LETTER TO THE EDITOR

Interpolatory Wavelet Packets

Sherman D. Riemenschneider and Zuowei Shen

Communicated by Charles K. Chui
Received July 29, 1999; revised September 10, 1999

Abstract—In this note, we present a construction of interpolatory wavelet packets. Interpolatory wavelet packets provide a finer decomposition of the 2^j-th dilate cardinal interpolation space and hence give a better localization for an adaptive interpolation. This can lead to a more efficient compression scheme which, in turn, provides an interpolation algorithm with a smaller set of data for use in applications.

Key Words: interpolation; splines; wavelets.

1. INTRODUCTION

We begin with a refinable continuous compactly supported or exponentially decaying function \( \varphi \) that satisfies

\[
\varphi(\alpha) = \delta(\alpha), \quad \alpha \in \mathbb{Z},
\]

(1.1)

The function \( \varphi \) satisfying (1.1) is called the fundamental function.

Since \( \varphi \) is fundamental, the values of the refinement mask \( m \) of \( \varphi \) are determined on the even integers:

\[
m(2\beta) = \varphi(\beta) = \delta(\beta), \quad \beta \in \mathbb{Z}.
\]

Compactly supported refinable fundamental functions with high order of smoothness were given in [3]. Exponentially decaying spline fundamental functions can be obtained from B-splines as follows: Let \( B_{2n} \) be the univariate B-spline of order \( 2n \) with integer knots. The function \( \varphi \) defined via its Fourier transform by

\[
\hat{\varphi} := \hat{B}_{2n} / \sum_{\beta} \hat{B}_{2n}(\cdot + 2\pi\beta)
\]

is a refinable continuous fundamental function with exponential decay.

\[1\] Research supported in part by NSERC Canada under Grant A7687 and by the National University of Singapore under Grant RP3981647.

\[2\] Department of Mathematics, West Virginia University, Morgantown, WV 26506. E-mail: sherm@math.wvu.edu.

\[3\] Department of Mathematics, National University of Singapore, Singapore 119260. E-mail: matzuows@leonis.nus.edu.sg.
Define a function $p_0 = \varphi$ and functions $p_n$ inductively as follows:

\[
p_{2n} := \sum_{\alpha \in \mathbb{Z}} m(\alpha) p_n(2 \cdot -\alpha) \\
p_{2n+1} := p_n(2 \cdot -1).
\]  

(1.2)

We denote $\mathcal{P}_n$ to be the closed shift invariant subspace in $C(\mathbb{R})$ generated by $p_n$; that is,

\[
\mathcal{P}_n := \left\{ f : f = \sum_{\alpha \in \mathbb{Z}} a(\alpha) p_n \cdot -\alpha, \ a \in \ell_0(\mathbb{Z}) \right\},
\]

where $\ell_0(\mathbb{Z})$ is the space of finitely supported sequences. The space $\mathcal{P}_0$ is a cardinal interpolation space, that is, the space spanned by translates of a fundamental solution for cardinal interpolation. The collection

\[
\{ p_n(2^k \cdot -\alpha) : n \in \mathbb{Z}_+, \ k \in \mathbb{Z}, \ \alpha \in \mathbb{Z} \}
\]

is the wavelet packets from which we will draw bases for our interpolation spaces.

The dilation operator $\sigma : \mathcal{P}_n \mapsto \sigma \mathcal{P}_n$ is defined by $\sigma f := \sqrt{2} f(2 \cdot )$. The space $\sigma^j \mathcal{P}_0$ is the $2^j$ dilate of the cardinal interpolation space. It is generated by $\varphi(2^j \cdot -\alpha), \ \alpha \in \mathbb{Z}$, which are the dyadic shifts of a fundamental solution for interpolation on the lattice $\mathbb{Z}/2^j$. Our goal is to decompose the spaces $\sigma \mathcal{P}_n$ and to find $\ell_{\infty}$-stable bases for the decomposition. For a given space $\mathcal{P}_n$, it is easy to show that (see [6, 7]) $\sigma \mathcal{P}_n = \mathcal{P}_{2n} \oplus \mathcal{P}_{2n+1}$, and $\{ p_{2n} \cdot -\alpha, \ p_{2n+1} \cdot -\alpha : \alpha \in \mathbb{Z} \}$ forms an $\ell_{\infty}$-stable basis for $\sigma \mathcal{P}_n$. Further, for each $j \in \mathbb{Z}$, the functions $\{ p_n(2^j \cdot -\alpha) : \alpha \in \mathbb{Z}, \ 2^{k-1} \leq n \leq 2^k - 1 \}$ form an $\ell_{\infty}$-stable basis for the space $\sigma^{k-1+j} \mathcal{P}_j$.

A collection $\mathcal{J}$ of pairs $(\ell, k), \ \ell \in \mathbb{Z}$ and $0 < k \in \mathbb{Z}$, disjointly covers the integer interval $[j_1..j_2]$ if $\bigcup \{ k + \ell : (\ell, k) \in \mathcal{J} \} = [j_1..j_2] \cap \mathbb{Z}$ and the representation $j = k + \ell$ is unique from $\mathcal{J}$. We have the following proposition:

**PROPOSITION 1.3.** For given $j_1 < j_2$ and any disjoint cover $\mathcal{J}$ of $[j_1..j_2] \cap \mathbb{Z}$, the functions

\[
\{ p_0(2^{j_1-1} \cdot -\alpha), \ p_n(2^k \cdot -\alpha) : 2^{k-1} \leq n \leq 2^k - 1, \ (\ell, k) \in \mathcal{J}, \ \alpha \in \mathbb{Z} \}
\]

form an $\ell_{\infty}$-basis for the space $\sigma^{k-1} \mathcal{P}_0$. 

Univariate orthogonal wavelet packets were introduced in [2] and their multivariate counterpart can be found in [6]. Recently, interpolatory wavelets, wavelet packets, and their applications were discussed in [1, 5, 7]. The interpolatory wavelet packets provided here, together with their decomposition and reconstruction algorithm, give wide choices of the decomposition of the lattice which, in turn, provides a possible way to interpolate scattered data. Applications are discussed in [4, 5].
2. DECOMPOSITION OF INTERPOLATION OPERATORS

The cardinal interpolation operator at the $2^j$th dyadic level can be written as

$$L_{2^j} f := \sum_{\alpha \in \mathbb{Z}} f(\alpha/2^j)\varphi(2^j : -\alpha).$$

This provides an approximation for the function $f$ which interpolates $f$ at $\mathbb{Z}/2^j$. Further, $\lim_j L_j f = f$ for any compactly supported continuous function $f$. To obtain higher accuracy, more interpolation points are needed. The interpolation operator $L_{2^j}$ gives each point the same priority regardless of the shape of the function. It is numerically more desirable to interpolate the function in a coarser grid where the function $f$ is flat and in a finer grid where it has large variation. However, to do this we need to know the shape of the function from a given set of data, because in most of the applications the function $f$ itself is not available. Decomposing the interpolation operator at the $2^j$ dyadic level into lower dyadic levels is one way to analyze the data structure and hence the shape of the function from the given set of data. Then, a quantization scheme designed according to the given practical problem is applied and finally the reconstruction algorithm leads to an interpolation of $f$ which approximates $f$ with the same order of the accuracy but using a smaller set of data. Similarly, this process also can be applied to decompose, compress, and reconstruct the sampled data at $2^j$th dyadic levels. Further, for the interpolatory wavelet packets, a standard subdivision scheme can provide a predication of $2^{j+1}$th dyadic level sample data from the sampled data at the $2^j$th dyadic level.

The cardinal interpolation operator at the $2^j$th dyadic level can be decomposed as a sum of elements from $\sigma^{j-1}P_0$ and $\sigma^{j-1}P_1$ as was given by Donoho [5]. Clearly, this decomposition can be iterated down to the $j_{\text{st}}$ level.

The wavelet packets given in this section provide a finer decomposition of $\sigma^j P_0$. These bases have descriptions as interpolation operators. This gives a finer decomposition of the interpolation operator and a more detailed analysis of the structure of the given set of data. This leads to a more effective compression and hence a better adapted interpolation.

The crucial proposition that follows uses a correspondence

$$r := \sum_{\ell=0}^{k-1} \eta_\ell 2^\ell \leftrightarrow r^* := \frac{1}{2^{k+1}} + \frac{1}{2} \sum_{\ell=0}^{k-1} \eta_\ell 2^{-\ell}, \quad \forall \eta_\ell \in \{0, 1\}$$

(2.1)

between the integers $r$ in $[0..2^k - 1]$ and points $r^*$ in the lattice $(2^{-k-1}\mathbb{Z} \setminus 2^{-k}\mathbb{Z})$ in $[0..1]$:

**Proposition 2.2.** For any $k \geq 0$, $\beta \in \mathbb{Z}$, $0 \leq s \leq 2^k - 1$, and $s$ and $s^*$ related by (2.1), we have

$$p_{2^k r}(\beta/2^k) = 0,$$

$$p_{2^k r}(\beta + \frac{1}{2^{k+1}} + \frac{1}{2}s^*) = \begin{cases} 0, & \text{if } s < r \text{ and } \beta \text{ arbitrary;} \\ \delta(\beta), & \text{if } s = r \text{ and } \beta \in \mathbb{Z}. \end{cases}$$
The proof is by induction on $k$. For $k = 0$, the only $r$ is $r = 0$. In that case, since $p_0 = \varphi$ is a fundamental function for cardinal interpolation from (1.2), we have

$$p_1(\beta) = p_0(2\beta - 1) = 0, \quad p_1\left(\beta + \frac{1}{2}\right) = p_0(2\beta + 1 - 1) = \delta(2\beta).$$

Now assume that the lemma holds for integers $< k$. Consider $2n = 2^k + 2r$. If $s = \sum_{\ell=0}^{k-1} \eta_\ell 2^\ell \leq 2r$, then for strict inequality we must have $\sum_{\ell=0}^{k-2} \eta_{\ell+1} 2^\ell < r$ while equality implies equality in the last inequality as well since $\eta_0 = 0$ in that case. Hence, by (1.2)

$$p_{2^k + 2r}(\beta/2^k) = \sum_{a \in \mathbb{Z}} m(a) p_{2^k - 1 + r}(\beta/2^{k-1} - a) = 0,$$

$$p_{2^k + 2r}\left(\beta + \frac{1}{2^{k+1}} + \frac{1}{2^{s+1}}\right) = \sum_{a \in \mathbb{Z}} m(a) p_{2^k - 1 + r}\left(2\beta + \frac{1}{2^k} + \eta_0 - \alpha + \frac{1}{2} \sum_{\ell=0}^{k-2} \eta_{\ell+1} 2^{-\ell}\right)$$

$$= \begin{cases} 0, & \text{if } s < 2r \text{ and } \beta \in \mathbb{Z}. \\ m(2\beta) = \delta(\beta), & \text{if } s = 2r. \end{cases}$$

Similarly, for $2n + 1 = 2^k + 2r + 1$, $s = \sum_{\ell=0}^{k-1} \eta_\ell 2^\ell \leq 2r + 1$, then we must have $\sum_{\ell=0}^{k-2} \eta_{\ell+1} 2^\ell < r$ while equality implies equality in the last inequality and $\eta_0 = 1$. Hence,

$$p_{2^k + 2r + 1}\left(\beta + \frac{1}{2^{k+1}} + \frac{1}{2^{s+1}}\right) = p_{2^k - 1 + r}\left(2\beta + \eta_0 - 1 + \frac{1}{2^k} + \frac{1}{2} \sum_{\ell=0}^{k-2} \eta_{\ell+1} 2^{-\ell}\right)$$

$$= \begin{cases} 0, & \text{if } s < 2r + 1 \text{ and } \beta \text{ arbitrary}; \\ \delta(\beta), & \text{if } s = 2r + 1. \end{cases}$$

We define interpolation operator $\mathcal{L}_{(2^k, r)}$ by

$$\mathcal{L}_{(2^k, r)} f := \sum_{\beta \in \mathbb{Z}} f\left(\beta + \frac{1}{2^{k+1}} + \frac{1}{2^{s+1}}\right)p_{2^k + r}(\cdot - \beta).$$

Then for $s, r \in \{0, 1, \ldots, 2^k - 1\}$, $\mathcal{L}_{(2^k, r)} f(\beta/2k) = 0$, $\forall \beta \in \mathbb{Z}$, and

$$\mathcal{L}_{(2^k, r)} f\left(\beta + \frac{1}{2^{k+1}} + \frac{1}{2^{s+1}}\right) = \begin{cases} 0, & \text{if } s < r \text{ and } \beta \in \mathbb{Z}; \\ f(\beta + \frac{1}{2^{s+1}} + \frac{\alpha}{2}), & \text{if } s = r \text{ and } \beta \in \mathbb{Z}. \end{cases}$$

The interpolation operator $\mathcal{L}_{2k}$ at the $2^k$th dyadic level can be decomposed between levels $k$ and $k - 1$ using the operators $\mathcal{L}_{(2^k, r)} f$. Here we give such a decomposition by the bases defined in Proposition 1.3 with $j_1 = j_2 = k$. Define

$$g_0 := (f - \mathcal{L}_{2^{k-1}} f) \quad \text{and} \quad g_r := g_{r-1} - \mathcal{L}_{(2^{k-1}, r)} g_{r-1}, \quad r = 1, \ldots, 2^{k-1} - 1.$$

Then from the uniqueness of the interpolation, it follows that

$$\mathcal{L}_{2^k} f = \mathcal{L}_{2^{k-1}} f + \sum_{r=0}^{2^{k-1}-1} \mathcal{L}_{(2^{k-1}, r)} g_r,$$

since both sides are in $\sigma^k P_0$ and interpolate $f$ at the points $\{\beta/2^k\}_{\beta \in \mathbb{Z}}$. 
REFERENCES