The pebbling number of squares of even cycles

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A pebbling move on a graph \( G \) consists of taking two pebbles off one vertex and placing one pebble on an adjacent vertex. The pebbling number of a connected graph \( G \), denoted by \( f(G) \), is the least \( n \) such that any distribution of \( n \) pebbles on \( G \) allows one pebble to be moved to any specified vertex by a sequence of pebbling moves. In this paper, we determine the pebbling numbers of squares of even cycles.

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1. Introduction

Pebbling of graphs was first introduced by Chung [1]. Consider a connected graph with a fixed number of pebbles distributed on its vertices. A pebbling move consists of the removal of two pebbles from a vertex and the placement of one of those pebbles on an adjacent vertex. The pebbling number of a vertex \( v \) in a graph \( G \) is the smallest number \( f(G, v) \) with the property that from every placement of \( f(G, v) \) pebbles on \( G \), it is possible to move a pebble to \( v \) by a sequence of pebbling moves. The pebbling number of a graph \( G \), denoted by \( f(G) \), is the maximum of \( f(G, v) \) over all the vertices of \( G \).

There are many known results about pebbling number (see [1,5,4,6,7,2,3]). If each vertex (except \( v \)) has at least one pebble, then no pebble can be moved to \( v \). Also, if \( v \) is of distance \( d \) from \( u \) and at most \( 2^d - 1 \) pebbles are placed on \( u \) and none elsewhere, then no pebble can be moved from \( u \) to \( v \). So it is clear that \( f(G) = \max[|V(G)|, 2^d] \), where \(|V(G)|\) is the number of vertices of \( G \), and \( D \) is the diameter of \( G \). Furthermore, \( f(K_n) = n \) and \( f(P_n) = 2^{n-1} \) (see [1]), where \( K_n \) denotes a complete graph with \( n \) vertices and \( P_n \) denotes a path with \( n \) vertices.

Throughout this paper, \( G \) denotes a simple connected graph with vertex set \( V(G) \) and edge set \( E(G) \). Let \( p \) be a distribution of pebbles on \( G \). Define \( p(H) \) to be the number of pebbles on a subgraph \( H \) of \( G \) and \( p(v) \) to be the number of pebbles on a vertex \( v \) of \( G \). Moreover, denote by \( \overrightarrow{p}(H) \) and \( \overrightarrow{p}(v) \) the number of pebbles on \( H \) and the number of pebbles on \( v \) after a specified sequence of pebbling moves, respectively. For \( uv \in E(G) \), \( u \rightarrow v \) refers to taking \( 2m \) pebbles off \( u \) and placing \( m \) pebbles on \( v \). Denote by \( \{v_0, v_1, \ldots, v_{n-1}\} \) (respectively, \( \{v_0, v_1, \ldots, v_{n-1}\} \) ) the path (respectively, cycle) with vertices \( v_0, v_1, \ldots, v_{n-1} \) in order.

Let \( G \) be a connected graph. For \( u, v \in V(G) \), we denote by \( d_G(u, v) \) the distance between \( u \) and \( v \) in \( G \). The \( k \)-th power of \( G \), denoted by \( G^k \), is the graph obtained from \( G \) by adding the edge \( uv \) to \( G \) whenever \( 2 \leq d_G(u, v) \leq k \) in \( G \). That is, \( E(G^k) = \{uv : 1 \leq d_G(u, v) \leq k\} \). Obviously, \( G^k \) is the complete graph whenever \( k \) is at least the diameter of \( G \). We now introduce a lemma which will be used in the subsequent proofs.

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Lemma 1 ([6]). \( f(p^2_{2k}) = 2^k, f(p^2_{2k+1}) = 2^k + 1. \)

In [6], Pachter et al. gave the pebbling numbers of squares of paths (see Lemma 1). Naturally, we want to know the pebbling number of \( C_n^2 \). In [8], the pebbling numbers of squares of odd cycles were obtained:

(i) for \( 2 < n < 6 \), \( f(C_{2n+1}^2) = 2n + 1; \)
(ii) for \( k \geq 3 \), \( f(C_{4k+3}^2) = 2^{k+1} + 1; \)
(iii) for \( k \geq 4 \), \( f(C_{2k+1}^2) = \left\lfloor \frac{2^{k+2}}{3} \right\rfloor + 1. \)

Motivated by this, we obtain the pebbling numbers of squares of even cycles in this paper.

2. Pebbling \( C_{2n}^2 \)

This section studies the pebbling number of \( C_{2n}^2 \). Let \( C_{2n} = \{v, a_1, \ldots, a_{n-1}, y, b_{n-1}, \ldots, b_1\} \). By symmetry, we may assume that \( v \) is the target vertex in \( C_{2n}^2 \) and \( p(v) = 0 \). First, we give the pebbling number of \( C_{2n}^2 \) for \( n \leq 6 \). See Theorems 2 and 3.

Theorem 2. For \( 2 \leq n \leq 5 \), \( f(C_{2n}^2) = 2n. \)

Proof. Let \( Q_a = \langle v, a_1, \ldots, a_{n-1} \rangle \) and \( Q_b = \langle v, b_1, \ldots, b_{n-1} \rangle \). For \( 2 \leq n \leq 5 \), we have \( f(Q_a^2) = f(Q_b^2) = n \) by Lemma 1. Clearly, \( f(C_{2n}^2) \geq 2n \). Now distribute 2n pebbles on \( C_{2n}^2 \). Without loss of generality, we assume that \( p(Q_a^2) \geq p(Q_b^2) \). Thus \( p(Q_a^2) \geq \left\lceil \frac{2n - p(Q_b^2)}{2} \right\rceil = n - \left\lceil \frac{p(Q_b^2)}{2} \right\rceil \). Since we can move \( \left\lceil \frac{p(Q_b^2)}{2} \right\rceil \) pebbles from \( y \) to \( a_{n-1} \), \( p(Q_a^2) \geq p(Q_b^2) + \left\lceil \frac{p(Q_b^2)}{2} \right\rceil \geq n \). The proof is completed. \( \square \)

Theorem 3. \( f(C_{12}^2) = 12 \).

Proof. Let \( Q_a = \langle v, a_1, a_2, a_3, a_4 \rangle \) and \( Q_b = \langle v, b_1, b_2, b_3, b_4 \rangle \). Moreover, let \( Q_a^+ = \langle v, a_1, a_2, a_3, a_4, a_5 \rangle \) and \( Q_b^+ = \langle v, b_1, b_2, b_3, b_4, b_5 \rangle \). By Lemma 1, we have \( f(Q_a^2) = f(Q_b^2) = 5 \) and \( f((Q_a^+)^2) = f((Q_b^+)^2) = 8 \). Clearly, \( f(C_{12}^2) \geq 12 \). For convenience, \( a_5 \) and \( b_5 \) are denoted by \( x \) and \( z \), respectively. Suppose that there are 12 pebbles distributed on the vertices of \( C_{12}^2 \), i.e.,

\[
p(Q_a^2) + p(Q_b^2) + p(x) + p(y) + p(z) = 12. \tag{1}
\]

We first consider the case \( p(x) + p(z) \geq 10 \). It suffices to show that, after some pebbling moves, \( \tilde{p}((Q_a^+)^2) \geq 8 \) or \( \tilde{p}((Q_b^+)^2) \geq 8 \). Without loss of generality, we may assume that \( p(x) \geq p(z) \). If \( p(x) > \max\{p(x), 5\} \), then \( p(x) + \left\lceil \frac{p(x)}{2} \right\rceil \geq 8 \), and this implies \( \tilde{p}((Q_a^+)^2) \geq 8 \). Now suppose that \( p(x) = p(z) = 5 \). If \( p(Q_a^2) = p(Q_b^2) = 0 \), then \( p(y) = 2 \). Moving one pebble from \( y \) to \( x \), we have \( p(x) > 5 \), and the previous case applies. Otherwise, suppose without loss of generality that \( Q^2_a \) has at least one pebble; now 2 pebbles can be moved from \( z \) to \( x \) to obtain \( \tilde{p}((Q_a^+)^2) \geq 8 \).

Next, we consider the case \( p(x) + p(z) < 10 \). Obviously, if \( \tilde{p}(Q_a^2) \geq 5 \) or \( \tilde{p}(Q_b^2) \geq 5 \), then we are done. If \( \tilde{p}(Q_a^2) < 5 \) and \( \tilde{p}(Q_b^2) < 5 \), then

\[
p(Q_a^2) + \left\lceil \frac{p(y)}{2} \right\rceil + \left\lceil \frac{p(x) + \frac{p(x)}{2}}{2} \right\rceil \leq 4 \tag{2}
\]

and

\[
p(Q_b^2) + \left\lceil \frac{p(y)}{2} \right\rceil + \left\lceil \frac{p(z) + \frac{p(z)}{2}}{2} \right\rceil \leq 4. \tag{3}
\]

(2) and (3) result from moving as many pebbles as possible from \( x, y, z \) to \( a_4 \) and \( b_4 \), respectively. We see that \( z \) can contribute pebbles not only to \( x \), but also to \( y \). If \( p(z) \geq 2 \), then we can move pebbles from \( z \) to \( x \) and \( y \) to make at least one of \( \tilde{p}(x) \) and \( \tilde{p}(y) \) be even. So (2) can be rewritten as \( p(Q_a^2) + \left\lceil \frac{p(x) + p(y) + \frac{p(x)}{2}}{2} \right\rceil \leq 4 \) for \( p(z) \geq 2 \).

For the case \( \min\{p(x), p(z)\} \geq 2 \), we have

\[
p(Q_a^2) + \left\lceil \frac{p(x) + p(y) + \frac{p(x)}{2}}{2} \right\rceil \leq 4 \quad \text{and} \quad p(Q_b^2) + \left\lceil \frac{p(z) + p(y) + \frac{p(z)}{2}}{2} \right\rceil \leq 4. \tag{4}
\]
Hence
\[ p(Q_4^2) + p(Q_5^2) + \left\lfloor \frac{p(x) + p(y) + \left\lfloor \frac{p(z)}{2} \right\rfloor}{2} \right\rfloor + \left\lfloor \frac{p(z) + p(y) + \left\lfloor \frac{p(x)}{2} \right\rfloor}{2} \right\rfloor \leq 8. \] (5)

Note that \( \left\lfloor \frac{k}{2} \right\rfloor \geq \frac{k-1}{2} \) for any integer \( k \). Using (5), we have
\[ p(Q_4^2) + p(Q_5^2) + p(y) + \frac{3}{4}(p(x) + p(z)) - \frac{3}{2} \leq 8. \] (6)

By (1) and (6), \( p(x) + p(z) \geq 10 \), and this is a contradiction.
For the case \( \max\{p(x), p(z)\} < 2 \), we have \( p(x) + p(z) < 2 \). According to (2) and (3),
\[ p(Q_4^2) + \left\lfloor \frac{p(y)}{2} \right\rfloor \leq 4 \quad \text{and} \quad p(Q_5^2) + \left\lfloor \frac{p(y)}{2} \right\rfloor \leq 4. \] (7)

Hence
\[ p(Q_4^2) + p(Q_5^2) + \left\lfloor \frac{p(y)}{2} \right\rfloor + \left\lfloor \frac{p(y)}{2} \right\rfloor \leq 8. \] (8)

Thus we have
\[ p(Q_4^2) + p(Q_5^2) + p(y) - 1 \leq 8. \] (9)

By (1) and (9), \( p(x) + p(z) \geq 3 \), and this is a contradiction.

The remaining case is \( p(z) < 2 \) and \( p(x) \geq 2 \) (or, similarly, \( p(x) < 2 \) and \( p(z) \geq 2 \) ). First suppose that \( p(z) = 0 \). Note that \( p(Q_4^2) \leq 4 \) and \( p(Q_5^2) \leq 4 \). Thus we have \( p(x) + p(y) \geq 4 \), and hence at least one pebble can be moved from \( x \) and \( y \) to \( b_4 \). If \( p(Q_4^2) = 4 \), then \( p(Q_5^2) = 5 \), and we are done. If \( p(Q_4^2) < 3 \), then \( p(Q_5^2) + p(x) + p(y) \geq 9 \), and we are done by Lemma 1.

Now suppose that \( p(z) = 1 \). If \( p(Q_4^2) = 4 \), then \( x \rightarrow z \rightarrow b_4 \) produces \( \tilde{p}(Q_5^2) = 5 \), and we are done. If \( p(Q_4^2) = 3 \), then \( p(Q_5^2) + p(x) + p(y) = 8 \). When \( p(Q_5^2) \geq 3 \), we can move pebbles from \( \{x, y\} \) to \( b_4 \) so that \( \tilde{p}(Q_5^2) \geq 5 \). When \( p(Q_5^2) < 2 \), we can move pebbles from \( \{x, y, z\} \) to \( b_4 \) so that \( \tilde{p}(Q_5^2) \geq 5 \). If \( p(Q_4^2) < 2 \), then \( p(Q_4^2) + p(x) + p(y) \geq 9 \), and we are done. \( \square \)

Next, we determine the pebbling number of \( G_2^2 \) for \( n \geq 7 \). We first prove a lemma about \( G_4^2 \), where \( k \geq 3 \). Let \( Q_k = \langle v, a_1, \ldots, a_{2k} \rangle \) and \( Q_k^\perp = \langle v, b_1, \ldots, b_{2k-2} \rangle \) be two subpaths of \( C_{4k+2} \). Moreover, let \( Q_k = \langle v, a_1, \ldots, a_{2k-2} \rangle \) and \( Q_k^\perp = \langle v, b_1, \ldots, b_{2k-2} \rangle \). By Lemma 1, \( f(Q_k) = f(Q_k^\perp) = 2^k + 1 \) and \( f((Q_k)^2) = f((Q_k^\perp)^2) = 2^{k-1} + 1 \).

**Lemma 4.** Let \( p \) be a distribution on \( G_4^2 \), where \( k \geq 3 \). If \( p(y) \geq 2^{k+1} - 2p(Q_4^2) \geq 2, \) then one pebble can be moved to \( v \).

**Proof.** If \( t = p(a_{2k}) \), \( r = p(a_{2k-1}) \) and \( s = p((Q_k^\perp)^2) \), then \( p(y) \geq 2^{k+1} - 2(r + s + t) \).

Case 1: If \( s = 0 \), then \( p(y) \geq 2^{k+1} - 2(r + t) \). First, suppose that \( r \) is even. We have
\[
\begin{align*}
& a_{2k-1} \xrightarrow{r} a_{2k-2} \\
& y \xrightarrow{2^{k-1} - r} a_{2k} \xrightarrow{2^{k-1} - \frac{r}{2}} a_{2k-2} \implies \tilde{p}(a_{2k-2}) = 2^{k-1}.
\end{align*}
\]

Second, suppose that \( r \) is odd. Since \( p(y) \geq 2 \), we have
\[
\begin{align*}
& y \xrightarrow{1} a_{2k-1} \xrightarrow{r+1} a_{2k-2} \\
& y \xrightarrow{2^{k-1} - r - 1} a_{2k} \xrightarrow{2^{k-1} - \frac{r+1}{2}} a_{2k-2} \implies \tilde{p}(a_{2k-2}) = 2^{k-1}.
\end{align*}
\]

Thus one pebble can be moved to \( v \), since \( d(a_{2k-2}, v) = k - 1 \).

Case 2: If \( s = 1 \), then \( p(y) \geq 2^{k+1} - 2(r + 1 + t) \), and there exists a vertex \( a_j \) of \( (Q_k^\perp)^2 \) with \( p(a_j) = 1 \). We first assume that \( j \) is even, and write \( d = d(a_{2k-2}, a_j) \), so \( d(a_j, v) = k - d - 1 \). We now move pebbles as follows:
\[
\begin{align*}
& a_{2k-1} \xrightarrow{1} a_{2k-2} \\
& y \xrightarrow{2^{k-1} - r - 1} a_{2k} \xrightarrow{2^{k-1} - \frac{r+1}{2}} a_{2k-2} \implies a_j \implies \tilde{p}(a_j) = 2^{k-d-1}.
\end{align*}
\]

Thus one pebble can be moved to \( v \).
Now assume that $j$ is odd, and let $d = d(a_{2k-1}, a_j)$, so $d(a_{j-1}, v) = k - d - 1$. If $r < 2^d$, then
\[
y \xrightarrow{2^d-r} a_{2k-1} \xrightarrow{2^d-1} a_{2k-3} \ldots \xrightarrow{1} a_j \xrightarrow{1} a_{j-1}
\]
and we can move one pebble to $v$.

Case 3: If $s \geq 2$, then for even $r$,
\[
\begin{align*}
a_{2k-1} &\xrightarrow{r} a_{2k-2} \\
y &\xrightarrow{2^r-s-1} a_{2k} \xrightarrow{2^r-1-\left\lfloor \frac{s}{2} \right\rfloor} a_{2k-2}
\end{align*}
\]
and we are done.

Theorem 5. For $k \geq 3$, $f(C_{4k+2}) = 2^{k+1}$.

Proof. Since the diameter of $C_{4k+2}$ is $k+1$, we have $f(C_{4k+2}) \geq 2^{k+1}$. Now we place $2^{k+1}$ pebbles on the vertices of $C_{4k+2}$. Without loss of generality, we may assume that $p(Q_{2}) \geq p(Q_{2})$. Next we consider the following cases.

Case 1: $p(y) = 0$ or 1.

In this case, $p(Q_{2}) \geq 2^k$. Obviously, if $p(Q_{2}) \geq 2^k + 1$, then we are done. Now we consider the case $p(Q_{2}) = 2^k$. This implies that $p(y) + p(Q_{2})^y \geq 2^k$. By Lemma 1, if $p(Q_{2})^y \geq 2^k-1+1$, then we are done. Next, if $p(Q_{2})^y \geq 2^k-1$, then $p(b_{2k-1}) + p(b_{2k}) + p(y) = 2^k - p(Q_{2})^y \geq 2^k-1 + 4(k \geq 3)$. Note that $f(P_{2}) = 4$, so one pebble can be moved from $b_{2k-1}, b_{2k}, y$ to $a_{2k}$ to produce $p(Q_{2}) = 2^k + 1$, and we are done.

Case 2: $p(y) = 2^{k+1} - q(q \leq 2^{k+1} - 2)$.

Obviously, $p(y) \geq 2$ and $p(Q_{2}) \geq \left\lceil \frac{q}{2} \right\rceil$. If $p(Q_{2}) \geq \left\lceil \frac{q}{2} \right\rceil + 1$, then move $2^k - \left\lceil \frac{q}{2} \right\rceil$ pebbles from $y$ to $a_{2k}$ so that $p(Q_{2}) \geq 2^{k+1}$, and we are done. If $p(Q_{2}) = \left\lceil \frac{q}{2} \right\rceil$, then $p(y) \geq 2^{k+1} - 2p(Q_{2})$. By Lemma 4, we are done.

Now we consider $C_{4k}$, where $k \geq 4$. Let $Q_{4} = (v, a_{1}, \ldots, a_{2k-2})$ and $Q_{B} = (v, b_{1}, \ldots, b_{2k-2})$. Moreover, let $Q_{4} = (v, a_{1}, \ldots, a_{2k-4})$ and $Q_{B} = (v, b_{1}, \ldots, b_{2k-4})$. By Lemma 1, $f(Q_{4}) = f(Q_{B}) = 2^{k-1} + 1$ and $f(Q_{4}) = f(Q_{B}) = 2^{k-2} + 1$. For convenience, $a_{2k-1}$ and $b_{2k-1}$ are denoted by $x$ and $z$, respectively. In order to determine the pebbling number of $C_{4k}$, we first give the following lemmas.

Lemma 6. Let $k \geq 4$ and $p$ be a distribution on $C_{4k}$. Let $m = p(Q_{2})$. If $p(x) + \left\lceil \frac{p(x)}{2} \right\rceil + \left\lceil \frac{p(y)}{2} \right\rceil \geq 2^k - 2m$, where $m = 0, 1, 2, 3$, then one pebble can be moved to $v$.

Proof. Note that $\left\lceil \frac{p(x)}{2} \right\rceil + \left\lceil \frac{p(y)}{2} \right\rceil$ pebbles can be moved from $y$ and $z$ to $x$ so that $\bar{p}(x) = p(x) + \left\lceil \frac{p(x)}{2} \right\rceil + \left\lceil \frac{p(y)}{2} \right\rceil$.

Case 1: If $m = 0$, then $\bar{p}(x) \geq 2^k$. Thus we can move one pebble to $v$, since $d(x, v) = k$.

Case 2: If $m = 1$, then $\bar{p}(x) \geq 2^k - 2$. Suppose that $a_{i}$ is a vertex of $Q_{2}$ with $p(a_{i}) = 1$. Let $d = d(x, a_{i})$, so $d(a_{i}, v) = k - d$. We have

\[
\begin{align*}
x &\xrightarrow{2^{k-1}-m} a_{2k-1} \xrightarrow{2^{k-2}-1} \ldots \xrightarrow{2^{k-d}-1} a_{j} \xrightarrow{2^{k-d}-1} \ldots \xrightarrow{1} v \\
x &\xrightarrow{2^{k-1}-m} a_{2k-3} \xrightarrow{2^{k-2}-1} \ldots \xrightarrow{2^{k-d}-1} a_{j} \xrightarrow{2^{k-d}-1} \ldots \xrightarrow{1} v
\end{align*}
\]
if $j$ is even.

\[
\begin{align*}
x &\xrightarrow{2^{k-1}-m} a_{2k-1} \xrightarrow{2^{k-2}-1} \ldots \xrightarrow{2^{k-d}-1} a_{j} \xrightarrow{2^{k-d}-1} \ldots \xrightarrow{1} v \\
x &\xrightarrow{2^{k-1}-m} a_{2k-3} \xrightarrow{2^{k-2}-1} \ldots \xrightarrow{2^{k-d}-1} a_{j} \xrightarrow{2^{k-d}-1} \ldots \xrightarrow{1} v
\end{align*}
\]
if $j$ is odd.
Case 3: If \( m = 2 \), then \( \tilde{p}(x) \geq 2^k - 4 \). For \( p(a_{2k-2}) = 0 \), we have \( p(Q_A^2 - a_{2k-2}) = 2 \). Thus

\[
\begin{align*}
\left\{ \begin{array}{c}
x \xrightarrow{2^{k-1-m}} a_{2k-2} \\ x \xrightarrow{2^{k-1-m-1}} a_{2k-3} \\
\end{array} \right. \Rightarrow \tilde{p}(Q_A^2 - a_{2k-2}) = 2^{k-1}.
\]

(B)

Suppose that \( p(a_{2k-2}) = 1 \). When \( p(a_{2k-2}) = 0 \), there exists a vertex \( a_j \) with \( p(a_j) = 1(1 \leq j \leq 2k - 4) \). Let \( d = d(x, a_j) \), so \( d(a_j, v) = k - d \). Now we can come back to (A). When \( p(a_{2k-2}) = 1 \), we have

\[
\begin{align*}
\left\{ \begin{array}{c}
x \xrightarrow{2^{k-3}} a_{2k-4} \\
\end{array} \right. \Rightarrow \tilde{p}(Q_A^2 - a_{2k-2}) = 2^{k-1}.
\]

(C)

For \( p(a_{2k-2}) = 2 \), \( 2^{k-1} - 2 \) pebbles can be moved from \( x \) to \( a_{2k-2} \) so that \( \tilde{p}(a_{2k-2}) = 2^{k-1} \). Thus we can move one pebble to \( v \), since \( d(a_{2k-2}, v) = k - 1 \).

Case 4: If \( m = 3 \), then \( \tilde{p}(x) \geq 2^k - 6 \). For \( p(a_{2k-2}) = 0 \), we have \( p(Q_A^2 - a_{2k-2}) = 3 \). Now we can come back to (B).

Suppose that \( p(a_{2k-2}) = 1 \). When \( p(a_{2k-2}) = 0 \), we have \( p((Q_A^2)^2) = 2 \), and move pebbles as follows:

\[
\begin{align*}
\left\{ \begin{array}{c}
x \xrightarrow{2^{k-4}} a_{2k-3} \\
\end{array} \right. \Rightarrow \tilde{p}((Q_A^2)^2) = 2^{k-2} + 1.
\]

When \( p(a_{2k-3}) = 1 \), there exists a vertex \( a_j \) with \( p(a_j) = 1(1 \leq j \leq 2k - 4) \). Let \( d = d(x, a_j) \), so \( d(a_j, v) = k - d \), and we have (A). When \( p(a_{2k-3}) = 2 \), we have (C).

Suppose that \( p(a_{2k-2}) = 2 \). When \( p(a_{2k-2}) = 0 \), there exists a vertex \( a_j \) with \( p(a_j) = 1(1 \leq j \leq 2k - 4) \). Let \( d = d(x, a_j) \), so \( d(a_j, v) = k - d \). If \( j \) is even, then it comes back to (A). If \( j \) is odd, then we first move one pebble from \( a_{2k-2} \) to \( a_{2k-3} \) before it comes back to (A). When \( p(a_{2k-3}) = 1 \), we move pebbles as follows:

\[
\begin{align*}
\left\{ \begin{array}{c}
x \xrightarrow{2^{k-3}} a_{2k-4} \\
\end{array} \right. \Rightarrow \tilde{p}(Q_A^2 - a_{2k-2}) = 2^{k-1}.
\]

For \( p(a_{2k-2}) = 3 \), \( 2^{k-1} - 3 \) pebbles can be moved from \( x \) to \( a_{2k-2} \) so that \( \tilde{p}(a_{2k-2}) = 2^{k-1} \). Thus we can move one pebble to \( v \), since \( d(a_{2k-2}, v) = k - 1 \).

Lemma 7. Let \( p \) be a distribution on \( C_{2k}^2 \), where \( k \geq 4 \). Let \( q = p(Q_A^2) + p(Q_B^2) \). If \( p(x) + p(y) + p(z) = 2 \left[ \frac{2^{k+1}}{3} \right] + 1 - q \), where \( q = 0, 1, 2, 3 \), then one pebble can be moved to \( v \).

Proof. Let \( h = p(y) \). Without loss of generality, suppose that \( p(Q_A^2) \geq p(Q_B^2) \).

Case 1: If \( q = 0 \), then \( p(Q_A^2) = p(Q_B^2) = 0 \) and

\[
p(x) + p(z) = 2 \left[ \frac{2^{k+1}}{3} \right] + 1 - h.
\]

(10)

First, suppose that \( h \) is odd. If \( p(x) < 2 \), then \( p(y) + p(z) \geq 2 \left[ \frac{2^{k+1}}{3} \right] \). Thus \( \left[ \frac{p(y)}{2} \right] + \left[ \frac{p(z)}{2} \right] \geq \frac{p(y) + p(z)}{2} - 1 \leq \left[ \frac{2^{k+1}}{3} \right] - 1 \geq 2^{k-1} + 2^{k-1} - 2 \geq 2^{k-1}(k \geq 4) \). Hence we can move \( 2^{k-1} \) pebbles from \( y \) to \( z \) to \( b_{2k-2} \), and we are done. Next suppose that \( p(x) \geq 2 \). If \( p(x) + \frac{p(z)}{2} + \frac{h-1}{2} \geq 2^k \), then \( p(x) + \left[ \frac{p(z)}{2} \right] + \left[ \frac{p(y)}{2} \right] \geq 2^k \). By Lemma 6, we are done. If \( p(x) + \frac{p(z)}{2} + \frac{h-1}{2} \leq 2^{k-1} \), then by (10), \( p(z) \geq \frac{2^{k+1}}{3} + 1 - h \). So \( p(z) + \frac{p(z)-2}{3} = \frac{p(z)+p(z)}{2} + \frac{p(y)}{2} - 1 \geq \left[ \frac{2^{k+1}}{3} \right] - \frac{h-1}{2} + \frac{1}{3} \frac{2^{k+1}}{3} - \frac{h-1}{2} - 1 \geq 2^k - (h+1) \). This implies that \( p(z) + \left[ \frac{p(z)-2}{3} \right] \geq 2^k - (h + 1) \). We now move pebbles as follows:

\[
\begin{align*}
\left\{ \begin{array}{c}
x \xrightarrow{\left[ \frac{p(z)-2}{3} \right]} b_{2k-2} \\
\end{array} \right. \Rightarrow \tilde{p}((Q_A^2)^2) = 2^{k-1} - (h + 1).
\]

(D)

Second, suppose that \( h \) is even. If \( p(x) + \frac{p(z)}{2} + \frac{h}{2} \geq 2^k \), then \( p(x) + \left[ \frac{p(z)}{2} \right] + \left[ \frac{p(y)}{2} \right] \geq 2^k \). By Lemma 6, we are done. If \( p(x) + \frac{p(z)}{2} + \frac{h}{2} \leq 2^{k-1} \), then \( p(z) \geq \frac{2^{k+1}}{3} + 2 - h \) by (10). So \( p(z) + \frac{p(z)}{2} \geq 2^k - h \). This implies that \( p(z) + \left[ \frac{p(z)}{2} \right] \geq 2^k - h \). We now move pebbles as follows:

\[
\begin{align*}
\left\{ \begin{array}{c}
x \xrightarrow{\frac{p(y)}{2}} z \\
\end{array} \right. \Rightarrow \tilde{p}(Q_A^2 - a_{2k-2}) = 2^{k-1} - h.
\]

(E)
Case 2: If \( q = 1 \), then \( p(Q^2_A) = 1 \), \( p(Q^2_B) = 0 \) and
\[
p(x) + p(z) = 2 \left[ \frac{2^{k+1}}{3} \right] - h. \quad (11)
\]

First, suppose that \( h \) is odd. If \( p(x) < 2 \), then \( p(y) + p(z) \geq 2 \left[ \frac{2^{k+1}}{3} \right] - 1 \). Thus \( \left[ \frac{p(y)}{2} \right] + \left[ \frac{p(z)}{2} \right] \geq \frac{p(y) + p(z) - 1}{2} = 2 \left[ \frac{2^{k+1}}{3} \right] - 2 \geq 2^{k-1} - \frac{2^{k-1} - 8}{3} = 2^{k-1}(k \geq 4) \). Hence we can move \( 2^{k-1} \) pebbles from \( y \) and \( z \) to \( b_{2k-2} \), and we are done.

Now suppose that \( p(x) \geq 2 \). If \( p(x) + \frac{p(y)}{2} + \frac{h-1}{2} \geq 2^{k} - 2 \), then \( p(x) + \left[ \frac{p(y)}{2} \right] + \left[ \frac{p(z)}{2} \right] \geq 2^{k} - 2 \). By Lemma 6, we are done. If \( p(x) + \frac{p(y)}{2} + \frac{h-1}{2} \leq 2^{k} - 3 \), then \( p(x) \geq \frac{2^{k+1}-2}{3} + 3 - h \) by (11). So \( p(x) + \frac{p(y)-2}{2} = \frac{p(x)+p(z)}{2} + \frac{p(z)}{2} - 1 \geq \frac{2^{k+1}}{3} - \frac{h+1}{2} + \frac{1}{2} \frac{2^{k+1}-2}{3} - \frac{h-3}{2} - 1 \geq 2^{k} - (h + 1) \). This implies that \( p(z) + \left[ \frac{p(x)-2}{2} \right] \geq 2^{k} - (h + 1) \), and it comes back to (D).

Second, suppose that \( h \) is even. If \( p(x) + \frac{p(y)}{2} + \frac{h}{2} \geq 2^{k} - 2 \), then \( p(x) + \left[ \frac{p(y)}{2} \right] + \left[ \frac{p(z)}{2} \right] \geq 2^{k} - 2 \). By Lemma 6, we are done. If \( p(x) + \frac{p(y)}{2} + \frac{h}{2} \leq 2^{k} - 3 \), then \( p(x) \geq \frac{2^{k+1}-2}{3} + 4 - h \) by (11). So \( p(x) + \frac{p(y)}{2} \geq 2^{k} - h + 1 \). This implies that \( p(z) + \left[ \frac{p(x)}{2} \right] \geq 2^{k} - h + 1 \). Hence we have (E).

Case 3: If \( q = 2 \), then
\[
p(x) + p(z) = 2 \left[ \frac{2^{k+1}}{3} \right] - 1 - h. \quad (12)
\]

Case 3.1: \( p(Q^2_A) = p(Q^2_B) = 1 \). There exists a vertex \( b_j \) of \( Q^2 \) with \( p(b_j) = 1 \), and let \( d = d(z, b_j) \), so \( d(b_j, v) = k - d \).

Suppose that \( h \) is odd. If \( p(x) + \frac{p(y)}{2} + \frac{h-1}{2} \geq 2^{k} - 2 \), then \( p(x) + \left[ \frac{p(y)}{2} \right] + \left[ \frac{p(z)}{2} \right] \geq 2^{k} - 2 \). By Lemma 6, we are done. Next, if \( p(x) + \frac{p(y)}{2} + \frac{h-1}{2} \leq 2^{k} - 3 \), then by (12), \( p(x) \geq \frac{2^{k+1}-2}{3} + 1 - h \). So \( p(x) + \frac{p(y)}{2} = \frac{p(x)+p(z)}{2} + \frac{p(z)}{2} \geq \frac{2^{k+1}}{3} - \frac{h+1}{2} + \frac{1}{2} \frac{2^{k+1}-2}{3} - \frac{h-3}{2} \geq 2^{k} - h - 1 \). This implies that \( p(z) + \left[ \frac{p(x)}{2} \right] \geq 2^{k} - h - 1 \). For even \( j \), we move pebbles as follows:

\[
y \frac{1}{2} \rightarrow b_{2k-2} \quad (F)
\]

For odd \( j \), we move pebbles as follows:

\[
z \frac{1}{2} \rightarrow b_{2k-3} \quad (G)
\]

\[
x \rightarrow z \frac{1}{2} \rightarrow b_{2k-2} \quad (G)
\]

\[
y \rightarrow b_{2k-2}
\]

\( \Rightarrow p(b_{j-1}) = 2^{k-d-1} \). Thus we can move one pebble to \( v \), since \( d(v, b_{j-1}) = k - d - 1 \).

Suppose that \( h \) is even. If \( p(x) + \frac{p(y)}{2} + \frac{h}{2} \geq 2^{k} - 2 \), then \( p(x) + \left[ \frac{p(y)}{2} \right] + \left[ \frac{p(z)}{2} \right] \geq 2^{k} - 2 \). By Lemma 6, we are done. Next, if \( p(x) + \frac{p(y)}{2} + \frac{h}{2} \leq 2^{k} - 3 \), then by (12), \( p(x) \geq \frac{2^{k+1}-2}{3} + 2 - h \). So \( p(x) + \frac{p(y)}{2} \geq 2^{k} - h - 1 \). This implies that \( p(z) + \left[ \frac{p(x)}{2} \right] \geq 2^{k} - h - 1 \). For even \( j \), it comes back to (F). For odd \( j \), it comes back to (G).

Case 3.2: \( p(Q^2_A) = 2 \) and \( p(Q^2_B) = 0 \).

Suppose that \( h \) is odd. If \( p(x) + \frac{p(y)}{2} + \frac{h-1}{2} \geq 2^{k} - 4 \), then \( p(x) + \left[ \frac{p(y)}{2} \right] + \left[ \frac{p(z)}{2} \right] \geq 2^{k} - 4 \). By Lemma 6, we are done. Next, if \( p(x) + \frac{p(y)}{2} + \frac{h-1}{2} \leq 2^{k} - 5 \), then \( p(x) \geq \frac{2^{k+1}-2}{3} + 6 - (h + 1) \) by (12). Thus \( p(x) + \frac{p(y)}{2} = \frac{p(x)+p(z)}{2} + \frac{p(z)}{2} \geq \frac{2^{k+1}}{3} - \frac{h+1}{2} + \frac{1}{2} \frac{2^{k+1}-2}{3} - \frac{h+1}{2} + 3 - \frac{h+1}{2} \geq 2^{k} - h + 1 \). This implies that \( p(z) + \left[ \frac{p(x)}{2} \right] \geq 2^{k} - h + 1 \). Hence we have (E).

Suppose that \( h \) is even. If \( p(x) + \frac{p(y)}{2} + \frac{h}{2} \geq 2^{k} - 4 \), then \( p(x) + \left[ \frac{p(y)}{2} \right] + \left[ \frac{p(z)}{2} \right] \geq 2^{k} - 4 \). By Lemma 6, we are done. Next, if \( p(x) + \frac{p(y)}{2} + \frac{h}{2} \leq 2^{k} - 5 \), then \( p(x) \geq \frac{2^{k+1}-2}{3} + 6 - h \) by (12). So \( p(x) + \frac{p(y)}{2} \geq 2^{k} - h + 1 \). This implies that \( p(z) + \left[ \frac{p(x)}{2} \right] \geq 2^{k} - h + 1 \), and it comes back to (E).
Case 4: If \( q = 3 \), then
\[
p(x) + p(z) = 2 \left[ \frac{2^{k+1}}{3} \right] - 2 - h.
\] (13)

Case 4.1: \( p(Q_A^4) = 2 \) and \( p(Q_B^4) = 1 \). There exists a vertex \( b_j \) of \( Q_b^2 \) with \( p(b_j) = 1 \), and let \( d = d(b_j, z) \), so \( d(v, b_j) = k - d \).

First, suppose that \( h \) is odd. If \( p(x) + \frac{p(x)}{2} + \frac{h-1}{2} \geq 2^k - 4 \), then \( p(x) + \left[ \frac{p(x)}{2} \right] + \frac{h-1}{2} \geq 2^k - 4 \), and we are done by Lemma 6. Next, if \( p(x) + \frac{p(x)}{2} + \frac{h-1}{2} \leq 2^k - 5 \), then \( p(z) \geq \frac{2^{k+1}-2}{3} + 4 - (h + 1) \) by (13). So \( p(z) + \frac{p(z)}{2} = \frac{p(x)+p(z)}{2} + \frac{p(z)}{2} \geq \left[ \frac{2^{k+1}}{3} \right] - \frac{h-1}{2} - 1 + \frac{1}{2} \left[ \frac{2^{k+1}-2}{3} + 4 - \frac{h-1}{2} \right] > 2^k - h - 1 \). This implies that \( p(z) + \left[ \frac{p(z)}{2} \right] > 2^k - h - 1 \). For even \( j \), we have (F).

Second, suppose that \( h \) is even. If \( p(x) + \frac{p(x)}{2} + \frac{h}{2} \geq 2^k - 4 \), then \( p(x) + \left[ \frac{p(x)}{2} \right] + \frac{h}{2} \geq 2^k - 4 \), and we are done by Lemma 6. Next, if \( p(x) + \frac{p(x)}{2} + \frac{h}{2} \leq 2^k - 5 \), then \( p(z) \geq \frac{2^{k+1}-2}{3} + 4 - h \) by (13). So \( p(z) + \frac{p(z)}{2} \geq 2^k - h \). This implies that \( p(z) + \left[ \frac{p(z)}{2} \right] \geq 2^k - h \). Thus we have (E).

Case 4.2: \( p(Q_A^4) = 3 \) and \( p(Q_B^4) = 0 \).

Suppose first that \( h \) is odd. If \( p(x) + \frac{p(x)}{2} + \frac{h-1}{2} \geq 2^k - 6 \), then \( p(x) + \left[ \frac{p(x)}{2} \right] + \frac{h-1}{2} \geq 2^k - 6 \) By Lemma 6, we are done. And if \( p(x) + \frac{p(x)}{2} + \frac{h-1}{2} \leq 2^k - 7 \), then \( p(z) \geq \frac{2^{k+1}}{3} - 8 = (h + 1) \) by (13). So \( p(z) + \frac{p(z)}{2} = \frac{p(x)+p(z)}{2} + \frac{p(z)}{2} \geq \left[ \frac{2^{k+1}}{3} \right] - \frac{h-1}{2} - 1 + \frac{1}{2} \left[ \frac{2^{k+1}-2}{3} - 8 + \frac{h-1}{2} \right] > 2^k - h + 1 \). Thus we have (E).

Suppose next that \( h \) is even. If \( p(x) + \frac{p(x)}{2} + \frac{h}{2} \geq 2^k - 6 \), then \( p(x) + \left[ \frac{p(x)}{2} \right] + \frac{h}{2} \geq 2^k - 6 \) By Lemma 6, we are done. And if \( p(x) + \frac{p(x)}{2} + \frac{h}{2} \leq 2^k - 7 \), then \( p(z) \geq \frac{2^{k+1}-2}{3} + 8 - h \) by (13). So \( p(z) + \frac{p(z)}{2} \geq 2^k - h + 2 \). This implies that \( p(z) + \left[ \frac{p(z)}{2} \right] \geq 2^k - h + 2 \). Thus we have (E).

**Theorem 8.** For \( k \geq 4 \), \( f(C_{4k}^2) = 2 \left[ \frac{2^{k+1}}{3} \right] + 1 \).

**Proof.** First, we claim that \( f(C_{4k}^2) > 2 \left[ \frac{2^{k+1}}{3} \right] \). Suppose that we have only 2 \( \left[ \frac{2^{k+1}}{3} \right] \) pebbles on \( x \) and \( \left[ \frac{2^{k+1}}{3} \right] \) pebbles on \( z \). We see that at most \( 2^{k-1} - 1 \) pebbles can be moved from \( x \) (or \( z \)) to \( a_{2k-2} \) (or \( b_{2k-2} \)). And we can move at most \( \frac{1}{2} \left[ \frac{2^{k+1}-1}{3} \right] \) pebbles from \( z \) to \( x \) so that \( x \) has at most \( \left[ \frac{2^{k+1}-1}{3} + \frac{1}{2} \left[ \frac{2^{k+1}-1}{3} \right] \right] \) \( 2^k \) pebbles. Thus no pebbles can be moved to \( v \).

Next, we place \( 2 \left[ \frac{2^{k+1}}{3} \right] + 1 \) pebbles on the vertices of \( C_{4k}^2 \). Without loss of generality, we may assume that \( p(Q_A^4) > p(Q_B^4) \). Let \( h = p(y) \) and \( q = p(Q_A^4) + p(Q_B^4) \). If \( q < 4 \), then we are done by Lemma 7. Next we consider the case \( q \geq 4 \). And we have
\[
p(x) + p(z) = 2 \left[ \frac{2^{k+1}}{3} \right] + 1 - q - h.
\] (14)

Case 1: \( h \) is odd. If \( p(z) < 2 \), then \( p(x) \geq 2 \left[ \frac{2^{k+1}}{3} \right] - q - h \) by (14). Since \( p(Q_A^4) \geq \frac{q}{2} \), so
\[
y \xrightarrow{\frac{h-1}{2}} a_{2k-2} \quad \left[ \frac{2^{k+1}}{3} \right] - \frac{2^{k+1}}{3} - 1 \xrightarrow{\frac{h-1}{2}} a_{2k-2}
\]

Thus we can assume that \( p(z) \geq 2 \).

Suppose that \( p(Q_A^4) = 0 \) and \( p(Q_B^4) = q \). If \( p(x) + \frac{h-1}{2} + p(z) \geq 2^k \), then \( p(x) + \left[ \frac{p(x)}{2} \right] + \frac{h-1}{2} \geq 2^k \), and we are done by Lemma 6. And if \( p(x) + \frac{h-1}{2} + p(z) \leq 2^k - 1 \), then \( p(x) \geq 2^k - 3q - (h + 1) \) by (14). So \( p(z) + p(x) = \frac{p(x)+p(z)}{2} + \frac{p(x)}{2} - 1 = \left[ \frac{2^{k+1}}{3} \right] - \frac{h-1}{2} + \frac{1}{2} \left[ \frac{2^{k+1}-1}{3} + 1 - q - \frac{h-1}{2} \right] - 1 \geq 2^k - \frac{3q}{2} - h - 1 \). This implies that \( p(z) + \frac{p(z)}{2} + p(x) \geq 2^k - \frac{3q}{2} - h - 1 \). We now move pebbles as follows:
\[
z \xrightarrow{\frac{h+1}{2}} a_{2k-2} \quad \left[ \frac{2^{k+1}-1}{3} - \frac{3q}{2} + \frac{h+1}{2} \right] \xrightarrow{\frac{h+1}{2}} a_{2k-2}
\]

Thus we have (E).
Suppose that \( p(Q^2_a) = 1 \) and \( p(Q^2_a) = q - 1 \). If \( \frac{p(x)}{2} + \frac{h}{2} + p(z) \geq 2^k - 2 \), then \( \left\lceil \frac{p(x)}{2} \right\rceil + \left\lceil \frac{p(y)}{2} \right\rceil + p(z) \geq 2^k - 2 \).

By Lemma 6, we are done. If \( \frac{p(x)}{2} + \frac{h}{2} + p(z) \leq 2^k - 3 \), then \( p(x) \geq \frac{2k+1-2}{3} + 6 - 2q - (h + 1) \) by (14). So \( \frac{p(x)}{2} + p(x) = \frac{p(x)+p(z)}{2} + \frac{p(x)}{2} - 1 \geq \left\lceil \frac{2k+1-2}{3} \right\rceil - q - \frac{h}{2} - \frac{1}{2} \frac{2k+1-2}{3} + 3 - q - \frac{h}{2} - 1 \geq 2^k - \frac{3q}{2} - h + 1 \). This implies that \( \left\lceil \frac{p(x)}{2} \right\rceil + p(x) \geq 2^k - \frac{3q}{2} - h + 1 \). We now move pebbles as follows:

\[
\begin{align*}
X \quad \frac{h+1}{2} & \rightarrow a_{2k-2} \\
Y \quad \frac{p(x)}{2} - \frac{h+1}{2} + 1 & \rightarrow a_{2k-2} \\
Z \quad X, Y & \rightarrow a_{2k-2}
\end{align*}
\]

Suppose that \( p(Q^2_a) \geq 2 \). Note that \( p(Q^2_a) \geq \frac{q}{2} \). If \( p(x) + \frac{p(y)}{2} \geq 2^k - q - h + 2 \), then \( p(x) + \left\lceil \frac{p(x)}{2} \right\rceil \geq 2^k - q - h + 1 \).

We now move pebbles as follows:

\[
\begin{align*}
X \quad \frac{h+1}{2} & \rightarrow a_{2k-2} \\
Y \quad \frac{p(x)}{2} - \frac{h+1}{2} + 1 & \rightarrow a_{2k-2} \\
Z \quad X, Y & \rightarrow a_{2k-2}
\end{align*}
\]

Next, if \( p(x) + \frac{p(x)}{2} \leq 2^k - q - h + 1 \), then \( p(z) \geq \frac{2k+1-2}{3} - 2 \) by (14). So \( p(z) + \frac{p(x)}{2} = \frac{p(x)+p(z)}{2} + \frac{p(x)}{2} \geq \left\lceil \frac{2k+1-2}{3} \right\rceil - q - \frac{h}{2} - \frac{1}{2} \frac{2k+1-2}{3} + 3 - q - \frac{h}{2} - 1 \geq 2^k - \frac{3q}{2} - h + 1 \). Thus we can assume that \( p(Q^2_a) \leq \frac{q}{2} + 1 \), so \( p(Q^2_a) \geq \frac{q}{2} - 1 \). Furthermore, if \( \frac{p(z)}{2} + \frac{h}{2} + p(z) \geq 2^k - 4 \), then \( p(z) + \frac{p(y)}{2} + p(z) \geq 2^k - 4 \). By Lemma 6, we are done since \( p(Q^2_a) \geq 2 \). And if \( \frac{p(x)}{2} + \frac{h}{2} + p(z) \leq 2^k - 5 \), then \( p(x) \geq \frac{2k+1-2}{3} + 9 - 2q - h \) by (14). So \( p(x) + \frac{p(x)}{2} = \frac{p(x)+p(z)}{2} + \frac{p(x)}{2} - 1 \geq \left\lceil \frac{2k+1-2}{3} \right\rceil - q - \frac{h}{2} - \frac{1}{2} \frac{2k+1-2}{3} + 4 - q - \frac{h}{2} - 1 \geq 2^k - 3 - \frac{3q}{2} - h \). This implies that \( p(x) + \left\lceil \frac{p(x)}{2} \right\rceil \geq 2^k - 3 - \frac{3q}{2} - h \). We now move pebbles as follows:

\[
\begin{align*}
X \quad \frac{h}{2} & \rightarrow a_{2k-2} \\
Y \quad \frac{p(x)}{2} - h + 2 & \rightarrow a_{2k-2} \\
Z \quad X, Y & \rightarrow a_{2k-2}
\end{align*}
\]

Case 2: \( h \) is even. Suppose that \( p(Q^2_a) = 0 \) and \( p(Q^2_a) = q \). If \( \frac{p(x)}{2} + \frac{h}{2} + p(z) \geq 2^k \), then \( \left\lceil \frac{p(x)}{2} \right\rceil + \left\lceil \frac{p(y)}{2} \right\rceil + p(z) \geq 2^k \), and we are done by Lemma 6. And if \( \frac{p(x)}{2} + \frac{h}{2} + p(z) \leq 2^k - 1 \), then \( p(x) \geq \frac{2k+1-2}{3} + 2 - 2q - h \) by (14). So \( \frac{p(x)}{2} + p(x) \geq 2^k - \frac{3q}{2} - h \).

This implies that \( \left\lceil \frac{p(x)}{2} \right\rceil + p(x) \geq 2^k - \frac{3q}{2} - h \). We now move pebbles as follows:

\[
\begin{align*}
X \quad \frac{h}{2} & \rightarrow a_{2k-2} \\
Y \quad \frac{p(x)}{2} - h & \rightarrow a_{2k-2} \\
Z \quad X, Y & \rightarrow a_{2k-2}
\end{align*}
\]

Suppose that \( p(Q^2_a) = 1 \) and \( p(Q^2_a) = q - 1 \). If \( \frac{p(x)}{2} + \frac{h}{2} + p(z) \geq 2^k - 2 \), then \( \left\lceil \frac{p(x)}{2} \right\rceil + \left\lceil \frac{p(y)}{2} \right\rceil + p(z) \geq 2^k - 2 \). By Lemma 6, we are done. And if \( \frac{p(x)}{2} + \frac{h}{2} + p(z) \leq 2^k - 3 \), then \( p(x) \geq \frac{2k+1-2}{3} + 6 - 2q - h \) by (14). So \( \frac{p(x)}{2} + p(x) \geq 2^k - \frac{3q}{2} - h + 2 \).

This implies that \( \left\lceil \frac{p(x)}{2} \right\rceil + p(x) \geq 2^k - \frac{3q}{2} - h + 2 \). We now move pebbles as follows:

\[
\begin{align*}
X \quad \frac{h}{2} & \rightarrow a_{2k-2} \\
Y \quad \frac{p(x)}{2} - h & \rightarrow a_{2k-2} \\
Z \quad X, Y & \rightarrow a_{2k-2}
\end{align*}
\]

Suppose that \( p(Q^2_a) \geq 2 \). Note that \( p(Q^2_a) \geq \frac{q}{2} \). If \( p(x) + \frac{p(x)}{2} \geq 2^k - q - h + 2 \), then \( p(x) + \left\lceil \frac{p(x)}{2} \right\rceil \geq 2^k - q - h + 2 \). We now move pebbles as follows:

\[
\begin{align*}
X \quad \frac{h}{2} & \rightarrow a_{2k-2} \\
Y \quad \frac{p(x)}{2} - h + 1 & \rightarrow a_{2k-2} \\
Z \quad X, Y & \rightarrow a_{2k-2}
\end{align*}
\]
Next, if \( p(x) + \frac{p(y)}{2} \leq 2^k - q - h + 1 \), then \( p(z) \geq \frac{2^{k+1} - 3}{3} - 2 \) by (14). So \( p(z) + \frac{p(y)}{2} + \frac{p(x)}{2} \geq \frac{2^{k+1}}{3} - \frac{q}{2} + \frac{h}{2} + \frac{1}{2} \). Thus \( \tilde{p}(Q_{2}^2) = \frac{p(x) + p(y) + p(z)}{2} + p(Q_{2}^2) \geq 2^{k-1} - \frac{q}{4} - 1 + p(Q_{2}^2) \). Obviously, if \( p(Q_{2}^2) \geq \frac{q}{4} + 2 \), then \( \tilde{p}(Q_{2}^2) \geq 2^{k-1} + 1 \), and we are done. Thus we can assume that \( p(Q_{2}^2) \leq \frac{q}{4} + 1 \), so \( p(Q_{2}^2) \geq \frac{3q}{4} - 1 \). Furthermore, if \( \frac{p(x)}{2} + \frac{h}{2} + p(z) \geq 2^k - 4 \), then \( p(x) + \frac{p(y)}{2} + p(z) \geq p(z) + 2^k - 4 \). By Lemma 6, we are done since \( p(Q_{2}^2) \geq 2 \).

If \( \frac{p(x)}{2} + \frac{h}{2} + p(z) \leq 2^k - 5 \), then \( p(x) \geq \frac{2^{k+1} - 3}{3} + 10 - 2q - h \) by (14). So \( p(x) + \frac{p(y)}{2} \geq 2^k + 4 - \frac{3q}{2} - h \). This implies that \( p(x) + \left[ \frac{p(y)}{2} \right] \geq 2^k + 4 - \frac{3q}{2} - h \). We have

\[
\begin{align*}
y \xrightarrow{\frac{p(y)}{2}} a_{2k-2} \\
z \xrightarrow{\frac{p(x)}{2} + \frac{h}{2} + 2} x \xrightarrow{2^{k-1} - \frac{3q}{2} - \frac{h}{2} + 2} a_{2k-2}
\end{align*}
\]

Therefore we are done. \( \square \)

Combining Theorems 2, 3, 5 and 8, we obtain the pebbling number of \( C_{2n}^2 \).

Acknowledgments

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References