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Relative compactness, cotopology and some other notions from the bitopological point of view

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Abstract

The paper consists of two sections. Section 1 is the introduction which, in addition to the auxiliary information, contains some interesting results on Baire-like properties. Section 2 deals with the bitopological essence of the notions of relative compactness and cotopology in general topology, C-relation, subordination of topologies and closed neighborhoods condition in analysis. A generalization of Choquet's theorem on Baire spaces is given and the sufficient conditions for families of (i, j)-nowhere dense sets to coincide with families of (i, j)-first category sets are established using a finite measure. A bitopological solution of one of Ulam's problems is obtained. The corresponding relations are almost always studied using essentially the bitopological modifications of regularity, which, as seen in various problems of general topology, analysis and potential theory, are the most natural forms of relations of two topologies defined on the same set.

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1. Introduction

In different areas of mathematics there are situations of both symmetric and nonsymmetric occurrence of two topologies on the same set. For example, concrete problems connected with nonsymmetric distance functions, quasi-uniformity, quasi-proximity, or-

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dered topological spaces, partially ordered sets and hence directed graphs, as well as semi-Boolean algebras and S-related topologies belong to the first situation, while the second situation underlies the notions of relative compactness, cotopology and C-relation in general topology, subordination of topologies and closed neighborhoods condition in analysis, initial and fine topologies in potential theory, cohomologies of spaces with two topologies in algebraic topology, etc.

By considering all the above cases we obtain a bitopology, i.e., an ordered pair of topologies (τ_1, τ_2) on a set X and a bitopological space (briefly, BS) is a set X equipped with two arbitrary topologies τ_1 and τ_2. In the sequel, if (X, τ_1, τ_2) is a BS and $\mathcal{P}$ is some topological property, then (i, j)-$\mathcal{P}$ denotes an analog of this property for τ_i with respect to τ_j, and p-$\mathcal{P}$ denotes the conjunction $(1,2)$-$\mathcal{P} \wedge (2,1)$-$\mathcal{P}$, i.e., p-$\mathcal{P}$ denotes an "absolute" bitopological analog of $\mathcal{P}$, where p is the abbreviation for "pairwise". Sometimes $(1,2)$-$\mathcal{P} \Longleftrightarrow (2,1)$-$\mathcal{P}$ (and thus $\Longleftrightarrow p$-$\mathcal{P}$), so that it suffices to consider one of these three bitopological analogs. Furthermore, there are certain cases for which it is not natural to consider p-$\mathcal{P}$ since $(1,2)$-$\mathcal{P}$ and $(2,1)$-$\mathcal{P}$ cannot represent all analogs of $\mathcal{P}$ for a simple reason that equivalent topological formulations in these cases do not remain equivalent when passing to their bitopological counterparts; in particular, this is observed in the case of Baire spaces [15,16]. Also note that (X, τ_i) has a property $\mathcal{P} \Longleftrightarrow (X, \tau_1, \tau_2)$ has a property i-$\mathcal{P}$, and d-$\mathcal{P} \Longleftrightarrow 1$-$\mathcal{P} \wedge 2$-$\mathcal{P}$, where d is the abbreviation for "double", and always $i, j \in \{1, 2\}$, $i \neq j$. Further, let (X, τ_1, τ_2) be any BS, $\mathcal{A} = \{A_s\}_{s \in S} \subset 2^X$ be any family and $A \subset X$ be any subset; then the conjugate family is $\operatorname{co}\mathcal{A} = \{X \setminus A_s\colon\ A_s \in \mathcal{A}\}$, $\tau_i \operatorname{cl} A$ and $\tau_i \operatorname{int} A$ denote respectively the closure and the interior of A in the topology τ_i.

The reasons connected in an obvious or veiled manner, on the one hand, with studying (X, τ_1, τ_2), where τ_1 and τ_2 are either independent of each other or interconnected by the inclusion, S-, C- and N-relations [39,42,15,16] or by their various combinations or by other relations, and, on the other hand, with applications of the theory of BS's, lead us to the following basic objectives as regards two general cases:

(1) to establish pairwise properties using the properties of τ_1 and τ_2 (or the properties of one of them) or other pairwise properties (or their combinations);
(2) to establish properties of τ_i using the properties of τ_j or pairwise properties (or their combinations).

As the study of various questions of the theory of BS's shows, (1) suggests a further development of the theory of BS's, while (2) is natural and typical of applications, especially when $i = 2$, $j = 1$, $\tau_1 \subset \tau_2$. Incidentally, note that by $(X, \tau_1 < \tau_2)$ will always be meant a BS (X, τ_1, τ_2) with $\tau_1 \subset \tau_2$.

This paper deals with bitopological characterizations of some principal notions of analysis and general topology, including a new characterization of almost (i, j)-Baire spaces obtained by generalizing the notion of a sifter introduced by Choquet [7]. We obtain a solution of one of Ulam's problems [17,40,43] which concerns the coincidence of the classes $\mathcal{H}(X, \tau)$ and $\mathcal{H}(X, \gamma)$ of all homeomorphisms of the topological spaces (X, τ) and (X, γ) onto themselves. Using a finite measure which is in agreement with the $(1,2)$-category for a special class of Baire BS's, we establish sufficient conditions for the four families of nowhere dense sets to coincide with the four families of first category sets for a BS $(X, \tau_1 < \tau_2)$. Thus in some cases the investigation of a set with two topologies interconnected by relations of "bitopological character" makes it possible to obtain the combinative effect, i.e., to get more information than in the case of considering the same set with each topology separately. We emphasize this fact since the formation and progress of the theory of BS's (as well as of other mathematical theories) are not isolated phenomena and acquire special importance in the light of applications of the obtained results. A broad range of bitopological applications is given in [13,15,16].

Since the study of applications of the theory of BS's demands special knowledge of bitopologies, we would like recall some notions from this theory needed for our purposes.

Many kinds of bitopological compactness imply even a greater variety of notions of bitopological local compactness. Their relations are indicated in [28].

Definition 1.1. Let (X, τ_1, τ_2) be a BS. Then:

(1) (X, τ_1, τ_2) is (i, j)-locally quasicompact (briefly, (i, j)-lqc) if each point $x \in X$ has an i-neighborhood $U(x)$ such that $\tau_j \operatorname{cl} U(x)$ is quasicompact [28].
(2) (X, τ_1, τ_2) is (i, j)-locally compact in Stoltenberg's sense (briefly, (i, j)-Slc) if each point $x \in X$ has an i-neighborhood $U(x)$ such that $\tau_j \operatorname{cl} U(x)$ is j-compact [37].
(3) (X, τ_1, τ_2) is (i, j)-locally compact in Reilly's sense (briefly, (i, j)-Rlc) if each point $x \in X$ has an i-neighborhood $U(x)$ such that $\tau_j \operatorname{cl} U(x)$ is FHP-compact, i.e., every family $\mathcal{U} = \{U_s\}_{s\in S}$ such that $\mathcal{U} \subset \tau_1 \cup \tau_2$, $X = \bigcup_{s\in S} U_s$ and $\mathcal{U} \cap \tau_i$ contains a nonempty set, has a finite subfamily [34].
(4) (X, τ_1, τ_2) is (i, j)-locally compact in Raghavan's and Reilly's sense (briefly, (i, j)-RRlc) if each point $x \in X$ has a j-neighborhood $U(x)$ which is i-compact [33].
(5) (X, τ_1, τ_2) is (i, j)-locally compact in Birsan's sense (briefly, (i, j)-Blc) if each point $x \in X$ has an i-neighborhood which is j-compact [6].

Following [29],

$$(i, j)\text{-Rlc} \Longleftarrow (i, j)\text{-lqc} \Longrightarrow (i, j)\text{-Slc} \Longrightarrow (i, j)\text{-Blc} \Longleftrightarrow (j, i)\text{-RRlc}.$$

The notion of a zero-dimensional BS was introduced by Reilly [35] on the basis of the idea of bitopological disconnectedness studied by Swart [38]. A systematic study of bitopological dimension functions was undertaken by Jelić [22,23], Ćirić [8] and us [10, 11,15]. As distinct from [22,23,8], the ideas set forth in [10,11,15] are essentially based on the notions of bitopological boundaries.

Definition 1.2. For any subset A of a BS (X, τ_1, τ_2) the (i, j)-boundaries of A are the sets $(i, j)\text{-}\operatorname{Fr} A = \tau_i \operatorname{cl} A \cap \tau_j \operatorname{cl}(X\backslash A)$ [10].

The notions of (i, j)-boundaries are highly important not only for defining and studying bitopological dimensions, but also for establishing the minimum principle for finely superharmonic functions [26].

Definition 1.3. Let (X, τ_1, τ_2) be a BS and n denote a nonnegative integer. We say that:

(1) $(i,j)\text{-}\operatorname{ind} X = -1 \Longleftrightarrow X = \emptyset$.
(2) $(i,j)\text{-}\operatorname{ind} X \leqslant n$ if for every point $x \in X$ and any neighborhood $U(x) \in \tau_i$ there exists a neighborhood $V(x) \in \tau_i$ such that $\tau_j \operatorname{cl} V(x) \subset U(x)$ and $(i,j)\text{-}\operatorname{ind}(j,i)\text{-}\operatorname{Fr} V(x) \leqslant n-1$.
(3) $(i,j)\text{-}\operatorname{ind} X = n$ if $(i,j)\text{-}\operatorname{ind} X \leqslant n$ and the inequality $(i,j)\text{-}\operatorname{ind} X \leqslant n-1$ does not hold.
(4) $(i,j)\text{-}\operatorname{ind} X = \infty$ if the inequality $(i,j)\text{-}\operatorname{ind} X \leqslant n$ does not hold for any n.

Therefore $p\text{-}\operatorname{ind} X \leqslant n \Longleftrightarrow (1,2)\text{-}\operatorname{ind} X \leqslant n \wedge (2,1)\text{-}\operatorname{ind} X \leqslant n$.

In particular, for $n = 0$ we obtain the notion of Reilly [35], i.e., $p\text{-}\operatorname{ind} X = 0 \Longleftrightarrow \tau_1$-open sets have a base consisting of τ_2-closed sets and τ_2-open sets have a base consisting of τ_1-closed sets.

The $\mathcal{A}$-insertion property of a topology τ on a set X was defined in [26] to establish the criterion of nonnormality of fine topologies and to characterize Baire one functions. Below we define bitopological modifications of this notion with an aim to apply them in characterizing the relations between topologies.

Definition 1.4. We say that a bitopology (τ_1, τ_2) on a set X has the (i,j)-$\mathcal{A}$-insertion property, where $\mathcal{A} \subset 2^X$ is any family if either of the following two equivalent conditions is satisfied:

(1) For every subset $A \subset X$ there exists a set $G \in \mathcal{A}$ such that $\tau_i \operatorname{int} A \subset G \subset \tau_j \operatorname{cl} A$.
(2) For every pair of sets (U, F), where $U \in \tau_i$, $F \in \operatorname{co}\tau_j$ and $U \subset F$, there exists a set $G \in \mathcal{A}$ such that $U \subset G \subset F$ [14].

It is obvious that if (τ_1, τ_2) on X (τ on X) has the (i,j)-$\mathcal{A}$-insertion properties ($\mathcal{A}$-insertion property), then $\emptyset, X \in \mathcal{A}$. It is likewise obvious that the antidiscrete topology on X possesses the $\mathcal{A}$-insertion property for any family $\mathcal{A} \subset 2^X$.

Remark 1.1. The following implications hold in a BS $(X, \tau_1 < \tau_2)$ for any family $\mathcal{A} \subset 2^X$:

$$\begin{array}{ccc} (\tau_1,\tau_2) \text{ has the 2-}\mathcal{A}\text{-insertion property} & \Longrightarrow & (\tau_1,\tau_2) \text{ has the } (1,2)\text{-}\mathcal{A}\text{-insertion property} \\ \Downarrow & & \Downarrow \\ (\tau_1,\tau_2) \text{ has the } (2,1)\text{-}\mathcal{A}\text{-insertion property} & \Longrightarrow & (\tau_1,\tau_2) \text{ has the 1-}\mathcal{A}\text{-insertion property.} \end{array}$$

By reducing the emphasis on points and focusing attention on the families of sets, namely, on the topologies, it is possible to consider the relations on a set. The coupling of topologies, i.e., the C-relation was defined by Weston [42] to generalize some well-known theorems on topological groups and linear spaces and to connect the same properties of the coupled topologies.

Definition 1.5. A topology τ_1 is coupled to a topology τ_2 on a set X (briefly, $\tau_1 C \tau_2$) if $\tau_1 \operatorname{cl} U \subset \tau_2 \operatorname{cl} U$ for every set $U \in \tau_1$.

From this definition we immediately find that if $\tau_1 = \operatorname{co}\tau_1$, then τ_1 is coupled to every topology on X, so that the antidiscrete topology on X as well as the discrete topology on X is coupled to every topology on X.

Remark 1.2. By [42], if τ_1 is coupled to τ_2 on X, then τ_1 is coupled to every topology on X smaller than τ_2. A topology can be coupled to a strictly larger topology and in that case the coupling is mutual. For example, the antidiscrete topology is mutually coupled to every topology on the same set.

In [42], more interest is shown in the coupling of topologies than in the situation $\tau_1 \operatorname{cl} U \subset \tau_2 \operatorname{cl} U$ for every set $U \in \tau_2$ (the N-relation in our terms). This preference is based on the reasoning as follows: the C-relation defines a partial order $\leqslant$ (in our notation $<_C$) on the set of all topologies on X by virtue of the equivalence $\tau_1 <_C \tau_2 \Longleftrightarrow \tau_1 C \tau_2$ and $\tau_1 \subset \tau_2$, and in his subsequent investigations J.D. Weston considered the cases where $\tau_1 <_C \tau_2$ and (X, τ_2) satisfies the conditions for which it is regular. If instead of the partial order $<_C$ we consider the relation $<_N$ (which is also a partial order) by analogy with $<_C$, then by virtue of Theorem 1 in [42] the conditions $\tau_1 <_N \tau_2$ and (X, τ_2) is regular (where the regularity of (X, τ_2) is not superfluous) imply that $\tau_1 = \tau_2$. As distinct from the above situation, we have the following simple

Example 1.1. Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{a\}, \{b, c\}, X\}$ and τ_2 be the discrete topology on X. Then $\tau_1 <_C \tau_2$ and (X, τ_2) is regular. However $\tau_1 \neq \tau_2$.

The coincidence of the topologies $\tau_1 <_C \tau_2$ demands a stronger requirement on (X, τ_1, τ_2), namely, if $(X, \tau_1 <_C \tau_2)$ is $(2, 1)$-regular (i.e., for each point $x \in X$ and each 2-open set $U \subset X$, $x \in U$, there exists a 2-open set $V \subset X$ such that $x \in V \subset \tau_1 \operatorname{cl} V \subset U$), then $\tau_1 = \tau_2$ [15].

Taking this fact into account, we have studied the so-called nearness of topologies, i.e., the N-relation in detail in [15].

Let us consider the real line $\mathbb{R}$ with the lower topology $\omega_1 = \{\emptyset, \mathbb{R}\} \cup \{(a, +\infty)\colon a \in \mathbb{R}\}$ and the upper topology $\omega_2 = \{\emptyset, \mathbb{R}\} \cup \{(-\infty, a)\colon a \in \mathbb{R}\}$. Then ω_i is not coupled to ω_j, but ω_i is near to ω_j. A nontrivial example of near topologies is given in [16].

It is well known (see, for example, [20]) that a subset A of a topological space (briefly, TS) (X, τ) can be of one category in (X, τ) and of another category in itself as a subspace of (X, τ), while for open subsets of (X, τ) these categories coincide. This is the principal factor in defining Baire spaces in various equivalent ways [20].

However, as illustrated by Example 1.5.1 in [15], unlike the topological case, a nonempty i-open subset of a BS (X, τ_1, τ_2) can be of one (i, j)-category in (X, τ_1, τ_2) and of another category in itself as a bitopological subspace of (X, τ_1, τ_2). These arguments are closely connected with the definitions of (i, j)-Baire spaces [12] and serve as a good introduction to the discussion in [15] and [16].

Definition 1.6. A subset A of a BS (X, τ_1, τ_2) is of (i, j)-first category in X if $A = \bigcup_{n=1}^{\infty} A_n$, where $\tau_i \operatorname{int} \tau_j \operatorname{cl} A_n = \emptyset$, i.e., A_n is (i, j)-nowhere dense ($A_n \in (i, j)$-$\mathcal{ND}(X)$) for every $n = \overline{1, \infty}$, and A is of (i, j)-second category in X if A is not of (i, j)-first category in X [12].

A subset A of X is of (i, j)-first ((i, j)-second) category if A is of (i, j)-first ((i, j)-second) category in itself.

The families of sets of (i,j)-first ((i,j)-second) categories in X are denoted by (i,j)-$\mathcal{C}atg_{\mathrm{I}}(X)$ ((i,j)-$\mathcal{C}atg_{\mathrm{II}}(X)$), while the statements $X \in (i,j)$-$\mathcal{C}atg_{\mathrm{I}}(X)$ ($X \in (i,j)$-$\mathcal{C}atg_{\mathrm{II}}(X)$) are abbreviated to X are of (i,j)-$\mathcal{C}atg\,\mathrm{I}$ (X are of (i,j)-$\mathcal{C}atg\,\mathrm{II}$).

The theory of bitopological Baire spaces thoroughly developed in [15,16] is closely associated with the Baire-like properties from [26] and therefore can be essentially used for future studies in analysis and general topology.

Definition 1.7. An (i,j)-Baire space (briefly, (i,j)-BrS) is a BS (X,τ_1,τ_2) such that $U \in \tau_i \backslash \{\emptyset\} \Longrightarrow U$ is of (i,j)-$\mathcal{C}atg\,\mathrm{II}$.

This definition immediately implies that if (X,τ_1,τ_2) is an (i,j)-BrS, then X is of (i,j)-$\mathcal{C}atg\,\mathrm{II}$.

Example 1.2. A natural BS $(\mathbb{R},\omega_1,\omega_2)$ is an (i,j)-BrS since for every set $U \in \tau_i \backslash \{\emptyset\}$ the bitopological subspace (U,ω_1',ω_2') contains no nonempty (i,j)-nowhere dense sets. By (ii) of Theorem 1.1.3 in [20] it is also clear that $(\mathbb{R},\omega_1,\omega_2)$ is an i-BrS.

Definition 1.8. An almost (i,j)-Baire space (briefly, A-(i,j)-BrS) is a BS (X,τ_1,τ_2) such that $U \in \tau_i \backslash \{\emptyset\} \Longrightarrow U \in (i,j)$-$\mathcal{C}atg_{\mathrm{II}}(X)$.

In [16], in particular, it is proved that for a BS $(X,\tau_1<\tau_2)$ the following equivalence and implications are correct: (X,τ_1,τ_2) is a $(1,2)$-BrS $\Longleftrightarrow$ (X,τ_1,τ_2) is an A-$(1,2)$-BrS $\Longrightarrow$ (X,τ_1,τ_2) is a 1-BrS, (X,τ_1,τ_2) is a 2-BrS $\Longrightarrow$ (X,τ_1,τ_2) is an A-$(2,1)$-BrS.

Definition 1.9. A BS (X,τ_1,τ_2) is an (i,j)-BrS in a strong sense (briefly, S-(i,j)-BrS) if $F \in \mathrm{co}\,\tau_i \backslash \{\emptyset\} \Longrightarrow F$ is of (i,j)-$\mathcal{C}atg\,\mathrm{II}$.

If the C-relation is hereditary under 1-closed subsets, then for a BS $(X,\tau_1 <_C \tau_2)$ we have (X,τ_1,τ_2) is a S-$(1,2)$-BrS $\Longrightarrow$ (X,τ_1,τ_2) is a $(1,2)$-BrS [16].

Definition 1.10. A BS $(X,\tau_1<\tau_2)$ is a $(1,2)$-strict Baire space, a $(2,1)$-weak Baire space, a 2-weak Baire space and a 1-strict Baire space, respectively (briefly, $(1,2)$-SBrS, $(2,1)$-WBrS, 2-WBrS and 1-SBrS, respectively) if $U \in \tau_2 \backslash \{\emptyset\} \Longrightarrow U \in (1,2)$-$\mathcal{C}atg_{\mathrm{II}}(X)$, $U \in \tau_1 \backslash \{\emptyset\} \Longrightarrow U \in (2,1)$-$\mathcal{C}atg_{\mathrm{II}}(X)$, $U \in \tau_1 \backslash \{\emptyset\} \Longrightarrow U \in 2$-$\mathcal{C}atg_{\mathrm{II}}(X)$ and $U \in \tau_2 \backslash \{\emptyset\} \Longrightarrow U \in 1$-$\mathcal{C}atg_{\mathrm{II}}(X)$, respectively [15].

The interrelations of the above-stated notions are collected in Theorem 1.1, where for the purpose of abbreviating the conditions (1)–(3), instead of writing spaces, we will indicate only the corresponding Baire and Baire-like properties.

Theorem 1.1. *The following implications hold for a* BS $(X,\tau_1<\tau_2)$:

$$(1)\quad \begin{array}{ccccccccc} \text{2-WBrS} & \Longleftarrow & \text{2-BrS} & \Longleftarrow & (1,2)\text{-SBrS} & \Longrightarrow & (1,2)\text{-BrS} & \Longrightarrow & \text{2-WBrS} \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ (2,1)\text{-WBrS} & \Longleftarrow & A\text{-}(2,1)\text{-BrS} & \Longleftarrow & \text{1-SBrS} & \Longrightarrow & \text{1-BrS} & \Longleftrightarrow & (2,1)\text{-WBrS}. \end{array}$$

We have for a BS $(X, \tau_1 <_C \tau_2)$:

$$\begin{array}{ccccccccc} \text{(2)} & \text{2-WBrS} & \Longleftarrow & \text{2-BrS} & \Longleftarrow & (1,2)\text{-SBrS} & \Longrightarrow & (1,2)\text{-BrS} & \Longrightarrow & \text{2-WBrS} \\ & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ & (2,1)\text{-WBrS} & \Longleftarrow & A\text{-}(2,1)\text{-BrS} & \Longleftrightarrow & \text{1-SBrS} & \Longrightarrow & \text{1-BrS} & \Longleftrightarrow & (2,1)\text{-WBrS}. \end{array}$$

We have for a BS $(X, \tau_1 <_N \tau_2)$:

$$\begin{array}{ccccccccc} \text{(3)} & \text{2-WBrS} & \Longleftarrow & \text{2-BrS} & \Longleftrightarrow & (1,2)\text{-SBrS} & \Longrightarrow & (1,2)\text{-BrS} & \Longrightarrow & \text{2-WBrS} \\ & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ & (2,1)\text{-WBrS} & \Longleftarrow & A\text{-}(2,1)\text{-BrS} & \Longleftrightarrow & \text{1-SBrS} & \Longrightarrow & \text{1-BrS} & \Longleftrightarrow & (2,1)\text{-WBrS}. \end{array}$$

Remark 1.3. Being an equivalence relation, the S-relation introduced in [39] expresses a close relationship between two topologies on a set, which implies that if one of the members of an S-equivalence class is a Baire space, then all members of this class are also Baire spaces. For the relations $\tau_1 <_S \tau_2 \Longleftrightarrow \tau_1 S \tau_2$ and $\tau_1 \subset \tau_2$, all the above-mentioned Baire-like properties coincide [16].

Example 1.3. Let $(\mathbb{R}, s, \tau)$ be a BS, where s is the half-open interval topology, i.e., the Sorgenfrey topology on $\mathbb{R}$ having basic open sets of the form $[a, b)$, while τ is the topology with basic open sets of the form $(a, b]$. It is clear that neither topology is finer than the other, $\inf(s, \tau) = s \cap \tau = \omega$ is the natural topology on $\mathbb{R}$, $\sup(s, \tau)$ is the discrete topology on $\mathbb{R}$. Moreover, $sS\tau$, $sS \inf(s, \tau)$ and $\tau S \inf(s, \tau)$. Hence, by Remark 1.3, the BS's $(\mathbb{R}, \omega <_S s)$ and $(\mathbb{R}, \omega <_S \tau)$ are 2-BrS's since the natural topology ω is Baire [39], and by (1) of Theorem 2.1.2 in [15], they are also 2-WBrS's; nevertheless $\omega \neq s$ and $\omega \neq \tau$.

Since an abstract fine topology τ_2 on a TS (X, τ_1) is any topology on X, finer than τ_1, the Baire-like properties formulated in [26] become as follows:

Theorem 1.2. *Let τ_2 be a fine topology on a* TS (X, τ_1). *Then*:

(1) (X, τ_2) *is a weak Baire space with respect to* $\tau_1 \Longleftrightarrow (X, \tau_1, \tau_2)$ *is an* A-(2, 1)-BrS.
(2) τ_2 *on a* TS (X, τ_1) *has the Slobodnik property* $\Longleftrightarrow (X, \tau_1, \tau_2)$ *is a* (1, 2)-BrS.
(3) τ_2 *on a* TS (X, τ_1) *has the property* $\mathcal{M} \Longleftrightarrow (X, \tau_1, \tau_2)$ *is a inebreak* (1, 2)-SBrS [16].

2. The naturality of relations between some principal concepts of general topology, analysis, and bitopology. Applications

As said in [4], to extend some results obtained for compact and metric spaces, various authors, in particular, Arhangelskiĭ [3], Filippov [18], Hodel [21], Juhász [24], Nagata [30], Ponomarev [32], used the idea of relating compact subsets to the topology of a space by means of a special cover or a family of covers. The notion of relative compactness based on the relation of two topologies on the same set and used by Z. Balogh instead of the above-mentioned idea clearly reveals, even at first glance, the bitopological essence of this notion.

Hence we are able to choose different kinds of bitopological local compactness leading to relative compactness. Further, using the method of application in the opposite direction, we come to interesting and important results related to the varieties of bitopological local compactness. Moreover, the strengthening of relative compactness makes it possible to connect the resulting strong relative compactness with a special-type local compactness through an equivalence relation.

Definition 2.1. In a BS (X, τ_1, τ_2) the topology τ_i is compact with respect to the topology τ_j if for every i-open cover $\mathcal{U}$ of X and for each point $x \in X$ there is a j-neighborhood of x covered by a finite subfamily of $\mathcal{U}$ [5].

Now Definition 1.1 readily implies

Theorem 2.1. *The following implications hold for a* BS (X, τ_1, τ_2):

$$(X, \tau_1, \tau_2) \text{ is } (j, i)\text{-lqc} \Longrightarrow (X, \tau_1, \tau_2) \text{ is } (j, i)\text{-Slc} \Longrightarrow (X, \tau_1, \tau_2) \text{ is } (j, i)\text{-Blc}$$
$$\Longleftrightarrow (X, \tau_1, \tau_2) \text{ is } (i, j)\text{-RRlc} \Longrightarrow \tau_i \text{ is compact with respect to } \tau_j.$$

Proof. By the implications given after Definition 1.1 it suffices to prove only the last implication of the theorem. Let $\mathcal{U} = \{U_s\}_{s \in S}$ be any i-open cover of X and let $x \in X$ be an arbitrary point. Then by (4) of Definition 1.1 there exists a j-neighborhood $U(x)$ which is i-compact. Clearly, $\mathcal{U} = \{U_s\}_{s \in S}$ is also an i-open cover of $U(x)$ and thus $\mathcal{U}$ has a finite subfamily $\mathcal{U}' = \{U_{s_k}\}_{k=1}^n$ such that $U(x) \subset \bigcup_{k=1}^n U_{s_k}$. Hence τ_i is compact with respect to τ_j. □

Clearly, using the relative compactness argument and the well-known notions from general topology, the bitopological assertion of Theorem 2.1 gives many interesting results from [4] and [5]. Let us now establish the conditions under which the inverse implication to the last implication in Theorem 2.1 is true.

Definition 2.2. A subset A of a BS (X, τ_1, τ_2) is said to be i-extendable if for every i-open cover $\mathcal{U}'$ of A there is an i-open cover $\mathcal{U}$ of X such that $\mathcal{U}' \subset \mathcal{U}$ and $U \cap A = \emptyset$ for each $U \in \mathcal{U} \setminus \mathcal{U}'$. In that case $\mathcal{U}$ is said to be i-extended from A.

It is obvious that every i-closed set F, $\emptyset \neq F \neq X$, is i-extendable, while there are no i-dense i-extendable subsets of X.

If in a BS (X, τ_1, τ_2) the topology τ_i is compact with respect to the topology τ_j, then for an arbitrary point $x \in X$ and every i-open cover $\mathcal{U}$ of X the j-neighborhood of x mentioned in Definition 2.1 will be denoted by $U_{\mathcal{U}}(x)$.

Definition 2.3. In a BS (X, τ_1, τ_2) the topology τ_i is strongly compact with respect to the topology τ_j if τ_i is compact with respect to τ_j and for every point $x \in X$ there is a j-open i-extendable neighborhood $U(x)$ such that $U(x) \subset U_{\mathcal{U}}(x)$ for every i-open cover $\mathcal{U}$ of X, i-extended from $U(x)$.

Theorem 2.2. *If in a* BS (X, τ_1, τ_2) *the topology* τ_i *is strongly compact with respect to the topology* τ_j*, then* (X, τ_1, τ_2) *is* (i, j)-RRlc ($\Longleftrightarrow (j, i)$-Blc).

Proof. Let $x \in X$ be any point and $U(x)$ an i-extendable j-neighborhood of x, whose existence follows from the conditions of the theorem. We are to prove that $U(x)$ is the required i-compact j-neighborhood of x. If $\mathcal{U}'$ is any i-open cover of $U(x)$, then there exists an i-open cover $\mathcal{U}$ of X, i-extended from $U(x)$ and containing $\mathcal{U}'$ as a subfamily. Since τ_i is strongly compact with respect to τ_j, there is a j-neighborhood $U_{\mathcal{U}}(x)$ such that $U(x) \subset U_{\mathcal{U}}(x)$ and $U_{\mathcal{U}}(x)$ is covered by a finite subfamily $\mathcal{U}'' \subset \mathcal{U}$. It is obvious that $U(x)$ is also covered by $\mathcal{U}''$ and $\mathcal{U}'' \subset \mathcal{U}'$ since $U \in \mathcal{U} \backslash \mathcal{U}'$ implies that $U \cap U(x) = \emptyset$. □

Corollary 1. *Let for any point* x *of a* BS (X, τ_1, τ_2) *there exists an* i*-extendable* j*-neighborhood* $U(x)$ *such that* $U(x) \subset U_{\mathcal{U}}(x)$ *for every* i*-open cover* $\mathcal{U}$ *of* X*,* i*-extended from* $U(x)$*. Then* (X, τ_1, τ_2) *is an* (i, j)-RRlc ($\Longleftrightarrow (j, i)$-Blc) *if and only if* τ_i *is compact with respect to* τ_j*.*

Proof. Follows directly from Theorems 2.1 and 2.2. □

Corollary 2. *If for a* p-T_2*, i.e.,* p*-Hausdorff* BS (X, τ_1, τ_2) *the topology* τ_i *is strongly compact with respect to the topology* τ_j*, then* $\tau_i \subset \tau_j$*, where* (X, τ_1, τ_2) *is* p-T_2 *if for each pair of distinct points* $x, y \in X$ *there exist disjoint neighborhoods* $U(x) \in \tau_1$ *and* $V(y) \in \tau_2$*.*

Proof. It suffices to use Proposition 10 from [6]. □

Furthermore, as we will see below, there are aspects of bitopological insertions closely connected with characterizations of the C-relation and, as a result, with different notions from analysis.

Theorem 2.3. *The following conditions are equivalent in a* BS (X, τ_1, τ_2):

(1) τ_1 *is coupled to* τ_2.
(2) (τ_1, τ_2) *has the* $(2, 1)$-τ_1*-insertion property.*
(3) *For every set* $U \in \tau_2$ *there exists a set* $V \in \tau_1$ *such that* $U \subset V$ *and* $\tau_1 \operatorname{cl} U = \tau_1 \operatorname{cl} V$.

Proof. (1) $\Leftrightarrow$ (2). Let $A \subset X$ be any set. Then by (3) of Theorem 2.2.1 in [15], $\tau_2 \operatorname{int} A \subset \tau_1 \operatorname{int} \tau_1 \operatorname{cl} A$ and if $V = \tau_1 \operatorname{int} \tau_1 \operatorname{cl} A$, (1) of Definition 1.4 implies that (τ_1, τ_2) has the $(2, 1)$-τ_1-insertion property. Conversely, if $A \subset X$ is any subset, then by (1) of Definition 1.4 there exists a set $V \in \tau_1$ such that $\tau_2 \operatorname{int} A \subset V \subset \tau_1 \operatorname{cl} A$ and therefore $\tau_2 \operatorname{int} A \subset \tau_1 \operatorname{int} \tau_1 \operatorname{cl} A$. To complete the proof, it remains to use (3) of Theorem 2.2.1 in [15].

(2) $\Leftrightarrow$ (3). If $U \in \tau_2$ is any set, then by (2) of Definition 1.4 there exists a set $V \in \tau_1$ such that $U \subset V \subset \tau_1 \operatorname{cl} U$ and hence $\tau_1 \operatorname{cl} U = \tau_1 \operatorname{cl} V$. Conversely, let $U \in \tau_2$, $F \in \operatorname{co} \tau_1$

and $U \subset F$. Then there exists a set $V \in \tau_1$ such that $U \subset V$ and $\tau_1 \operatorname{cl} U = \tau_1 \operatorname{cl} V$. Therefore $U \subset V \subset \tau_1 \operatorname{cl} V = \tau_1 \operatorname{cl} U \subseteq F$, so that $U \subset V \subset F$. □

Corollary 1. *The following conditions are equivalent for a* BS (X, τ_1, τ_2):

(1) $\tau_1 <_C \tau_2$.
(2) τ_1 *and* τ_2 *are weakly connected in the sense of* [27].

Proof. Follows directly from Definition 1 in [27] and (3) of Theorem 2.3. □

Corollary 2. *A locally convex linear* TS (X, τ_1) *is barrelled if and only if the bitopology* (τ_1, τ_2) *has the* $(2,1)$-τ_1*-insertion property for any locally convex topology* τ_2 *admitted by* X.

Proof. It suffices to use Theorem 2.3 and the arguments from [42]. □

Corollary 3. *Every locally convex linear* TS *of the second category is barrelled.*

Proof. Let (X, τ_1) be any locally convex linear TS and (X, τ_1) be of $\mathcal{C}atg\,$II. If τ_2 is any locally convex topology on X, then the conditions of Theorem 3 from [42] are satisfied and hence, by virtue of the same theorem and Theorem 2.3, the bitopology (τ_1, τ_2) on X has the $(2,1)$-τ_1-insertion property. Thus it remains to use Corollary 2. □

Corollary 4. *A barrelled linear* TS (X, τ_1) *is of* $\mathcal{C}atg\,$II (*or metrizable*) *if there is a locally convex topology* τ_2 *on* X*, larger than* τ_1*, such that* (X, τ_2) *is of* $\mathcal{C}atg\,$II (*or metrizable*).

Proof. Let (X, τ_1) be a barrelled linear TS and τ_2 be a locally convex topology on X, larger than τ_1. Then by Corollary 2, the bitopology (τ_1, τ_2) has the $(2,1)$-τ_1-insertion property and it remains to use Theorem 6 (or Theorem 7) from [42]. □

It should however be noted that by an example from [41], (X, τ_1) can be a barrelled linear TS of the first category although (X, τ_2) is a linear TS of the second category with $\tau_1 \subset \tau_2$.

Definition 2.4. Let τ_1 and τ_2 be two locally convex topologies on a linear space X. Then the topology τ_2 is subordinate to the topology τ_1 if τ_2 is finer than τ_1 and there exists a base of 2-neighborhoods of 0 (zero element) consisting of 1-closed convex circled sets [19].

The significance of the notion of subordination is confirmed by the results from [19].

Theorem 2.4. *Let* τ_1 *and* τ_2 *be two locally convex topologies on a linear space* X*. Then* τ_2 *is subordinate to* τ_1 *if and only if the* BS (X, τ_1, τ_2) *is* p*-regular and* (τ_1, τ_2) *has the* $(1,2)$-τ_2*-insertion property.*

Proof. First we assume that τ_2 is subordinate to τ_1. Then $\tau_1 \subset \tau_2$ and therefore $\tau_2\,\mathrm{cl}\,U \subset \tau_1\,\mathrm{cl}\,U$ for each $U \in \tau_2$, so that τ_2 is coupled to τ_1. Hence, by Corollary 2 of Theorem 2.3, (τ_1, τ_2) has the $(1,2)$-τ_2-insertion property. Moreover, (X, τ_2) has a local base consisting of 1-closed sets. Therefore (X, τ_1, τ_2) is $(2,1)$-regular. On the other hand, (X, τ_1) is also a locally convex linear TS having a local base consisting of 1-closed sets [25, 6.5], i.e., (X, τ_1) is regular. It is clear that (X, τ_1, τ_2) is $(1,2)$-regular (i.e., for each point $x \in X$ and each 1-open set $U \subset X$, $x \in U$, there exists a 1-open set $V \subset X$ such that $x \in V \subset \tau_2\,\mathrm{cl}\,V \subset U$) since $\tau_1 \subset \tau_2$ and hence (X, τ_1, τ_2) is p-regular.

Conversely, assume that a BS (X, τ_1, τ_2), where τ_1 and τ_2 are two locally convex topologies on a linear space X, is p-regular and (τ_1, τ_2) has the $(1,2)$-τ_2-insertion property. Hence, by (2) of Theorem 2.3, τ_2 is coupled to τ_1 and by Corollary 3 of Theorem 2.2.1 in [15] we have $\tau_1 \subset \tau_2$. For every neighborhood $V(0) \in \tau_2$ choose a neighborhood $U(0) \in \tau_2$ such that $\tau_1\,\mathrm{cl}\,U(0) \subset V(0)$. Since (X, τ_2) is locally convex, by [25, 6.5] there exists a set $F \in \mathrm{co}\,\tau_2$ with the property $0 \in F \subset U(0)$. Therefore $\tau_1\,\mathrm{cl}\,F \subset \tau_1\,\mathrm{cl}\,U(0) \subset V(0)$, where the set $\tau_1\,\mathrm{cl}\,F$ is convex circled because F is convex circled [36, Proposition 4]. Thus for every neighborhood $V(0) \in \tau_2$ there exists a convex circled set $\Phi = \tau_1\,\mathrm{cl}\,F$ such that $0 \in \Phi \subset V(0)$ and therefore τ_2 is subordinate to τ_1. □

Note that [19] contains many interesting examples from analysis illustrating the situation described by Theorem 2.4.

Further, we will consider the bitopological essence of the notion of cotopology. To this end, it is appropriate to give a quotation from [1]: "Cotopology may be roughly defined as the part of topology in which cospaces of a space X are used to study the properties of X". In the context of this statement and our arguments (see page 2 of this paper) we can state that bitopology may be roughly defined as the part of topology in which BS's can also be used to study the properties of the corresponding TS's.

Definition 2.5. Let (X, τ_2) be a TS. A topology τ_1 on X is called a cotopology of the topology τ_2 and (X, τ_1) is a cospace of (X, τ_2) if the following conditions are satisfied:

(1) τ_1 is weaker than τ_2.
(2) For each point $x \in X$ and any 2-closed neighborhood $M(x)$ there is a 1-closed neighborhood $N(x)$ such that $N(x) \subset M(x)$ [1].

It is not difficult to verify that if a BS (X, τ_1, τ_2) is 2-regular, then the above condition (2) can be replaced by the following equivalent condition:

(2′) Each point $x \in X$ has a 2-neighborhood base whose elements are 1-closed.

We have thus obtained the following simple, but important result.

Theorem 2.5. *In a* BS $(X, \tau_1 < \tau_2)$ *the topology* τ_1 *is a cotopology of the regular topology* τ_2 *if and only if* (X, τ_1, τ_2) *is* $(2,1)$*-regular.*

Corollary 1. *If for a* BS $(X, \tau_1 < \tau_2)$ *the dimension* $(2,1)$-$\mathrm{ind}\,X$ *is finite, then the topology* τ_1 *is a cotopology of the regular topology* τ_2.

Proof. Follows directly from (1) of Proposition 3.1.1 in [15]. □

Corollary 2. *If* (X, τ_1, τ_2) *is a* p*-regular* BS *and* (τ_1, τ_2) *has the* $(1,2)$-τ_2*-insertion property, then the topology* τ_1 *is a cotopology of the regular topology* τ_2*.*

Proof. Indeed, if (τ_1, τ_2) has the $(1,2)$-τ_2-insertion property, then by Theorem 2.3, τ_2 is coupled to τ_1. Hence, following Corollary 3 of Theorem 2.2.1 in [15], $\tau_1 \subset \tau_2$ since (X, τ_1, τ_2) is $(1,2)$-regular. Thus it remains to use Theorem 2.5. □

Corollary 3. *Let* τ_1 *and* τ_2 *be two locally convex topologies on a linear space* X*. Then* τ_2 *is subordinate to* τ_1 *if and only if the topology* τ_1 *is the cotopology of the regular topology* τ_2*.*

Proof. Let the locally convex topology τ_2 be subordinate to the locally convex topology τ_1 on the linear space X. Clearly, τ_1 and τ_2 are both regular. By Theorem 2.4, (X, τ_1, τ_2) is p-regular and (τ_1, τ_2) has the $(1,2)$-τ_2-insertion property. Hence, by Corollary 2, the topology τ_1 is the cotopology of the regular topology τ_2.

On the other hand, let τ_1 and τ_2 be two locally convex topologies on the linear space X and τ_1 be the cotopology of the (always regular) topology τ_2. Since (X, τ_1, τ_2) is always 1-regular and $\tau_1 \subset \tau_2$, (X, τ_1, τ_2) is also $(1,2)$-regular. Since (X, τ_1, τ_2) is 2-regular, it is also $(2,1)$-regular by Theorem 2.5. By Definition 1.5, $\tau_1 \subset \tau_2 \Longrightarrow \tau_2 C \tau_1$ and, following Theorem 2.3, (τ_1, τ_2) has the $(1,2)$-τ_2-insertion property. Thus it remains to apply Theorem 2.4. □

Corollary 4. *Let* τ_1 *and* τ_2 *be two locally convex topologies on a linear space* X*. Then the following conditions are equivalent*:

(1) τ_2 *is subordinate to* τ_1*.*
(2) (τ_1, τ_2) *satisfies the closed neighborhoods condition in the sense of* [9].
(3) τ_1 *is a cotopology of* τ_2*.*

Proof. It suffices to recall that (τ_1, τ_2) satisfies the closed neighborhoods condition if $\tau_1 \subset \tau_2$ and τ_2 has a base of zero element, consisting of 1-closed convex sets, and to use condition 6.5 from [25]. □

Corollary 5. *Let* $(X, \tau_1 < \tau_2)$ *be a* $(2,1)$*-regular* BS*. Then the following statements hold*:

(1) *If* (X, τ_1, τ_2) *is* 2-T_2 *and* 2*-locally compact, then* (X, τ_1, τ_2) *is* 1*-compact.*
(2) *If* (X, τ_1, τ_2) *is* 1*-compact, then* (X, τ_1, τ_2) *is a* 2-BrS *and thus an* A-$(2,1)$-BrS*, a* 2-WBrS *and a* $(2,1)$-WBrS.

Proof. Following Theorem 2.5, τ_1 is a cotopology of the regular topology τ_2. Hence (1) follows from [2] and, by (1) of Theorem 1.1, (2) follows from Theorem 2.9 in [20] since $(X, \tau_1 < \tau_2)$ is 2-quasi regular (i.e., for every set $U \in \tau_2 \setminus \{\emptyset\}$ there is a set $V \in \tau_2 \setminus \{\emptyset\}$ such that $\tau_2 \operatorname{cl} V \subset U$). □

Corollary 6. *A metrizable* TS (X, τ_2) *is topologically complete if and only if there exists a topology* τ_1 *on* X*, weaker than* τ_2*, for which* (X, τ_1, τ_2) *is* $(2, 1)$*-regular and* 1*-compact.*

Proof. It suffices to use Theorem 2.5 together with Theorem 1 from [1]. □

A characterization of almost (i, j)-Baire spaces, different from that given in Theorem 4.1.2 [15], is based on

Definition 2.6. A $\bigcap_j$-sifter on a BS (X, τ_1, τ_2) is a binary relation $\sqsubset_j$ on the family $\mathcal{A}_0(X) = \{A = U \cap V \neq \emptyset\colon\ U \in \tau_1,\ V \in \tau_2\}$, satisfying the following conditions:

(1) $A_1 \sqsubset_j A_2 \Longrightarrow A_1 \subset A_2$.
(2) For each $A \in \mathcal{A}_0(X)$ there is $U \in \tau_j \setminus \{\emptyset\}$ such that $U \sqsubset_j A$.
(3) $A_1' \subset A_1 \sqsubset_j A_2 \subset A_2' \Longrightarrow A_1' \sqsubset_j A_2'$.
(4) If a sequence $(A_n)_{n=1}^{\infty} \subset \mathcal{A}_0(X)$ and $A_{n+1} \sqsubset_j A_n$ for every $n = \overline{1, \infty}$, then $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$.

It is obvious that every $\bigcap_j$-sifter on $\mathcal{A}_0(X)$ is a j-sifter on the family of all nonempty j-open sets [7] and the result of Choquet [7] together with (5) of Theorem 4.1.3 [15] leads to the following implications:

$$\text{there is a } \textstyle\bigcap_2\text{-sifter on } (X, \tau_1 < \tau_2) \Longrightarrow \text{there is a 2-sifter on } (X, \tau_1 < \tau_2)$$
$$\Longrightarrow (X, \tau_1 < \tau_2) \text{ is a 2-BrS} \Longrightarrow (X, \tau_1 < \tau_2) \text{ is an } A\text{-}(2,1)\text{-BrS}.$$

In the general case we have

Theorem 2.6. *If there exists a* $\bigcap_j$*-sifter on a* BS (X, τ_1, τ_2)*, then* (X, τ_1, τ_2) *is an* A-(j, i)-BrS.

Proof. By (2) of Theorem 4.1.2 in [15] it is sufficient to prove that $A_n \in \tau_i \cap j\text{-}\mathcal{D}(X)$ for each $n = \overline{1, \infty}$, where $j\text{-}\mathcal{D}(X) = \{A \in 2^X\colon\ \tau_j \operatorname{cl} A = X\} \Longrightarrow \bigcap_{n=1}^{\infty} A_n \in j\text{-}\mathcal{D}(X)$. Let $U \in \tau_j \setminus \{\emptyset\}$ be any set and let us prove that $U \cap (\bigcap_{n=1}^{\infty} A_n) \neq \emptyset$. Clearly, for $U_1 = U$ we have $\emptyset \neq U_1 \cap A_1 \in \mathcal{A}_0(X)$. By (2) of Definition 2.6 there is a set $U_2 \in \tau_j \setminus \{\emptyset\}$ such that $U_2 \sqsubset_j U_1 \cap A_1$. Therefore, by the same condition and the fact that $A_n \in j\text{-}\mathcal{D}(X)$ for each $n = \overline{1, \infty}$, one can define a sequence of j-open nonempty sets (U_n) such that $U_1 = U$ and $U_{n+1} \sqsubset_j U_n \cap A_n$ for each $n = \overline{1, \infty}$. Thus $U_{n+1} \subset U_{n+1} \sqsubset_j U_n \cap A_n \subset U_n$ and, by (3) of Definition 2.6, $U_{n+1} \sqsubset_j U_n$. Therefore (4) of Definition 2.6 gives $\bigcap_{n=1}^{\infty} U_n \neq \emptyset$. On the other hand, we have $U_{n+1} \sqsubset_j U_n \cap A_n$ and, by (1) of Definition 2.6, $U_2 \subset A_1$, $U_3 \subset A_2, \ldots$. Hence

$$\bigcap_{n=2}^{\infty} U_n \subset \bigcap_{n=1}^{\infty} A_n \quad \text{and} \quad \bigcap_{n=1}^{\infty} U_n = U \cap \left(\bigcap_{n=2}^{\infty} U_n\right) \subset U \cap \left(\bigcap_{n=1}^{\infty} A_n\right). \qquad \square$$

Theorem 2.6 together with (4) of Theorem 4.1.3 in [15] implies a more general result than that of G. Choquet.

Corollary. *For a* BS $(X, \tau_1 < \tau_2)$ *the following implications hold*:

$$\begin{array}{ccc} \text{there exists a } \bigcap_1\text{-sifter on } (X,\tau_1<\tau_2) & \Longrightarrow & (X,\tau_1<\tau_2) \text{ is an } A\text{-}(1,2)\text{-BrS} \\ & & \Updownarrow \\ \Big\Downarrow & & (X,\tau_1<\tau_2) \text{ is a } (1,2)\text{-BrS} \\ & & \Downarrow \\ \text{there exists a 1-sifter on } (X,\tau_1<\tau_2) & \Longrightarrow & (X,\tau_1<\tau_2) \text{ is a 1-BrS.} \end{array}$$

Hence it follows that for BS's of the type $(X, \tau_1 < \tau_2)$, having a $\bigcap_1$-sifter and a $\bigcap_2$-sifter, the results obtained respectively for $(1, 2)$-BrS and A-$(2, 1)$-BrS in [12,15,16], are valid.

In 4.A.6 of [26] the sufficient conditions are found for which $\tau_1 S \tau_2$ in a BS $(X, \tau_1 < \tau_2)$. Below we prove the same result under weaker conditions.

Theorem 2.7. *If* $(X, \tau_1 < \tau_2)$ *is a* 2*-quasiregular and* $(1, 2)$-SBrS*, then we have* $\tau_1 S \tau_2$ *and therefore* $(X, \tau_1 < \tau_2)$ *is a* 1*-Blumberg space* $\Longrightarrow$ $(X, \tau_1 < \tau_2)$ *is a* 2*-Blumberg space, where by* [20], $(X, \tau_1 < \tau_2)$ *is an* i*-Blumberg space if for any* i*-real function* f *on* X *there is an* i*-dense subset* $D \subset X$ *such that the restriction* $f|_D$ *is continuous.*

Proof. By (3) of Theorem 1.2, $(X, \tau_1 < \tau_2)$ is a $(1, 2)$-SBrS $\Longleftrightarrow$ τ_2 on (X, τ_1) has the property $\mathcal{M}$. Thus, by (3) of 4.A.6 in [26] we have $\tau_2 \backslash \{\emptyset\} \subset (1, 2)$-$\mathcal{SD}(X) = 2^X \setminus (1, 2)$-$\mathcal{ND}(X)$. Now let $U \in \tau_2 \backslash \{\emptyset\}$ be any set. Then there is a set $V \in \tau_2 \backslash \{\emptyset\}$ such that $\tau_2 \operatorname{cl} V \subset U$ and therefore $\emptyset \neq \tau_1 \operatorname{int} \tau_2 \operatorname{cl} V \subset U$. Thus, by (2) of Corollary 2 of Theorem 2.1.1 in [15] we have $\tau_1 S \tau_2$. The rest is an immediate consequence of 4.B.4 of [26] as well as of (2) Theorem 2.1.1 in [15] since $\tau_1 S \tau_2 \Longrightarrow 1$-$\mathcal{D}(X) = 2$-$\mathcal{D}(X)$. □

Assume that Φ is a convex cone of nonnegative lower semicontinuous functions on a TS (X, τ_1) and τ_2 is a fine topology on X defined by the cone Φ. Then the obtained BS in potential theory is denoted by $(X, \tau_1 <_\Phi \tau_2)$ [15].

Corollary. *If a* 1*-Tychonoff* BS $(X, \tau_1 <_\Phi \tau_2)$ *is a* $(2, 1)$-SBrS*, then* $(X, \tau_1 <_\Phi \tau_2)$ *is a* 1*-Blumberg space* $\Longrightarrow$ $(X, \tau_1 <_\Phi \tau_2)$ *is a* 2*-Blumberg space.*

Proof. By Corollary 1 of Theorem 7.2.1 in [15], a BS $(X, \tau_1 <_\Phi \tau_2)$ is $(2, 1)$-completely regular and hence 2-quasiregular. □

Now, to establish the conditions of coincidence of the families (i, j)-$\mathcal{ND}(X)$ and (i, j)-$\mathcal{C}atg_{\mathrm{I}}(X)$, let us consider finite measures on BS's. As we will see below, these finite measures are closely related to the topologies τ_1 and τ_2, too, and therefore to the operators $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}$, where $\tau_1 \subset \tau_2$ and $\mathbf{n}_1(A) = \tau_1 \operatorname{cl} A \backslash \tau_2 \operatorname{cl} A$, $\mathbf{n}_2(A) = \tau_2 \operatorname{int} A \backslash \tau_1 \operatorname{int} A$, $\mathbf{n}(A) = \mathbf{n}_1(A) \cup \mathbf{n}_2(A)$ for every set $A \subset X$.

Definition 2.7. A subset A of a BS (X, τ_1, τ_2) has the (i, j)-Baire property if it can be represented as $A = U \triangle C$, where $U \in \tau_j$ and $C \in (i, j)$-$\mathcal{C}atg_{\mathrm{I}}(X)$.

The families of all subsets of (X, τ_1, τ_2), having the (i, j)-Baire properties, are denoted by (i, j)-$\mathcal{B}(X)$, i.e., (i, j)-$\mathcal{B}(X) = \{A \in 2^X\colon\ A = U \triangle C,\ U \in \tau_j,\ C \in (i, j)\text{-}\mathcal{C}atg_{\mathrm{I}}(X)\}$.

It is obvious that, for a BS $(X, \tau_1 < \tau_2)$ and a set $U \in \tau_2$ the set $\tau_1 \operatorname{cl} U \setminus U \in (1, 2)\text{-}\mathcal{N}\mathcal{D}(X) \subset (1, 2)\text{-}\mathcal{C}atg_{\mathrm{I}}(X)$ and therefore $\tau_1 \operatorname{cl} U = U \cup (\tau_1 \operatorname{cl} U \setminus U) = U \triangle (\tau_1 \operatorname{cl} U \setminus U) \in (1, 2)\text{-}\mathcal{B}(X)$.

It is clear that for a BS $(X, \tau_1 < \tau_2)$ the following inclusions hold:

$$\begin{array}{ccc} (2,1)\text{-}\mathcal{B}(X) & \subset & 2\text{-}\mathcal{B}(X) \\ \cap & & \cap \\ 1\text{-}\mathcal{B}(X) & \subset & (1,2)\text{-}\mathcal{B}(X). \end{array}$$

Theorem 2.8. *For a* BS $(X, \tau_1 < \tau_2)$ *the conditions below are satisfied*:

(1) $A \in (1, 2)\text{-}\mathcal{B}(X) \Longleftrightarrow A = F \triangle D$, *where* $F \in \operatorname{co}\tau_2$, $D \in (1, 2)\text{-}\mathcal{C}atg_{\mathrm{I}}(X) \Longrightarrow X \setminus A \in (1, 2)\text{-}\mathcal{B}(X)$.
(2) $(1, 2)\text{-}\mathcal{B}(X)$ *is a* σ*-algebra generated by the union* $\tau_2 \cup (1, 2)\text{-}\mathcal{C}atg_{\mathrm{I}}(X)$.
(3) $A \in (1, 2)\text{-}\mathcal{B}(X) \Longleftrightarrow A = G \cup E$, *where* $G \in 2\text{-}\mathcal{G}_\delta(X) = \{A \in 2^X\colon\ A \text{ is a } 2\text{-}\mathcal{G}_\delta\text{-set}\}$, $E \in (1, 2)\text{-}\mathcal{C}atg_{\mathrm{I}}(X)$ *and* $G \cap E = \emptyset \Longleftrightarrow A = F \setminus E$, *where* $F \in 2\text{-}\mathcal{F}_\sigma(X) = \{A \in 2^X\colon\ A \text{ is a } 2\text{-}\mathcal{F}_\sigma\text{-set}\}$ *and* $E \in (1, 2)\text{-}\mathcal{C}atg_{\mathrm{I}}(X)$.
(4) $A \in (1, 2)\text{-}\mathcal{B}(X) \Longleftrightarrow A = V \triangle M$, *where* $V \in (2, 1)\text{-}\mathcal{O}\mathcal{D}(X)$, *i.e.*, $V = \tau_2 \operatorname{int} \tau_1 \operatorname{cl} V$, $M \in (1, 2)\text{-}\mathcal{C}atg_{\mathrm{I}}(X)$ *and for a* $(1, 2)$-SBrS *this representation is unique.*

Proof. (1) First let us prove the equivalence. If $A = U \triangle C$, where $U \in \tau_2$, $C \in (1, 2)\text{-}\mathcal{C}atg_{\mathrm{I}}(X)$, then $N = \tau_2 \operatorname{cl} U \setminus U \in 2\text{-}\mathcal{N}\mathcal{D}(X) \subset (1, 2)\text{-}\mathcal{C}atg_{\mathrm{I}}(X)$ and, by (1) Theorem 1.1.3 in [15], $D = N \triangle C \in (1, 2)\text{-}\mathcal{C}atg_{\mathrm{I}}(X)$. Let $F = \tau_2 \operatorname{cl} U$. Then $A = U \triangle C = (\tau_2 \operatorname{cl} U \triangle N) \triangle C = \tau_2 \operatorname{cl} U \triangle (N \triangle C) = F \triangle D$. Conversely, let $A = F \triangle D$, where $F \in \operatorname{co}\tau_2$, $D \in (1, 2)\text{-}\mathcal{C}atg_{\mathrm{I}}(X)$, and let $U = \tau_2 \operatorname{int} F$. Then $N = F \setminus U \in 2\text{-}\mathcal{N}\mathcal{D}(X) \subset (1, 2)\text{-}\mathcal{N}\mathcal{D}(X)$, $C = D \triangle \in (1, 2)\text{-}\mathcal{C}atg_{\mathrm{I}}(X)$ and $A = F \triangle D = (U \triangle N) \triangle D = U \triangle C$.

The implication directly follows from the fact that $X \setminus (U \triangle C) = (X \setminus U) \triangle C$ and the above equivalence.

(2) To prove that $(1, 2)\text{-}\mathcal{B}(X)$ is a σ-algebra, by the implication in (1) it suffices to prove that $(1, 2)\text{-}\mathcal{B}(X)$ is closed under countable unions. Let $A_n = U_n \triangle C_n$, where $U_n \in \tau_2$ and $C_n \in (1, 2)\text{-}\mathcal{C}atg_{\mathrm{I}}(X)$ for each $n = \overline{1, \infty}$, and let $A = \bigcup_{n=1}^{\infty} A_n$, $U = \bigcup_{n=1}^{\infty} U_n$, $C = \bigcup_{n=1}^{\infty} C_n$. Then $U \in \tau_2$, $C \in (1, 2)\text{-}\mathcal{C}atg_{\mathrm{I}}(X)$ by (1) of Theorem 1.1.3 in [15], and $U \setminus C \subset A \subset U \cup C$. Therefore $U \triangle A = (U \setminus A) \cup (A \setminus U) \subset C$, i.e., $U \triangle A \in (1, 2)\text{-}\mathcal{C}atg_{\mathrm{I}}(X)$ so that $A = U \triangle (U \triangle A) \in (1, 2)\text{-}\mathcal{B}(X)$.

Clearly, $\tau_2 \cup (1, 2)\text{-}\mathcal{C}atg_{\mathrm{I}}(X) \subset (1, 2)\text{-}\mathcal{B}(X)$ and if $\mathcal{A}$ is any σ-algebra containing $\tau_2 \cup (1, 2)\text{-}\mathcal{C}atg_{\mathrm{I}}(X)$, then it is not difficult to see that $(1, 2)\text{-}\mathcal{B}(X) \subset \mathcal{A}$.

(3) Let $A = U \triangle C$, where $U \in \tau_2$, $C \in (1, 2)\text{-}\mathcal{C}atg_{\mathrm{I}}(X)$. Then by (3) of Theorem 1.1.3 in [15] there exists a set $Q \in 2\text{-}\mathcal{F}_\sigma(X) \cap (1, 2)\text{-}\mathcal{C}atg_{\mathrm{I}}(X)$ such that $C \subset Q$. Clearly, $Q = \bigcup_{n=1}^{\infty} A_n$, where $A_n \in \operatorname{co}\tau_2 \cap (1, 2)\text{-}\mathcal{N}\mathcal{D}(X)$ and $G = U \setminus Q \in 2\text{-}\mathcal{G}_\delta(X)$. Now we have $A = U \triangle C = U \triangle (C \cap Q) = ((U \setminus Q) \cup (U \cap Q)) \triangle (C \cap Q) = ((U \setminus Q) \triangle (U \cap Q)) \triangle (C \cap Q) = G \triangle ((U \cap Q) \triangle (C \cap Q)) = G \triangle ((U \triangle C) \cap Q) = G \triangle E$, where $G \in 2\text{-}\mathcal{G}_\delta(X)$, $E \in (1, 2)\text{-}\mathcal{C}atg_{\mathrm{I}}(X)$ and $G \cap E = \emptyset$ so that $A = G \cup E$.

(4) Let $A = U \triangle C$, $U \in \tau_2$ and $C \in (1, 2)\text{-}\mathcal{C}atg_{\mathrm{I}}(X)$. Then, by (3) of Proposition 1.3.1 in [15], $U = V \setminus \tau_2 \operatorname{cl} B$, where $V = \tau_2 \operatorname{int} \tau_1 \operatorname{cl} U \in (2, 1)\text{-}\mathcal{O}\mathcal{D}(X)$ and $B = V \setminus U \in$

$(1,2)$-$\mathcal{ND}(X)$. Therefore $A = U \triangle C = (V \triangle (V \setminus U)) \triangle C = V \triangle (B \triangle C) = V \triangle M$, where $M \in (1,2)$-$\mathcal{C}atg_{\mathrm{I}}(X)$.

Thus it remains only to prove that for a $(1,2)$-SBrS this representation is unique. Indeed, let $A = V \triangle M = W \triangle N$, where $V \in (2,1)$-$\mathcal{OD}(X)$, $W \in \tau_2$, $M, N \in (1,2)$-$\mathcal{C}atg_{\mathrm{I}}(X)$. Then $W \setminus \tau_2 \operatorname{cl} V \subset W \setminus V \subset W \triangle V = M \triangle N \in (1,2)$-$\mathcal{C}atg_{\mathrm{I}}(X)$. Since $W \setminus \tau_2 \operatorname{cl} V \in \tau_2 \cap (1,2)$-$\mathcal{C}atg_{\mathrm{I}}(X)$ and $(X, \tau_1 < \tau_2)$ is a $(1,2)$-SBrS, we have $W \setminus \tau_2 \operatorname{cl} V = \emptyset$, i.e., $W \subset \tau_2 \operatorname{cl} V$ so that $W \subset \tau_2 \operatorname{int} \tau_1 \operatorname{cl} V = V$. Therefore in the representation $A = V \triangle M$, the 2-open set $V \in (2,1)$-$\mathcal{OD}(X)$ is maximal and if V and W are both $(2,1)$-open domains, then $V \subset W$ and $W \subset V$, i.e., $V = W$ and $M = N$. □

Definition 2.8. Let $(X, \tau_1 < \tau_2)$ be a BS and μ be a finite measure on a σ-algebra $(1,2)$-$\mathcal{B}(X)$. Then μ is said to be in agreement with the $(1,2)$-category if $\mu(A) = 0 \Longleftrightarrow A \in (1,2)$-$\mathcal{C}atg_{\mathrm{I}}(X)$.

Theorem 2.9. *Let $(X, \tau_1 < \tau_2)$ be a $(2,1)$-regular $(1,2)$-SBrS and μ be a finite measure on $(1,2)$-$\mathcal{B}(X)$ which is in agreement with the $(1,2)$-category. Then for each 2-open set G and each $\varepsilon > 0$ there exists a 1-closed set F such that $F \subset G$, $\mu(F) > \mu(G) - \varepsilon$ and for each 2-closed set F there exists a 1-open set G such that $F \subset G$, $\mu(G) < \mu(F) + \varepsilon$.*

Proof. Let $\mathcal{U} = \{U\}$ be a maximal family of nonempty disjoint 2-open sets such that $\tau_1 \operatorname{cl} U \subset G$ for each $U \in \mathcal{U}$. Since $(X, \tau_1 < \tau_2)$ is a $(1,2)$-SBrS, $U \in \mathcal{U} \Longrightarrow U \in (1,2)$-$\mathcal{C}atg_{\mathrm{II}}(X) \Longrightarrow \mu(U) > 0$ and therefore the family $\mathcal{U}$ is countable at most, i.e., $\mathcal{U} = \{U_n\}_{n=1}^{\infty}$. Then $V = \bigcup_{n=1}^{\infty} U_n \subset G$. Let us prove that $G \subset \tau_2 \operatorname{cl} V$. Indeed, if $G \cap (X \setminus \tau_2 \operatorname{cl} V) \neq \emptyset$, then by the $(2,1)$-regularity of $(X, \tau_1 < \tau_2)$ there is a set $H \in \tau_2 \setminus \{\emptyset\}$ such that $\tau_1 \operatorname{cl} H \subset G \cap (X \setminus \tau_2 \operatorname{cl} V)$, i.e., $V \in \mathcal{U}$ and $V \neq U_n$ for each $n = \overline{1, \infty}$, which contradicts the maximality of $\mathcal{U}$. Hence $G \setminus V \subset \tau_2 \operatorname{cl} V \setminus V \in 2$-$\mathcal{ND}(X) \subset (1,2)$-$\mathcal{ND}(X) \subset (1,2)$-$\mathcal{C}atg_{\mathrm{I}}(X)$ and therefore $\mu(G \setminus V) = 0$, i.e., $\mu(G) = \mu(\bigcup_{n=1}^{\infty} U_n)$. For each $U \in \tau_2$, $\tau_1 \operatorname{cl} U \setminus U \in (1,2)$-$\mathcal{ND}(X) \subset (1,2)$-$\mathcal{C}atg_{\mathrm{I}}(X)$. Since $\mu(G) = \mu(\bigcup_{n=1}^{\infty} U_n)$, for each $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $\sum_{k=1}^{n} \mu(U_k) > \mu(G) - \varepsilon$. But if $F = \bigcup_{k=1}^{n} \tau_1 \operatorname{cl} U_k = \tau_1 \operatorname{cl} \bigcup_{k=1}^{n} U_k$, then $\mu(F) = \mu((\tau_1 \operatorname{cl} \bigcup_{k=1}^{n} U_k \setminus \bigcup_{k=1}^{n} U_k) \cup \bigcup_{k=1}^{n} U_k) = \mu(\bigcup_{k=1}^{n} U_k) = \sum_{k=1}^{n} \mu(U_k) > \mu(G) - \varepsilon$.

The rest of the proof can be obtained by passing to complements. □

Corollary. *Under the hypotheses of Theorem* 2.9 *we have*:

(1) $(X, \tau_1 < \tau_2)$ *is a* $(1,2)$-SBrS $\Longleftrightarrow$ $(X, \tau_1 < \tau_2)$ *is a* 2-BrS $\Longleftrightarrow$ $(X, \tau_1 < \tau_2)$ *is an* A-$(2,1)$-BrS $\Longleftrightarrow$ $(X, \tau_1 < \tau_2)$ *is a* $(1,2)$-BrS $\Longleftrightarrow$ $(X, \tau_1 < \tau_2)$ *is a* 2-WBrS $\Longleftrightarrow$ $(X, \tau_1 < \tau_2)$ *is a* 1-BrS $\Longleftrightarrow$ $(X, \tau_1 < \tau_2)$ *is a* $(2,1)$-WBrS $\Longleftrightarrow$ $(X, \tau_1 < \tau_2)$ *is a* 1-SBrS.

(2) $(2,1)$-$\mathcal{ND}(X) = 2$-$\mathcal{ND}(X) = 1$-$\mathcal{ND}(X) = (1,2)$-$\mathcal{ND}(X) = (2,1)$-$\mathcal{C}atg_{\mathrm{I}}(X) = 2$-$\mathcal{C}atg_{\mathrm{I}}(X) = 1$-$\mathcal{C}atg_{\mathrm{I}}(X) = (1,2)$-$\mathcal{C}atg_{\mathrm{I}}(X)$.

(3) $A \in (1,2)$-$\mathcal{B}(X) \Longrightarrow \mu(A) = \mu(\tau_1 \operatorname{cl} A) = \mu(\tau_2 \operatorname{int} A) = \mu(\tau_2 \operatorname{cl} A) = \mu(\tau_1 \operatorname{int} A)$ *and* $\mu(\mathbf{n}(A)) = 0$.

(4) $(2,1)$-$\mathcal{B}(X) = 2$-$\mathcal{B}(X) = 1$-$\mathcal{B}(X) = (1,2)$-$\mathcal{B}(X)$.

Proof. (1) Following Theorem 2.7, we have $\tau_1 <_S \tau_2$ and therefore, by (4) of Theorem 4.4.13 in [15], $(X, \tau_1 < \tau_2)$ satisfies all the equivalent Baire-like properties.

(2) Since $\tau_1 <_S \tau_2$, by (1) of Theorem 2.1.2 in [15], $(2,1)\text{-}\mathcal{ND}(X) = 2\text{-}\mathcal{ND}(X) = 1\text{-}\mathcal{ND}(X) = (1,2)\text{-}\mathcal{ND}(X)$ so that $(2,1)\text{-}\mathcal{C}atg_{\mathrm{I}}(X) = 2\text{-}\mathcal{C}atg_{\mathrm{I}}(X) = 1\text{-}\mathcal{C}atg_{\mathrm{I}}(X) = (1,2)\text{-}\mathcal{C}atg_{\mathrm{I}}(X)$ and hence $(1,2)\text{-}\mathcal{B}(X) = 2\text{-}\mathcal{B}(X)$. Moreover, the conditions of Theorem 22.1 in [1] are fulfilled for the topology τ_2 since $(X, \tau_1 < \tau_2)$ is $(2,1)$-regular $\Longrightarrow (X, \tau_1 < \tau_2)$ is 2-regular and thus Theorem 22.2 in [31] gives that $2\text{-}\mathcal{ND}(X) = 2\text{-}\mathcal{C}atg_{\mathrm{I}}(X)$. The rest of the proof is obvious.

(3) The conditions of Theorem 22.3 in [31] for the topology τ_2 are satisfied and hence $\mu(A) = \mu(\tau_2 \operatorname{cl} A) = \mu(\tau_2 \operatorname{int} A)$. On the other hand, since $\tau_1 <_S \tau_2$, by (b) of 4.A.2 in [26] and (2) above we obtain $\mathbf{n}_1(A), \mathbf{n}_2(A) \in 1\text{-}\mathcal{ND}(X) \cap 2\text{-}\mathcal{ND}(X) = 1\text{-}\mathcal{ND}(X) = 2\text{-}\mathcal{ND}(X) = (1,2)\text{-}\mathcal{C}atg_{\mathrm{I}}(X)$ and therefore Definition 2.8 gives that $\mu(\mathbf{n}_1(A)) = \mu(\mathbf{n}_2(A)) = 0$. Thus $\mu(\tau_2 \operatorname{cl} A) = \mu(\tau_2 \operatorname{cl} A \cup \mathbf{n}_1(A)) = \mu(\tau_1 \operatorname{cl} A)$, $\mu(\tau_2 \operatorname{int} A) = \mu(\tau_1 \operatorname{int} A \cup \mathbf{n}_2(A)) = \mu(\tau_1 \operatorname{int} A)$. The equality $\mu(\mathbf{n}(A)) = 0$ is obvious.

(4) From (2) above we immediately obtain $(2,1)\text{-}\mathcal{B}(X) = 1\text{-}\mathcal{B}(X) \subset 2\text{-}\mathcal{B}(X) = (1,2)\text{-}\mathcal{B}(X)$. Thus it remains only to prove that $2\text{-}\mathcal{B}(X) \subset 1\text{-}\mathcal{B}(X)$. Let $A \in 2\text{-}\mathcal{B}(X)$, i.e., $A = U \triangle C$, where $U \in \tau_2$, $C \in 2\text{-}\mathcal{C}atg_{\mathrm{I}}(X) = 1\text{-}\mathcal{C}atg_{\mathrm{I}}(X)$. But $U \in \tau_2 \setminus \{\emptyset\}$ and $\tau_1 <_S \tau_2 \Longrightarrow \tau_1 \operatorname{int} U \neq \emptyset$. Therefore $A = U \triangle C = (\tau_1 \operatorname{int} U \cup (U \setminus \tau_1 \operatorname{int} U)) \triangle C = (\tau_1 \operatorname{int} U \triangle (U \setminus \tau_1 \operatorname{int} U)) \triangle C = \tau_1 \operatorname{int} U \triangle ((U \setminus \tau_1 \operatorname{int} U) \triangle C) = V \triangle D$, $(U \setminus \tau_1 \operatorname{int} U) \triangle C \in 1\text{-}\mathcal{C}atg_{\mathrm{I}}(X)$ since $U \setminus \tau_1 \operatorname{int} U = n_2(U) \in (1,2)\text{-}\mathcal{C}atg_{\mathrm{I}}(X) = 1\text{-}\mathcal{C}atg_{\mathrm{I}}(X)$. □

Remark 2.1. Theorem 2.9 and its Corollary hold for a 1-Tychonoff $(1,2)$-SBrS $(X, \tau_1 <_\Phi \tau_2)$ since, by Corollary 1 of Theorem 7.2.1 in [15], $(X, \tau_1 <_\Phi \tau_2)$ is $(2,1)$-completely regular and therefore $(2,1)$-regular.

Finally, given a topological space (X, τ), let $\mathcal{H}(X, \tau)$ be the class of all homeomorphisms of (X, τ) onto itself. In 1948 Everett and Ulam [17] (see also [40]) posed the following problem: when and how can a new topology γ be constructed on (X, τ) such that $\mathcal{H}(X, \tau) = \mathcal{H}(X, \gamma)$? Among the (partial or complete) answers to Ulam's problem we use the result of Lee [43], according to which on a locally compact space (X, τ) there exists a coarse topology γ such that $\mathcal{H}(X, \tau) = \mathcal{H}(X, \gamma)$. This result is based on the following simple but nevertheless important

Lemma 2.1. *Let (X, τ) be a topological space, and let $P(V)$ be a topological property possessed by certain subsets V of X. If $\gamma = \{V\colon\ P(V)\}$ is a topology on X, then $\mathcal{H}(X, \tau) \subset \mathcal{H}(X, \gamma)$* [43, *Lemma* 1].

Lemma 2.2. *Let τ and γ be two topologies for X such that $U \in \tau \Longleftrightarrow U \cup V \in \gamma$ for all nonempty V in γ. Then $\mathcal{H}(X, \gamma) \subset \mathcal{H}(X, \tau)$* [43, *Lemma* 2].

As we will see below, certain bitopological conditions are imposed on the considered BS, which ensure a satisfactory solution of the problem.

It is worth noting here that the requirement $\tau_1 \setminus \{\emptyset\} = \tau_2 \setminus \{\emptyset\} \cap 2\text{-}\mathcal{C}atg_{\mathrm{II}}(X)$, in general, is stronger than the one that $(X, \tau_1 < \tau_2)$ be a 2-WBrS. Indeed, $\tau_1 \setminus \{\emptyset\} = \tau_2 \setminus \{\emptyset\} \cap 2\text{-}\mathcal{C}atg_{\mathrm{II}}(X) \Longrightarrow (X, \tau_1 < \tau_2)$ is a 2-WBrS, but by Example 1.4, the BS's $(\mathbb{R}, \omega <_S s)$

and $(\mathbb{R}, \omega <_S \tau)$ are 2-WBrS's for which $\omega\backslash\{\emptyset\} \subset s\backslash\{\emptyset\} = (s\backslash\{\emptyset\}) \cap 2\text{-}\mathcal{C}atg_{\mathrm{II}}(\mathbb{R})$ and $\omega\backslash\{\emptyset\} \subset \tau\backslash\{\emptyset\} = (\tau\backslash\{\emptyset\}) \cap 2\text{-}\mathcal{C}atg_{\mathrm{II}}(\mathbb{R})$.

Theorem 2.10. *If $(X, \tau_1 < \tau_2)$ is a* BS *such that $\tau_1\backslash\{\emptyset\} = (\tau_2\backslash\{\emptyset\}) \cap 2\text{-}\mathcal{C}atg_{\mathrm{II}}(X)$ and for each point $x \in X$ there is a neighborhood $U(x) \in \tau_2\backslash 1\text{-}\mathcal{D}(X)$, then $\mathcal{H}(X, \tau_1) = \mathcal{H}(X, \tau_2)$ and $(X, \tau_1 < \tau_2)$ is a* 2-WBrS.

Proof. Let $P(V)$ mean that $V \in \tau_2$ and $V \in 2\text{-}\mathcal{C}atg_{\mathrm{II}}(X)$ ($\Longleftrightarrow V$ is of $2\text{-}\mathcal{C}atg\,\mathrm{II}$). Then the family $\{V\colon\ V = \emptyset \text{ or } P(V)\}$ is exactly the topology τ_1. Hence, by Lemma 2.1, $\mathcal{H}(X, \tau_2) \subset \mathcal{H}(X, \tau_1)$. If $(X, \tau_1 < \tau_2)$ is a 2-BrS, then $\tau_1 = \tau_2$. Therefore we may assume that $(X, \tau_1 < \tau_2)$ is not a 2-BrS.

Now let $U \in \tau_2$ and $V \in \tau_1\backslash\{\emptyset\}$. Then $U \cup V \in \tau_2$ and, by (4) of Theorem 1.1.3 in [15], $U \cup V \in 2\text{-}\mathcal{C}atg_{\mathrm{II}}(X) \Longleftrightarrow U \cup V$ is of $2\text{-}\mathcal{C}atg\,\mathrm{II}$. Therefore $U \cup V \in \tau_1$.

Furthermore, let $U \mathrel{\overline{\in}} \tau_2$ and $x \in U\backslash \tau_2 \operatorname{int} U$. By condition, there is a 2-open neighborhood $V(x)$ such that $\tau_1 \operatorname{cl} V(x) \neq X$. It is obvious that $x \in \tau_2 \operatorname{cl}(\tau_1 \operatorname{cl} V(x)\backslash U)$ since the contrary means that there is a neighborhood $W(x) \in \tau_2$ such that $W(x) \cap (\tau_1 \operatorname{cl} V(x)\backslash U) = \emptyset$, i.e., $W(x) \cap V(x) \subset U$ so that $x \in \tau_2 \operatorname{int} U$. Let $E = X\backslash(\tau_1 \operatorname{cl} V(x)\backslash U) = (X\backslash\tau_1 \operatorname{cl} V(x)) \cup U$. Then $x \in E$ but $E \mathrel{\overline{\in}} \tau_2$ since $x \in \tau_2 \operatorname{cl}(\tau_1 \operatorname{cl} V(x)\backslash U)$. Therefore for $U \mathrel{\overline{\in}} \tau_2$ there is $V = X\backslash\tau_1 \operatorname{cl} V(x) \in \tau_1$ such that $U \cup V \mathrel{\overline{\in}} \tau_2$ and hence $U \cup V \mathrel{\overline{\in}} \tau_1$ since $\tau_1 \subset \tau_2$.

Thus by Lemma 2.2 we obtain $\mathcal{H}(X, \tau_1) \subset \mathcal{H}(X, \tau_2)$ and therefore $\mathcal{H}(X, \tau_1) = \mathcal{H}(X, \tau_2)$.

The rest follows from Definition 1.10 because $\tau_1\backslash\{\emptyset\} \subset 2\text{-}\mathcal{C}atg_{\mathrm{II}}(X)$. □

Corollary. *If $(X, \tau_1 < \tau_2)$ is a* 2-WBrS *such that $((\tau_2\backslash\tau_1) \cup \{X\}) \cap 2\text{-}\mathcal{C}atg_{\mathrm{II}}(X) = \{X\}$ and for each point $x \in X$ there is a neighborhood $U(x) \in \tau_2\backslash 1\text{-}\mathcal{D}(X)$, then $\mathcal{H}(X, \tau_1) = \mathcal{H}(X, \tau_2)$.*

Proof. Follows directly from Definition 1.10. □

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