilde{X}, \Delta A, \Delta P \) be real matrices, \( \eta(\tilde{X}) \) be the backward error defined by (4.1), and let the scalars \( \gamma, \tau, \mu \) be defined by (4.8). If \( \gamma < \frac{\tau}{4\mu \alpha^2} \), then we have
\[ 0 < l(\gamma) \leq \eta(\tilde{X}) \leq u(\gamma), \quad (4.18) \]
where
\[ u(\gamma) = \frac{2\gamma \tau}{\tau^2 + 4\gamma \tau \mu \alpha^2}, \quad l(\gamma) = \gamma - \frac{\mu \alpha^2}{\tau} u^2(\gamma). \quad (4.19) \]

**Remark 4.1.** The functions \( u(\gamma) \) and \( l(\gamma) \) defined by (4.19) have the Taylor expansions
\[ u(\gamma) = \gamma + \frac{\mu \alpha^2}{\tau} \gamma^2 + O(\gamma^3) \]
and
\[ l(\gamma) = \gamma - \frac{\mu \alpha^2}{\tau} \gamma^2 + O(\gamma^3), \]
respectively. Consequently, when \( \gamma \) is sufficiently small, we have
\[ \gamma - \frac{\mu \alpha^2}{\tau} \gamma^2 \lesssim \eta(\tilde{X}) \lesssim \gamma + \frac{\mu \alpha^2}{\tau} \gamma^2. \quad (4.20) \]

4.2. The complex case

Let
\[ (I \otimes (\tilde{X}^{-1} A)^*) = U_1 + i\Omega_1, \quad ((\tilde{X}^{-1} A)^T \otimes I) \Pi = U_2 + i\Omega_2, \]
where $\Pi$ is still the vec-permutation. Then Eq. (4.3) can be written as

$$T_c g = \begin{bmatrix} r \\ s \end{bmatrix} - \begin{bmatrix} a \\ b \end{bmatrix}.$$  

Completely similar to the proof of Theorem 4.1 we can prove the following theorem.

**Theorem 4.2.** Suppose that not all the matrices $A, P, \tilde{X}, uDeltayA, uDeltayP$ are real, and let $\eta(\tilde{X})$ be the backward error defined by (4.1). If $\gamma_c < \tau c / 4 \mu_1^2$, then we have

$$0 < l(\gamma_c) \leq \eta(\tilde{X}) \leq u(\gamma_c),$$

where

$$\gamma_c = \left\| T_c^+ \begin{bmatrix} r \\ s \end{bmatrix} \right\|_2, \quad \tau_c = \left\| T_c^+ \right\|^{-1}_2, \quad \mu = \left\| \tilde{X}^{-1} \right\|_2,$$

$$u(\gamma_c) = \frac{2 \gamma_c \tau_c}{\tau_c + \sqrt{\tau_c^2 - 4 \gamma_c \tau_c \mu \alpha^2}}, \quad l(\gamma_c) = \gamma_c - \frac{\mu \alpha^2}{\tau_c} u^2(\gamma_c).$$

### 5. Numerical examples

To illustrate the results of the previous sections, in this section three simple examples are given, which were carried out using MATLAB on a PC Pentium III/500 computer, with machine epsilon $\varepsilon = 2.2 \times 10^{-16}$.

**Example 5.1.** Consider the NMEQ (1.1) with the coefficient matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

where $0 < a < 1$. As has been pointed out in Remark 2.2, in this case the NMEQ (1.1) has the maximal solution

$$X_L = \begin{bmatrix} 1 & 0 \\ 0 & 1 - a^2 \end{bmatrix}.$$
Table 1

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\xi_*/|X_L|_F$</th>
<th>$\tilde{\xi}$</th>
<th>$c_{\text{rel}}(X_L)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$0.1077 \times 10^{-7}$</td>
<td>$0.1228 \times 10^{-6}$</td>
<td>1.3198</td>
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<tr>
<td>3</td>
<td>$0.1192 \times 10^{-7}$</td>
<td>$0.1167 \times 10^{-4}$</td>
<td>1.3198</td>
</tr>
<tr>
<td>5</td>
<td>$0.1193 \times 10^{-7}$</td>
<td>$0.0012$</td>
<td>1.3198</td>
</tr>
<tr>
<td>7</td>
<td>$0.1193 \times 10^{-7}$</td>
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<td>1.3198</td>
</tr>
<tr>
<td>9</td>
<td>$0.1193 \times 10^{-7}$</td>
<td>$11.6683$</td>
<td>1.3198</td>
</tr>
</tbody>
</table>

Take $a = 0.5 - 10^{-k}$, and suppose that the perturbations in the coefficient matrices are

$$ \Delta A = \begin{bmatrix} 0.9501 & 0.6068 \\ 0.2311 & 0.4860 \end{bmatrix} \times 10^{-9}, \quad \Delta P = \begin{bmatrix} -0.4326 & -0.7701 \\ 0.7701 & 0.2877 \end{bmatrix} \times 10^{-8}. $$

Some numerical results on the relative perturbation bounds $\xi_*/\|X_L\|_F$ and $\tilde{\xi}$ are shown in Table 1, where $\xi_*$ is as in (2.13) with the unitary invariant norm $\|\cdot\|_F$, $\tilde{\xi}$ is the relative perturbation bound given by Theorem 3.1 of [9]. The relative condition number $c_{\text{rel}}(X_L)$ is given in Remark 3.2.

The results listed in Table 1 show that the relative perturbation bound $\xi_*/\|X_L\|_F$ is fairly sharp, while the bound $\tilde{\xi}$ given by [9] is conservative.

On the other hand, take $a = 0.99$, and suppose that the perturbations in the coefficient matrices are

$$ \Delta A = \begin{bmatrix} 0.9501 & 0.6068 \\ 0.2311 & 0.4860 \end{bmatrix} \times 10^{-k}, \quad \Delta P = \begin{bmatrix} -0.4326 & -0.7701 \\ 0.7701 & 0.2877 \end{bmatrix} \times 10^{-k}. $$

In this case the relative condition number is $c_{\text{rel}}(X_L) = 2.892$, which is computed by the formula given as in Remark 3.2. This shows that the maximal solution $X_L$ is well-conditioned. Since the condition (3.1) of Theorem 3.1 in [9] is violated, $\tilde{\xi}$ becomes negative, and so we can not use it as an perturbation bound. However, as shown in Table 2, in such a case $\xi_*/\|X_L\|_F$ can still give quit sharp perturbation bounds.

**Example 5.2.** Consider the NMEQ (1.1) with the coefficient matrices

$$ A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}, \quad P = X + A^*X^{-1}A, $$

where $X = \text{diag}(1, 2, 3)$, which ensures that the maximal solution of the associate NMEQ (1.1) exists. It is easy to verify that $X$ is just the maximal solution and

Table 2

<table>
<thead>
<tr>
<th>$k$</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi_*/|X_L|_F$</td>
<td>$4.4319 \times 10^{-6}$</td>
<td>$4.4100 \times 10^{-7}$</td>
<td>$4.4079 \times 10^{-8}$</td>
<td>$4.4076 \times 10^{-9}$</td>
<td>$4.4076 \times 10^{-10}$</td>
</tr>
</tbody>
</table>
\( \rho(X^{-1}A) = 1 \). This implies that the linear operator \( L \) defined by (2.5) is singular, and so, theoretically, the expression of the relative condition number \( c_{rel}(X) \) given in Remark 3.1 is undefined. However, using it we found that \( c_{rel}(X) = 9.9573 \times 10^{16} \), and moreover, the MATALAB gave a warning: Matrix is close to singular or badly scaled. This reveals that in this case the maximal solution \( X \) is seriously ill-conditioned.

Let now
\[
\tilde{X} = X + \begin{bmatrix} 0.5 & -0.1 & 0.2 \\ -0.1 & 0.3 & 0.6 \\ 0.2 & 0.6 & -0.4 \end{bmatrix} \times 10^{-j}
\]
be an approximate solution to the NMEQ (1.1). Take \( \alpha = \|A\|_F \) and \( \rho = \|P\|_F \) in Theorem 4.1. Some numerical results on lower and upper bounds for the backward error \( \eta(X) \) are displayed in Table 3.

From the results listed in Table 3 we see that the backward error of \( \tilde{X} \) decreases as the error \( \|\tilde{X} - X\|_F \) decreases, and moreover, we see that for smaller \( \gamma \) (e.g., \( \gamma < 10^{-4} \)) we can get a quite better estimate for the backward error \( \eta(\tilde{X}) \) by taking \( \gamma \) as an approximation to \( u(\gamma) \) or \( l(\gamma) \). This example also shows that the backward error \( \eta(\tilde{X}) \) for an approximate solution \( \tilde{X} \) seems to be independent of the conditioning of the maximal solution \( X \).

**Example 5.3.** Consider the NMEQ (1.1) with the coefficient matrices \( A = n \times n \) Hilbert matrix, and \( P = X + A^T X^{-1} A \), where \( X \) is an \( n \times n \) symmetric tridiagonal matrix with diagonal elements 2 and subdiagonal elements 1. By Theorem 3.4 of [1], the maximal solution \( X_L \) to the NMEQ (1.1) exists. Let \( \tilde{X} \) be the computed maximal solution by using the MATLAB function \texttt{dare} (i.e., \( \tilde{X} = \texttt{dare}(0, I, P, 0, A^T) \)). Some numerical results on backward errors are shown in Table 4, where the values of \( \eta_{abs}(\tilde{X}) \) and \( \eta_{rel}(\tilde{X}) \) approximate the exact values up to about 15 significant digits. (Note. The approximate values of \( \eta_{abs}(\tilde{X}) \) and \( \eta_{rel}(\tilde{X}) \) are obtained from the computed upper bounds \( u(\gamma) \) and lower bounds \( l(\gamma) \).)

The values of the backward errors listed in Table 4 show that the computed maximal solution \( \tilde{X} \) is the exact Hermitian solution to a slightly perturbed NMEQ (1.1), in other words, the computation has proceeded quite stably.

<table>
<thead>
<tr>
<th>( j )</th>
<th>( |\tilde{X} - X|_F )</th>
<th>( \gamma )</th>
<th>( l(\gamma) )</th>
<th>( u(\gamma) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1149</td>
<td>0.0181</td>
<td>0.0177</td>
<td>0.0185</td>
</tr>
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<td>0.1805 \times 10^{-5}</td>
<td>0.1806 \times 10^{-5}</td>
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<td>0.1149 \times 10^{-6}</td>
<td>0.1806 \times 10^{-7}</td>
<td>0.1806 \times 10^{-7}</td>
<td>0.1806 \times 10^{-7}</td>
</tr>
<tr>
<td>9</td>
<td>0.1149 \times 10^{-8}</td>
<td>0.1806 \times 10^{-9}</td>
<td>0.1806 \times 10^{-9}</td>
<td>0.1806 \times 10^{-9}</td>
</tr>
</tbody>
</table>
Table 4

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\eta_{\text{abs}}(\tilde{X})$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>$0.291 \times 10^{-14}$</td>
<td>$0.537 \times 10^{-15}$</td>
</tr>
<tr>
<td>10</td>
<td>$0.564 \times 10^{-14}$</td>
<td>$0.794 \times 10^{-15}$</td>
</tr>
<tr>
<td>15</td>
<td>$0.821 \times 10^{-14}$</td>
<td>$0.906 \times 10^{-15}$</td>
</tr>
<tr>
<td>20</td>
<td>$0.105 \times 10^{-13}$</td>
<td>$0.972 \times 10^{-15}$</td>
</tr>
<tr>
<td>25</td>
<td>$0.149 \times 10^{-13}$</td>
<td>$0.121 \times 10^{-14}$</td>
</tr>
</tbody>
</table>

Acknowledgment

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References