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Disjoint sets of distinct sum sets[1]

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Abstract

An (h, J)-distinct sum set is a set of J integers such that all sums of h elements (repetitions allowed) are distinct. An (h, I, J)-set of disjoint distinct sum sets is a set of I disjoint (h, J)-distinct sum sets with positive elements. A number of constructions of such sets are given.

Keywords: Distinct sum set; Intermodulation interference

1. Introduction

Babcock [2] studied radio systems having frequencies without intermodulation interference. To avoid intermodulation interference of order $2h-1$ and less, his construction required sets such that all sums of h elements from the set are distinct. In our notation such a set of size J is called an (h, J)-DS or an $(h, 1, J)$-DDS. It is also known as a finite B_h-set. Such sets have been studied in a number of contexts, and for $h = 2$ also under various other names, see e.g. [1, 3, 5, 14, 16, 19].

A generalization of the problem was considered by Chen [9]. He considered a mobile radio system for a collection of I areas, and without intermodulation interference of order up to $2h-1$ within each area. His construction requires a set of I disjoint (h, J)-distinct sum sets with positive elements (in our notation: an (h, I, J)-DDS). In this paper we give a number of constructions of DDS.

Let

$$C(h, J) = \left\{ \bar{x} = (x_1, x_2, \ldots, x_J) \mid x_j \text{ nonnegative integers and } \sum_{j=1}^{J} x_j = h \right\}.$$

An (h, J)-distinct sum set (DS) is a set $A = \{a_j \mid 1 \leqslant j \leqslant J\}$ of distinct integers such that

$$\text{if } \bar{x}, \bar{y} \in C(h, J) \quad \text{and} \quad \sum_{j=1}^{J} x_j a_j = \sum_{j=1}^{J} y_j a_j, \quad \text{then } \bar{x} = \bar{y}.$$

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Note that any permutation of a sequence in $C(h, J)$ also belongs to $C(h, J)$. Hence the definition of DS does not depend on a particular ordering of the elements in A.

An (h, I, J)-set of Disjoint Distinct Sum sets (DDS) is a set

$$\Delta = \{\Delta_1, \Delta_2, \ldots, \Delta_I\}$$

of I disjoint (h, J)-DS with positive elements. We denote the elements of Δ_i by $a_{i1}, a_{i2}, \ldots, a_{iJ}$, and usually we assume that they are sorted in increasing order, i.e.

$$1 \leqslant a_{i1} < a_{i2} < \cdots < a_{iJ}.$$

Let

$$v = v(\Delta) = \max\{a_{ij} \mid 1 \leqslant i \leqslant I,\ 1 \leqslant j \leqslant J\}.$$

For the application we want an (h, I, J)-DDS with v as small as possible. Let

$$N_h(I, J) = \min\{v(\Delta) \mid \Delta \text{ is an } (h, I, J) - \text{DDS}\}.$$

To determine $N_h(I, J)$ in general seems to be a very hard problem. $N_2(1, J)$ is known only for $J \leqslant 18$ where it has been determined by computer search. Even less is known about $N_h(I, J)$ in general.

In [9, 17] we gave a number of results on $(2, I, J)$-DDS. Some of these results generalizes immediately or with minor modifications to general h. The generalizations are given below without proofs. We also give some new general results.

There are a number of known lower bounds on $N_h(1, J)$, see [9]. New bounds that improve Chen's bounds asymptotically were given by Chen, Li and Kløve [10] and, independently by Jia [15] for even h and S. Chen [6] for odd h.

For general I we have the following simple lower bounds on $N_h(I, J)$.

Proposition 1. *For all h, I, J we have*

(i) $N_h(I, J) \geqslant N_h(1, J) + I - 1$,

(ii) $N_h(I, J) \geqslant IJ$.

If Δ is an (h, I, J)-DDS such that $v(\Delta) = IJ$, we call Δ perfect. We note that

$$N_h(1, J) + I - 1 < IJ \quad \text{if and only if} \quad I \geqslant \frac{N_h(1, J)}{J - 1}. \tag{1}$$

Hence the first bound in Proposition 1 is best for small I and the second for large I. In particular, we see that perfect (h, I, J)-DDS can exist only for I sufficiently large. Below we show that for all J there exists an $\imath_h(J)$ such that perfect (h, I, J)-DDS do exist for all $I \geqslant \imath_h(J)$.

2. Constructions and upper bounds

There are several known constructions of DS, but not many for DDS in general. One class of constructions was given by Xin [21]. We will now give some new constructions of DDS and the corresponding upper bounds on $N_h(I, J)$. In particular we are interested in those (I, J) for which $N_h(I, J) = IJ$. i.e. for which perfect (I, J)-DDS exist. Our first construction is the immediate generalization of Construction 1 in [17].

Construction 1. *Let*

$$\Delta = \{\{a_{ij} \mid 1 \leqslant j \leqslant J\} \mid 1 \leqslant i \leqslant I\}$$

be an (h, I, J)-DDS such that $v(\Delta) = N_h(I, J)$ and

$$\Delta' = \{\{a'_{ij} \mid 1 \leqslant j \leqslant J\} \mid 1 \leqslant i \leqslant I'\}$$

an (h, I', J)-DDS such that $v(\Delta') = N_h(I', J)$. Then

$$\Gamma = \Delta \cup \{\{N_h(I, J) + a'_{ij} \mid 1 \leqslant j \leqslant J\} \mid 1 \leqslant i \leqslant I'\}$$

is an $(h, I + I', J)$-DDS and $v(\Gamma) = N_h(I, J) + N_h(I', J)$.

From Construction 1 we get the following bound:

Proposition 2. *For all I, I', $N_h(I + I', J) \leqslant N_h(I, J) + N_h(I', J)$.*

For completeness we give simple construction of the best possible $(h, I, 1)$-DDS $(h, I, 2)$-DDS and $(h\ I, 3)$-DDS.

Construction 2. (i) $\{\{i\} \mid 1 \leqslant i \leqslant I\}$ *is an $(h, I, 1)$-DDS.*

(ii) $\{\{i, I + i\} \mid 1 \leqslant i \leqslant I\}$ *is an $(h, I, 2)$-DDS.*

(iii) $\{\{i, \lfloor (h + 3)/2 \rfloor + i, h + 1 + i\} \mid 1 \leqslant i \leqslant I\}$ *is an $(h, I, 3)$-DDS if $I \leqslant \lfloor h/2 \rfloor$.*

(iv) $\{\{i, I + 1 + i, 2I + i\} \mid 1 \leqslant i \leqslant I - 1\} \cup \{\{I, I + 1, 3I\}\}$ *is an $(h, I, 3)$-DDS if $I > \lfloor h/2 \rfloor$.*

Proof. The first two are trivial and the last two follow from the following lemma.

Lemma 1. *If $0 \leqslant a \leqslant b$, $b \geqslant h + 1$, $\gcd(a, b) = 1$, and $i > 0$, then $\{i, a + i, b + i\}$ is an $(h, 3)$-DS.*

Proof. Suppose $(x_1, x_2, x_3), (y_1, y_2, y_3) \in C(h, 3)$ such that

$$x_1 i + x_2(a + i) + x_3(b + i) = y_1 i + y_2(a + i) + y_3(b + i). \tag{2}$$

Then

$$x_2 a + x_3 b = y_2 a + y_3 b. \tag{3}$$

In particular $x_2 a \equiv y_2 a \pmod b$. Hence $x_2 \equiv y_2 \pmod b$. Since $b > h$ we have $x_2 = y_2$, and so $x_3 = y_3$ by (3). Finally, $x_1 = y_1$ since $x_1 + x_2 + x_3 = y_1 + y_2 + y_3$. □

Proposition 3. *We have*

$$N_h(I, 1) = I,$$

$$N_h(I, 2) = 2I,$$

$$N_h(I, 3) = \begin{cases} h + 1 + I & \text{if } I \leqslant \lfloor \frac{h}{2} \rfloor, \\ 3I & \text{otherwise.} \end{cases}$$

Proof. The Proposition follows from Construction 2 and Proposition 1 except for $J = 3$ and $I \leqslant \lfloor h/2 \rfloor$. In the remaining case Construction 2 and Proposition 1 give

$$N_h(1, 3) + I - 1 \leqslant N_h(I, 3) \leqslant h + 1 + I.$$

In [19] it is shown that $N_h(1, 3) = h + 2$. Hence $N_h(I, 3) = h + 1 + I$. □

Our next construction of DDS will be based on a generalization of sonar sequences introduced by Golomb and Taylor [13]. The construction is a generalization of Construction 4 in [17].

Construction 3. *Let* $\bar{a} = (a_1, a_2, \ldots, a_J)$ *be a sequence of nonnegative integers such that*

$$\text{if } \bar{x}, \bar{y} \in C(h, J), \ \sum_{j=1}^{J} x_j a_j = \sum_{j=1}^{J} y_j a_j \ \text{ and } \left| \sum_{j=1}^{J} j x_j - \sum_{j=1}^{J} j y_j \right| < h, \ \text{ then } \bar{x} = \bar{y}. \tag{4}$$

Let

$$I > I_0 = \max\left\{ \sum_{j=1}^{J} x_j a_j - \sum_{j=1}^{J} y_j a_j \,|\, \bar{x}, \bar{y} \in C(h, J), \bar{x} \neq \bar{y} \right\}.$$

Define $\Delta = \{\{a_{ij} \,|\, 1 \leqslant j \leqslant J\} \,|\, 1 \leqslant i \leqslant I\}$ *by*

$$a_{ij} \equiv a_j + i \pmod I, \ (j - 1)I < a_{ij} \leqslant jI.$$

Then Δ *is a perfect* (h, I, J)*-DDS.*

Any (h, J)-DS clearly satisfies (4). In [17] we constructed sequences which satisfy (4) for $h = 2$ and which have a much smaller largest element then any (h, J)-DS. Construction to general h have been discussed in [18].

Already Construction 3 is sufficient to prove the existence of perfect (h, I, J)-DDS for I sufficiently large. Define $\iota_h(J)$ as the minimal integer such that $N_h(I, J) = IJ$ for all $I \geqslant \iota_h(J)$.

Construction 3 proves that $\iota_h(J) \leqslant hN_h(1, J)$. However, our next construction will improve this bound.

Construction 4. *Let v be a positive integer and let $\bar{a} = (a_2, a_3, \ldots, a_J)$ be a sequence such that*

$$\text{if } \bar{x}, \bar{y} \in C(h, J) \quad \text{and} \quad \sum_{j=2}^{J} x_j a_j \equiv \sum_{j=2}^{J} y_j a_j \pmod{v}, \quad \text{then } \bar{x} = \bar{y}.$$

Let $a_1 = a_2$ and let I be a multiple of v. Define

$$\Delta = \{\{a_{ij} \mid 1 \leqslant j \leqslant J\} \mid 1 \leqslant i \leqslant I\}$$

by

$$a_{ij} \equiv a_j + i \pmod{I}, \quad (j-1)I < a_{ij} \leqslant jI.$$

Then Δ is a perfect (h, I, J)-DDS.

Proof. If $i \neq i'$, then

$$a_{ij} - a_{i'j} \equiv i - i' \not\equiv 0 \pmod{I}.$$

Hence $a_{ij} \neq a_{i'j}$. Further, if $j < j'$, then $a_{ij} \leqslant jI \leqslant (j'-1)I < a_{i'j'}$ for all i, i'. Therefore, the elements are distinct. We show that the sums are distinct.

Let $\bar{x}, \bar{y} \in C(h, J)$ be such that $\sum_{j=1}^{J} x_j a_{ij} = \sum_{j=1}^{J} y_j a_{ij}$. Since $a_1 = a_2$ we get

$$0 = \sum_{j=1}^{J} x_j a_{ij} - \sum_{j=1}^{J} y_j a_{ij}$$

$$\equiv (x_1 + x_2)a_2 + \sum_{j=3}^{J} x_j a_j - (y_1 + y_2)a_2 - \sum_{j=3}^{J} y_j a_j \pmod{I}.$$

In particular, this is true modulo v. Hence.

$$x_1 + x_2 = y_1 + y_2 \tag{5}$$

and

$$x_j = y_j \quad \text{for } 3 \leqslant j \leqslant J.$$

Further we note that $a_{i2} = a_{i1} + I$ and so

$$x_1 a_{i1} + x_2(a_{i1} + I) - y_1 a_{i1} - y_2(a_{i1} + I) = \sum_{j=1}^{J} x_j a_{ij} - \sum_{j=1}^{J} y_j a_{ij} = 0.$$

Combining this with (5) we get $x_1 = y_1$ and $x_2 = y_2$. Hence $\bar{x} = \bar{y}$. □

Let $(a_2, a_3, \ldots, a_J)$ be an $(h, J-1)$-DS, where

$$1 = a_2 < a_3 < \cdots < a_J = N_h(1, J-1),$$

and let $v = hN_h(1, J-1) - h + 1$. Then the conditions of Construction 4 are satisfied. Choosing $I = v$, we get the following proposition.

Proposition 4. *For all $J \geqslant 2$ we have*

$$\iota_h(J) \leqslant hN_h(1, J-1) - h + 1.$$

When I is small compared to $N_h(1, J-1)$, perfect (h, I, J)-DDS do not exist. Our final two constructions give some upper bounds on $N_h(I, J)$ for small I.

First, we construct an $(h, q-1, q)$-DDS by a modification of a construction of Bose [4]. The construction is a generalization of the Construction 7 in [17].

Construction 5. *Let q be a prime power, and let $\mathrm{GF}(q) = \{f_0, f_1, \ldots, f_{q-1}\}$ where $f_0 = 0$. Let $p(x) = x^h - \sum_{k=1}^{h} c_k x^{h-k}$ be a primitive polynomial over $\mathrm{GF}(q)$ and α a root of $p(x) = 0$. For each $b \in \mathrm{GF}(q)^* = \mathrm{GF}(q) \setminus \{0\}$, let*

$$\Delta_b = \{a \mid 1 \leqslant a \leqslant q^h - 1,\ \alpha^a - b\alpha \in \mathrm{GF}(q)\}.$$

Then $\{\Delta_b \mid b \in \mathrm{GF}(q)^*\}$ is an $(h, q-1, q)$-DDS.

Proof. Clearly, there are $(q-1)$ sets Δ_b, they are disjoint, and each contains q elements:

a_{bj} may be defined by $\alpha^{a_{bj}} = b\alpha + f_j$

For $\bar{x} \in C(h, q)$ let $\sigma_l(\bar{x})$, $0 \leqslant l \leqslant h$, be defined by

$$\prod_{j=1}^{q} (Z + f_j)^{x_j} = \sum_{l=0}^{h} \sigma_l(\bar{x}) Z^{h-l}.$$

Then $\bar{x}$ is uniquely determined by $(\sigma_0(\bar{x}), \sigma_1(\bar{x}), \ldots, \sigma_h(\bar{x}))$.

We show that the sums are distinct. Let $\bar{x}, \bar{y} \in C(h, q)$ and suppose that $\sum_{j=1}^{q} x_j a_{bj} = \sum_{j=1}^{q} y_j a_{bj}$.

Then

$$0 = \alpha^{\sum_{j=1}^{q} x_j a_{bj}} - \alpha^{\sum_{j=1}^{q} y_j a_{bj}}$$

$$= \prod_{j=1}^{q} (b\alpha + f_j)^{x_j} - \prod_{j=1}^{q} (b\alpha + f_j)^{y_j}$$

$$= \sum_{l=0}^{h} \{\sigma_l(\bar{x}) - \sigma_l(\bar{y})\} (b\alpha)^{h-l}.$$

Since α is a primitive root of a polynomial of degree h and $\sigma_0(\bar{x}) = (\sigma_0(\bar{y}) = 1$, this implies that $\sigma_l(\bar{x}) = \sigma_l(\bar{y})$ for $0 \leqslant l \leqslant h$. Hence $\bar{x} = \bar{y}$. □

Proposition 5. *If q is a prime power, then $N_h(q-1, q) \leqslant q^h - 1$.*

Construction 6. *Let $A = \{a_1, a_2, \ldots, a_n\}$ be a set of integers and v an integer such that if*

$$\bar{x}, \bar{y} \in C(h, n) \quad \textit{and} \quad \sum_{j=1}^{n} x_j a_j \equiv \sum_{j=1}^{n} y_j a_j \ (\mathrm{mod}\ v), \textit{ then } \bar{x} = \bar{y}.$$

For $\bar{b} \in C(h-1, n)$ *and* $1 \leqslant i \leqslant n$ *let*

$$a(\bar{b}, i) \equiv a_i - \sum_{j=1}^{n} b_j a_j \pmod{v}, \quad 0 \leqslant a(\bar{b}, i) \leqslant v - 1,$$

$$\Gamma(\bar{b}) = \{a(\bar{b}, i) \mid b_i = 0\},$$

$$s(\bar{b}) = \#\{j \mid b_j \neq 0\}.$$

Note that $\#\Gamma(\bar{b}) = n - s(\bar{b})$. *Let* r *be such that* $1 \leqslant r \leqslant \min\{h-1, n-1\}$ *and let*

$$\Delta = \{\Delta(\bar{b}) \mid s(\bar{b}) \leqslant r\}$$

where $\Delta(\bar{b})$ *is any subset of* $\Gamma(\bar{b})$ *containing* $n - r$ *elements.*

Then Δ *is an* $(h, I_r, n-r)$*-DDS and* $v(\Delta) \leqslant v - 1$, *where*

$$I_r = \sum_{k=1}^{r} \binom{n}{k} \binom{h-2}{k-1}.$$

Proof. First we show that the elements are distinct. Suppose that $a(\bar{b}, i) = a(\bar{c}, l)$. Then

$$a_i - \sum_{j=1}^{n} b_j a_j \equiv a_l - \sum_{m=1}^{n} c_m a_m \pmod{v}$$

and so

$$a_i + \sum_{m=1}^{n} c_m a_m \equiv a_l + \sum_{j=1}^{n} b_j a_j \pmod{v}.$$

Since $b_i = 0$ this is possible only if $l = i$ and $c_j = b_j$ for all j.

Next consider sums of elements of $\Delta(\bar{b})$. If $\bar{x}, \bar{y} \in C(h, n)$ and

$$\sum_{i=1}^{n} x_i a(\bar{b}, i) = \sum_{i=1}^{n} y_i a(\bar{b}, i)$$

then

$$\begin{aligned}
\sum_{i=1}^{n} x_i a_i &= \sum_{i=1}^{n} x_i \left(a_i - \sum_{j=1}^{n} b_j a_j \right) + h \sum_{j=1}^{n} b_j a_j \\
&\equiv \sum_{i=1}^{n} x_i a(\bar{b}, i) + h \sum_{j=1}^{n} b_j a_j \\
&\equiv \sum_{i=1}^{n} y_i a(\bar{b}, i) + h \sum_{j=1}^{n} b_j a_j \\
&\equiv \sum_{i=1}^{n} y_i \left(a_i - \sum_{j=1}^{n} b_j a_j \right) + h \sum_{j=1}^{n} b_j a_j \\
&\equiv \sum_{i=1}^{n} y_i a_i \pmod{v}
\end{aligned}$$

and so $x_i = y_i$ for all i.

Since $\Delta(\bar{b})$ contains $n-r$ elements, this proves that Δ is an $(h, I_r, n-r)$-DDS where

$$I_r = \sum_{k=1}^{r} \#\{\bar{b} \mid s(\bar{b}) = k\}$$

$$= \sum_{k=1}^{r} \sum_{a_{i_1}, a_{i_2}, \ldots, a_{i_k}} \#C(h-1, k)$$

$$= \sum_{k=1}^{r} \binom{n}{k}\binom{h-2}{k-1}.$$

Finally, $1 \leqslant a(\bar{b}, i) \leqslant v-1$ for all i such that $a(\bar{b}, i) \in \Delta(\bar{b})$. Hence $v(\Delta) \leqslant v-1$. □

Proposition 6. *If q is a prime power, then*

(i) *For $1 \leqslant r \leqslant \min\{h-1, q-1\}$ we have*

$$N_h\left(\sum_{k=1}^{r} \binom{q}{k}\binom{h-2}{k-1}, q-r\right) \leqslant q^h - 2.$$

(ii) *For $1 \leqslant r \leqslant \min\{h-1, q\}$ we have*

$$N_h\left(\sum_{k=1}^{r} \binom{q+1}{k}\binom{h-2}{k-1}, q+1-r\right) \leqslant \frac{q^{h+1}-1}{q-1} - 1.$$

Proof. Bose and Chowla [5] proved that there exist sets A with the modular distinct sum property assumed in Construction 6 when q is a prime power and $n = q$, $v = q^h - 1$ and also $n = q+1$, $v = (q^{h+1}-1)/(q-1)$. The bounds on H given in Proposition 6 therefore follows from Construction 6. □

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