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Rep-tiling for triangles

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Abstract

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In this paper we prove that one can only tile a triangle with tiles all congruent to each other and similar to the original triangle when k^2, l^2+k^2, or $3k^2$ tiles are used. The result is based on the geometry of packing and a result of I. Niven's on rational trigonometric values. In addition we describe how to tile most triangles.

Introduction

Golomb [2] introduced the notion of 'rep-tiling' in which a plane figure is partitioned into replicas of the original. In particular, we say that a triangle has been *n-tiled* if it can be partitioned into n pairwise congruent triangles each similar to the original. Of particular interest to us in Golomb's article was the statement that a triangle can be n-tiled if and only if n is in the form k^2, k^2+l^2 or $3k^2$. In this paper we give a complete proof of Golomb's theorem and use the results from the proof to show exactly what the triangles that can be $3k^2$ or (k^2+l^2)-tiled must look like. We are grateful to the referee for pointing out the existence of a brief sketch of a proof published by Posner [4]. We begin by first restating the simple construction arguments that lead to k^2, $3k^2$, and (k^2+l^2)-tilings.

1. Constructions

Construction of a k^2-tiling. Take any triangle and divide each of its sides into k pieces of equal length. Then draw the line segments joining the corresponding points as in Fig. 1.

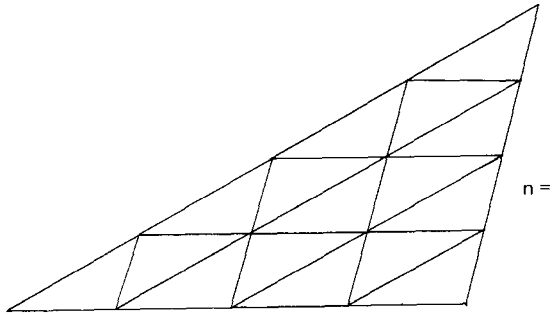

Fig. 1.

Construction of a $3k^2$-tiling. First 3-tile a 30-60-90 triangle as in Fig. 2. Then k^2-tile each of the three tiles.

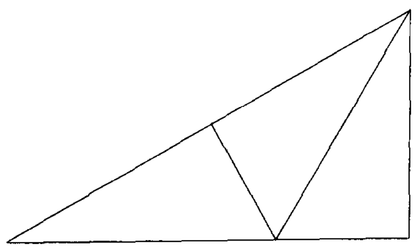

Fig. 2.

Construction of a (k^2+l^2)-tiling. This construction is a bit more complicated than the previous two. Draw a right triangle with shorter sides of lengths k and l and draw the altitude to the hypotenuse as indicated in Fig. 3.

Now l^2-tile the triangle on the left and k^2-tile the triangle on the right. The area of a tile for the triangle on the left is:

$$L=\frac{ah}{2k^2}=\frac{a}{2k^2}\left(\frac{bk}{l}\right)=\frac{ab}{2kl},$$

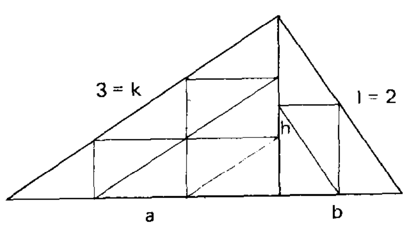

Fig. 3.

while the area of a tile for the triangle on the right is

$$R = \frac{bh}{2l^2} = \frac{b}{2l^2}\left(\frac{al}{k}\right) = \frac{ab}{2kl}.$$

2. Main result

Theorem (the main result). *There exists a triangle that can be n-tiled if and only if* $n = k^2$, $n = k^2 + l^2$, *or* $n = 3k^2$.

The proof of the if part of this implication was given in section one. For the only if part, the proof will consist of a result of I. Niven's on rational trigonometric values, two geometric lemmas, and the observation that if n is not a perfect square, 1 and $\sqrt{n}$ are linearly independent over the rationals. First we need to set up some notation.

After scaling we may label the sides of the tiles 1, x, and y where $1 \leqslant x \leqslant y$ and corresponding angles α, β, δ, with $\alpha \leqslant \beta \leqslant \delta$. The large triangle will then have sides of length $\sqrt{n}$, $\sqrt{n}\,x$, and $\sqrt{n}\,y$ with the same angles (see Fig. 4).

If a triangle is tiled, each side of the large triangle must be packed with sides of the tiles. This leads to three linear equations:

$$a + bx + cy = \sqrt{n}, \tag{1}$$

$$d + ex + fy = \sqrt{n}\,x, \tag{2}$$

$$g + hx + iy = \sqrt{n}\,y. \tag{3}$$

Here all the coefficients are nonnegative integers. We shall call these the *packing equations*.

The result of Niven we will need is the following [3, p. 41].

Lemma 1. *If* α *is a rational multiple of* π *and* $\cos(\alpha)$, $\sin(\alpha)$ *or* $\tan(\alpha)$ *is rational, then* α *is* 0, $\pi/6$, $\pi/4$, *or* $\pi/3 \bmod(\pi/2)$.

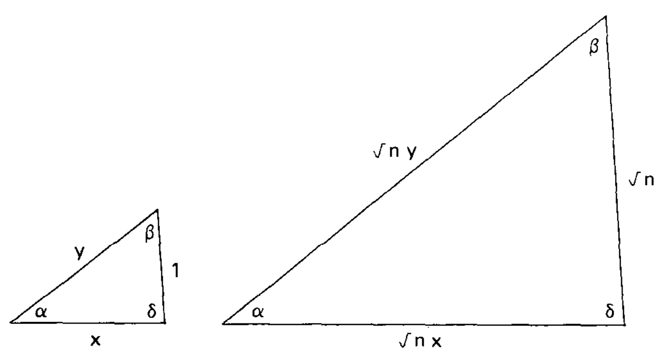

Fig. 4.

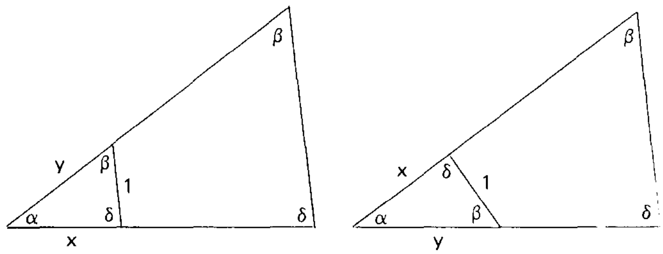

Fig. 5.

The two geometric lemmas we need are the following.

Lemma 2. *In the packing equations, either e and i are nonzero or f and h are nonzero.*

Lemma 3. *In the packing equations, if* $i = 0$ *then* $\delta = \pi/2$ *or* $\delta = \beta$.

Proof of Lemma 2. Since α is the smallest angle, the α angle in the large triangle can only be filled with the α angle of a tile. That forces the tile to be placed in one of the two ways pictured in Fig. 5.

Proof of Lemma 3. If $i = 0$, then only sides of tiles of length 1 and x pack the side of length $\sqrt{n}\,y$. Therefore the δ angle always touches the $\sqrt{n}\,y$ side. Either δ packs α, δ packs β, or δ meets δ somewhere in between (see Fig. 6). If δ packs α, then $\alpha = \beta = \delta$. If δ packs β, then $\delta = \beta$. If δ meets δ, then $\delta + \delta + \theta = \pi$ where θ is some combination of δ, β and α. Use $\alpha \leqslant \beta \leqslant \delta$ and $\alpha + \beta + \delta = \pi$ (since they are angles of a triangle) along with the previous equation to obtain: $\alpha - \theta = \delta - \beta \geqslant 0$. But since α is the smallest angle, θ is either equal to α or θ is zero. If $\theta = \alpha$, then $\delta = \beta$. If $\theta = 0$, then δ is $\pi/2$. □

Now we will complete the proof of the main theorem in three steps found in Lemmas 4, 5, and 7.

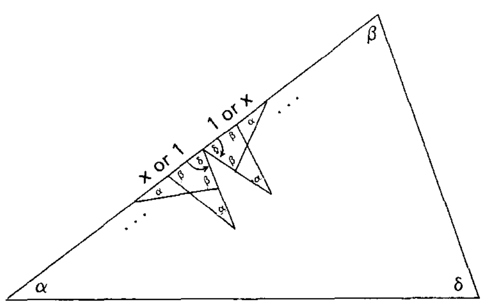

Fig. 6.

Lemma 4. *If a triangle is n-tiled and x is rational, then* $n = k^2$ *or* $n = k^2 + l^2$.

Proof of Lemma 4. Assume that x is rational and n is not a perfect square. From the first two packing equations, if either c or f is zero, $\sqrt{n}$ would be rational, hence n would be a perfect square. Therefore both c and f are not zero.

Use Equations (1) and (2) to eliminate y and obtain:

$$fa - cd + (fb - ec)x = \sqrt{n}\,(f - cx).$$

Since the left side is rational, the right side must be, too. Therefore, $x = f/c$. Plug x into equation (1) and solve for y to obtain:

$$y = -\frac{(ca + fb)}{c^2} + \frac{1}{c}\sqrt{n}.$$

Plug these values for x and y into Equation (3). Equating irrational parts then gives the following equation:

$$ci = -(ca + fb).$$

Since a, b, and i are non-negative and c and f are positive, $i = a = b = 0$. This makes $y = (1/c)\sqrt{n}$. From Lemma 3, $i = 0$ implies $\delta = \pi/2$ or $\delta = \beta$. If $\delta = \beta$, then $x = y$. But since $y = (1/c)\sqrt{n}$ and x is rational, $\sqrt{n}$ is rational contradicting the assum;ʻion that n is not a perfect square. If δ is $\pi/2$, then $y^2 = x^2 + 1$. This gives

$$\frac{n}{c^2} = \frac{f^2}{c^2} + 1 \quad \text{or} \quad n = f^2 + c^2. \quad \square$$

Lemma 5. *If a triangle is n-tiled and y is rational, then* $n = k^2$ *or* $n = 3k^2$.

Proof of Lemma 5. Assume y is rational and n is not a perfect square. Repeating the argument in the proof of the previous lemma switching y with x and Equation (2) with Equation (3) yields the following information: b and h are not zero, $a = c = e = 0$, $x = (1/b)\sqrt{n}$ and $y = h/b$. Now plug these values of x and y into equation (3). The rational part of this equation yields:

$$bg + ih = 0.$$

Since g and i are nonnegative and b and h are positive, $g = i = 0$. From Lemma 3, since $i = 0$ either $\delta = \pi/2$ or $\delta = \beta$. If $\delta = \beta$ then $x = y$ and as before this contradicts the assumption that n is not a perfect square. Therefore δ is $\pi/2$. Since i and g are zero, the $\sqrt{n}\,y$ side is packed entirely with sides of tiles of length x. Therefore angle β in the large triangle must be packed entirely with α's. Since δ is $\pi/2$ and β is a multiple of α, α must be a rational multiple of π. Also since $\sin(\alpha)$ is $1/y$, $\sin(\alpha)$ is rational. Therefore by Lemma 1, $\alpha = \pi/6$. Hence $x = \sqrt{3}$.

Therefore

$$\sqrt{3} = \frac{1}{b}\sqrt{n} \quad \text{or} \quad n = 3b^2. \quad \square$$

The last part of the proof of the theorem involves showing that when x and y are irrational, n must be a perfect squre. In order to do this we need to introduce some more notation. If n is not a perfect square then

$$\mathbb{Q}(\sqrt{n}) = \{w\colon w = u + v\sqrt{n},\ u,\ v \text{ in } \mathbb{Q}\}$$

is an extension of the rationals with basis 1 and $\sqrt{n}$. Any number in $\mathbb{Q}(\sqrt{n})$ can then be written uniquely as $u + v\sqrt{n}$ where u and v are rational numbers. We shall call u the *rational part* of w and denote it $\mathrm{Ra}(w)$ and v the *irrational part* of w and denote it $\mathrm{Ir}(w)$. Finally we shall denote the *conjugate of* w, $u - v\sqrt{n}$, as $\bar{w}$.

Observe that we may solve for x and y in Equations (1) and (2) in terms of the integers a through f and $\sqrt{n}$. Therefore x and y are in $\mathbb{Q}(\sqrt{n})$.

Lemma 6. *If $\sqrt{n}$, x, and y are irrational then either*:

$$\mathrm{Ra}(x) < 0, \quad \mathrm{Ir}(x) > 0, \quad \mathrm{Ra}(y) > 0, \quad \mathrm{Ir}(y) < 0, \quad \textit{and} \quad \mathrm{Ir}(\bar{x}y) < 0$$

or

$$\mathrm{Ra}(x) > 0, \quad \mathrm{Ir}(x) < 0, \quad \mathrm{Ra}(y) < 0, \quad \mathrm{Ir}(y) > 0, \quad \textit{and} \quad \mathrm{Ir}(\bar{x}y) > 0.$$

Proof of Lemma 6. The rational part of Equation (1) is $a + b\,\mathrm{Ra}(x) + c\,\mathrm{Ra}(y) = 0$. Therefore either $\mathrm{Ra}(x)$, $\mathrm{Ra}(y)$, or both are negative.

Assume that $\mathrm{Ra}(x) < 0$. Since $x > 0$, $\mathrm{Ir}(x) > 0$. The irrational part of Equation (2) is $e\,\mathrm{Ir}(x) + f\,\mathrm{Ir}(y) = \mathrm{Ra}(x)$. Since $\mathrm{Ir}(x) > 0$ and $\mathrm{Ra}(x) < 0$, $\mathrm{Ir}(y) < 0$. Again since $y > 0$ and $\mathrm{Ir}(y) < 0$, $\mathrm{Ra}(y) > 0$.

Now since $\mathrm{Ra}(x) < 0$ and $\mathrm{Ir}(x) > 0$, $\bar{x} < 0$. Since $\mathrm{Ra}(y) > 0$ and $\mathrm{Ir}(y) < 0$, $\bar{y} > 0$. Therefore $\bar{x}y < 0$ and $x\bar{y} > 0$. Putting these two together, we obtain:

$$2\sqrt{n}\,\mathrm{Ir}(\bar{x}y) = \bar{x}y - x\bar{y} < 0.$$

The proof of the second case is obtained by switching y with x and substituting Equation (3) for Equation (2). $\square$

Lemma 7. *If x and y are irrational then $n = k^2$.*

Proof of Lemma 7. Assume n is not a perfect square. Using the packing equations, multiply Equation (2) by $\bar{y}$ and equate irrational parts. This yields:

$$-d\,\mathrm{Ir}(y) + e\,\mathrm{Ir}(x\bar{y}) = \mathrm{Ra}(x\bar{y}).$$

Now multiply Equation (3) by $\bar{x}$ and equate the irrational parts. This yields:

$$-g\,\mathrm{Ir}(x) + i\,\mathrm{Ir}(\bar{x}y) = \mathrm{Ra}(\bar{x}y).$$

Now using the relationship $\mathrm{Ra}(\bar{x}y) = \mathrm{Ra}(x\bar{y})$ and $\mathrm{Ir}(\bar{x}y) = -\mathrm{Ir}(x\bar{y})$, we can combine the two equations above to obtain:

$$-g\,\mathrm{Ir}(x) + d\,\mathrm{Ir}(y) + (e+i)\mathrm{Ir}(\bar{x}y) = 0.$$

From Lemma 6, $g = d = e = i = 0$.

Since $d = e = 0$, $fy = \sqrt{n}\,x$. Therefore $f\,\mathrm{Ra}(y) = n\,\mathrm{Ir}(x)$ and $f\,\mathrm{Ir}(y) = \mathrm{Ra}(x)$. We may put these two equations together to yield:

$$f^2\,\mathrm{Ra}(y)\mathrm{Ir}(y) = n\,\mathrm{Ra}(x)\mathrm{Ir}(x).$$

Since $i = 0$, Lemma 3 gives $\delta = \pi/2$ or $\delta = \beta$. If $\delta = \beta$ then $x = y$ contradicting Lemma 6. If $\delta = \pi/2$ then $y^2 = x^2 + 1$. The irrational part of this equation yields:

$$2\,\mathrm{Ra}(y)\mathrm{Ir}(y) = 2\,\mathrm{Ra}(x)\mathrm{Ir}(x).$$

Since both the rational and irrational parts of x and y are nonzero from Lemma 6, we may combine this equation with the previous equation giving $n = f^2$, contradicting the original assumption. □

Conclusion

First note that our list of admissible integers does not exhaust the integers.

Corollary 1. *The set of integers n for which there is no triangle which can be n-tiled is infinite.*

The corollary follows by noting that our result shows that n is admissible only if n is the sum of three or fewer squares and then by applying Lagrange's four-square theorem. The latter includes the result that no integer of the form $4^r(8s+7)$ can be written as the sum of three or fewer squares [1, theorem 12-5]. Note that the set of non-admissible integers is larger than this and starts 6, 7, 11, ...

We can also draw a few conclusions from the last section on what the triangles for specific n-tilings look like.

Corollary 2. *If a triangle is $(k^2 + l^2)$-tiled when $k^2 + l^2$ is not a perfect square, then the triangle is a right triangle.*

Note that there is a unique triangle for each pair of integers k and l. However, some integers can be written as the sum of squares of integers in several different ways. For example, 65 is $1^2 + 8^2$ and $7^2 + 4^2$. Therefore there may be several right triangles that are n-tilable when $n = k^2 + l^2$.

Corollary 3. *If a triangle is $3k^2$-tiled, then the triangle is a* 30-60-90 *triangle.*

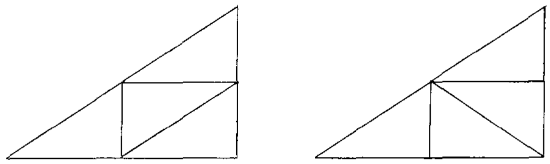

Fig. 7.

We cannot say exactly what form a tiling may take. Actually there may be several ways to n-tile a triangle. Any right triangle may be 4-tiled in either of the two ways in Fig. 7.

This means that one could 4^r-tile any right triangle by first 4-tiling it in either of the ways above and then 4-tiling each tile in either of the two ways above and so on.

The authors have found 117 ways to 12-tile a 30-60-90 triangle!

Finally, if we relax the condition that n-tiling requires congruent tiles and instead ask whether for any integer n there exists a triangle that can be tiled with n tiles all *similar* to the original, we find that indeed the answer is yes.

Corollary 4. *For any integer n, there exists a triangle that can be tiled with n tiles each similar to the original triangle.*

Proof of Corollary 4. By Lagrange's four-square theorem, we may represent any integer n as the sum of one, two, three or four squares [1, theorem 12-7]. The main theorem covers the first two cases. If $n = j^2 + k^2 + l^2$ where j, k, and l are all nonzero, first 3-tile a 30-60-90 triangle and then j^2-tile the first tile, k^2-tile the second, and l^2-tile the third. If $n = j^2 + k^2 + l^2 + m^2$ where j, k, l, and m are all nonzero, 4-tile any triangle and then j^2-tile the first tile, k^2-tile the second, l^2-tile the third and m^2-tile the fourth. □

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