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Finite-Amplitude Disturbances in Self-Gravitating Media. II*

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This paper deals with finite-amplitude axisymmetric disturbances in a self-gravitating fluid column of finite radius R. It is shown that the cutoff wavelength λ_{nl} above which gravitational breakup occurs now depends on the relative amplitude ϵ/R of the initial perturbation. Actually, for small- but finite-amplitude disturbances, $\lambda_{nl} = \lambda_l \; (1 - 0.34368\, \epsilon^2/R^2)$, where λ_l $(=5.8898R)$ designates the cutoff wavelength predicted in the linear approximation.

I. INTRODUCTION

The response of a self-gravitating, incompressible cylinder to small axisymmetric disturbances was investigated by Chandrasekhar and Fermi [2] by means of an energy principle. Soon afterwards, Oganesian [9] was the first to perform a detailed normal mode analysis for both axisymmetric and non-axisymmetric perturbations. These *linearized* results show that a breakup occurs when the wavelengths of axisymmetric deformations exceed (approximately) the circumference of the cylinder; the latter is stable for all non-axisymmetric disturbances (cf. also Chandrasekhar [1]).

The foregoing question also resembles the problem of the breakup of a liquid jet held together in its equilibrium position by capillary forces. More recent experiments were performed by Crane, Birch, and McCormack [3], and Donnelly and Glaberson [4]. A qualitative agreement was found between the experimental results and the linearized theory devised by Rayleigh [10]. However, quasiperiodic flow patterns were observed, thus indicating that *finite-amplitude* motions should be considered. A recent work by Nayfeh [8] shows that this is indeed the case.

The main purpose of the present paper is to determine, for a self-gravitating cylinder, the nonlinear cutoff wavelength above which no stable flow pattern exists. Actually, this work ensues from our interest in the appearance of

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condensations within astronomical bodies. The nonlinear gravitational instability of a plane–parallel fluid slab of finite thickness was considered at length in a previous paper (Tassoul and Dedic [11], referred to hereinafter as "Paper I"). In this paper, our problem, very much akin to the capillary instability of a liquid jet, is solved by means of a singular perturbation technique devised by Nayfeh [6, 7]. The basic results are summarized in the closing section.

II. Formulation

Consider an infinite cylinder of an incompressible, inviscid fluid. The self-gravitating system, imbedded *in vacuo*, is characterized by a constant material density ρ, and a radius R. Let us now assume that, at $t = 0$ (say), the surface $r = R$ is deformed so that it becomes

$$r = R_0 + \epsilon \cos kz, \tag{1}$$

where z is measured along the axis of symmetry; ϵ and k denote, respectively, the amplitude and the wave number of our initial disturbance (without loss of generality, we can assume that k is positive). By virtue of the conservation of mass, the constant R_0 is related to the equilibrium radius R by the relation

$$R_0^2 + \tfrac{1}{2}\epsilon^2 = R^2. \tag{2}$$

In order to discuss the nonlinear "sausage" instability of the cylindrical configuration, we will now describe the motion of its surface by means of the Lagrangian displacement $\xi(z, t)$. Hence, in view of Eqs. (1) and (2), we can assume that[1]

$$\xi(z, 0) = (1 - \tfrac{1}{2}\epsilon^2)^{1/2} - 1 + \epsilon \cos kz; \tag{3}$$

in addition, we shall let

$$\xi_{,t}(z, 0) = 0, \tag{4}$$

at our arbitrarily chosen initial instant $t = 0$.

By virtue of the foregoing assumptions, we know that at every subsequent instant t (> 0) the motion can be described by a *potential flow*. Hence, the basic equations of hydrodynamics can be rewritten in the form

$$\nabla^2\Phi = 0, \tag{5}$$

$$\nabla^2 V = 1, \qquad \text{and} \qquad \nabla^2 W = 0. \tag{6}$$

[1] From now on, all physical variables are normalized by means of the characteristic length R, and the characteristic time $(4\pi G\rho)^{-1/2}$, where G is the constant of gravitation. In particular, the wave number k is measured in the unit R^{-1}. Moreover, an index after a comma will denote a derivative.

The functions Φ, V, and W denote, respectively, the velocity potential, and the (internal and external) gravitational potentials (cf. footnote 1). The fluid velocity can be derived at once from Φ; the pressure is given by the Bernoulli theorem.

Furthermore, we must impose appropriate boundary conditions; they are given by

$$\Phi_{,t} + \tfrac{1}{2}(\Phi_{,r}^2 + \Phi_{,z}^2) + V = 0, \tag{7}$$

$$V_{,r} = W_{,r}, \qquad \text{and} \qquad V_{,z} = W_{,z}, \tag{8}$$

and

$$\xi_{,t} + \xi_{,z}\Phi_{,z} - \Phi_{,r} = 0, \tag{9}$$

on the *variable* surface $r = 1 + \xi(z, t)$. Equation (7) imposes the vanishing of the pressure on the moving boundary. Continuity of gravity is provided by conditions (8). Finally, Eq. (9) expresses the fact that the limiting surface retains its material identity in the course of time.

III. Perturbation Technique Near the Linear Cutoff

Let us first mention that by linearizing the foregoing equations we recover the well-known solution

$$\xi(z, t) = \epsilon \cos \sigma_0 t \cos kz, \tag{10}$$

where

$$\sigma_0^2(k) = k(I_1(k)/I_0(k))\,[\tfrac{1}{2} - I_0(k)\,K_0(k)], \tag{11}$$

which introduces modified Bessel functions. It is apparent from Eq. (11) that stable flows only exist when $k \geqslant k_l$, where k_l is solution of the transcendental equation

$$\tfrac{1}{2} - I_0(k_l)\,K_0(k_l) = 0, \tag{12}$$

viz., $k_l = 1.0668$. On the contrary, when $k < k_l$, we have $\sigma_0^2 < 0$; and, hence, exponentially growing motions occur. This can most readily be seen from the fact that, in the immediate vicinity of the linear cutoff,

$$\sigma_0^2(k) = a(k - k_l) + O[(k - k_l)^2], \tag{13}$$

where $a = 0.23434$; to be specific, we have

$$a = k_l[I_1(k_l)\,K_1(k_l) - \tfrac{1}{2}(I_1^2(k_l)/I_0^2(k_l))]. \tag{14}$$

These results were established by Chandrasekhar and Fermi [2].

If exception is made for geometric differences, the results obtained in Paper I (Appendix A) can easily be extended to show that, in the nonlinear regime, quasiperiodic flows always occur in the range $k > k_l + O(\epsilon^2)$; by the same token, no small-amplitude motion exists, when $k < k_l - O(\epsilon^2)$ and $\epsilon \neq 0$. However, no firm inference can be drawn in the range $| k - k_l | \lesssim O(\epsilon^2)$; henceforth, particular interest is attached to the behavior of the system in the latter domain.

Let us now investigate small—but finite—amplitude flows in the range $| k - k_l | \lesssim O(\epsilon^2)$. Second-order asymptotic expansions can be obtained by means of the multiple time scales method [6, 7]. We can first magnify the domain by putting

$$\alpha(k) = (k - k_l)/\epsilon^2, \tag{15}$$

which is now of $O(1)$. In addition, let us introduce the *two* time scales $t_0 = t$ and $t_1 = \epsilon t$. If Ψ denotes an arbitrary function of time, we thus have

$$\Psi_{,t} = \Psi_{,0} + \epsilon\Psi_{,1}\,. \tag{16}$$

(In the remainder of the present paper, indices "0" and "1" after a comma designate a derivative with respect to t_0 and t_1, respectively). Finally, it is convenient to let

$$y = kz = (k_l + \alpha\epsilon^2)\, z. \tag{17}$$

In agreement with our perturbation technique, we will next assume that

$$\xi = \epsilon\xi_1 + \epsilon^2\xi_2 + \epsilon^3\xi_3 + \cdots, \tag{18}$$

$$\Phi = -\tfrac{1}{4}\, t_0 + \epsilon\Phi_1 + \epsilon^2\Phi_2 + \epsilon^3\Phi_3 + \cdots, \tag{19}$$

$$V = \tfrac{1}{4}\, r^2 + \epsilon V_1 + \epsilon^2 V_2 + \epsilon^3 V_3 + \cdots, \tag{20}$$

and

$$W = \tfrac{1}{2}\log r + \epsilon W_1 + \epsilon^2 W_2 + \epsilon^3 W_3 + \cdots, \tag{21}$$

where all functions depend on r, y, t_0, and t_1. If we now insert expansions (18)–(21) into Eqs. (3)–(6), and equate the coefficients of equal powers of ϵ, we obtain a set of equations that can be solved successively. As far as conditions (7)–(9) are concerned, it is convenient to relate their values on $r = 1 + \xi(z, t)$ to the surface $r = 1$ by means of various Taylor expansions.

a. *First-order equations* (order ϵ)

We have

$$\xi_1(y, 0) = \cos y, \qquad \text{and} \qquad \xi_{1,0}(y, 0) = 0; \tag{22}$$

$$\Phi_{1,rr} + (1/r)\,\Phi_{1,r} + k_l^2\Phi_{1,yy} = 0, \tag{23}$$

and two equations similar to Eq. (23), in which Φ_1 is replaced, successively, by V_1 and W_1; finally, on $r = 1$, we have

$$\xi_{1,0} - \Phi_{1,r} = 0, \tag{24}$$

$$\Phi_{1,0} + V_1 + \tfrac{1}{2}\,\xi_1 = 0, \tag{25}$$

$$V_{1,y} - W_{1,y} = 0, \tag{26}$$

and

$$V_{1,r} - W_{1,r} + \xi_1 = 0. \tag{27}$$

b. *Second-order equations* (order ϵ^2)

We have

$$\xi_2(y, 0) = -\tfrac{1}{4}, \qquad \text{and} \qquad \xi_{2,0}(y, 0) = -\xi_{1,1}(y, 0); \tag{28}$$

$$\Phi_{2,rr} + (1/r)\,\Phi_{2,r} + k_l^2\Phi_{2,yy} = 0, \tag{29}$$

and two equations similar to Eq. (29), in which Φ_2 is replaced, successively, by V_2 and W_2; finally, on $r = 1$, we have

$$\xi_{2,0} - \Phi_{2,r} = -\xi_{1,1} + \xi_1\Phi_{1,rr} - k_l^2\xi_{1,y}\Phi_{1,y}\,, \tag{30}$$

$$\Phi_{2,0} + V_2 + \tfrac{1}{2}\,\xi_2 = -\Phi_{1,1} - \xi_1 V_{1,r} - \tfrac{1}{4}\,\xi_1{}^2 - \xi_1\Phi_{1,0r} - \tfrac{1}{2}\,\Phi_{1,r}^2 - \tfrac{1}{2}\,k_l{}^2\Phi_{1,y}^2\,, \tag{31}$$

$$V_{2,y} - W_{2,y} = -\xi_1 V_{1,ry} + \xi_1 W_{1,ry}\,, \tag{32}$$

and

$$V_{2,r} - W_{2,r} + \xi_2 = -\xi_1 V_{1,rr} + \xi_1 W_{1,rr} + \tfrac{1}{2}\,\xi_1{}^2. \tag{33}$$

c. *Third-order equations* (order ϵ^3)

We have

$$\xi_3(y, 0) = 0, \qquad \text{and} \qquad \xi_{3,0}(y, 0) = -\xi_{2,1}(y, 0); \tag{34}$$

$$\Phi_{3,rr} + (1/r)\,\Phi_{3,r} + k_l{}^2\Phi_{3,yy} = -2\alpha k_l\Phi_{1,yy}\,, \tag{35}$$

$$V_{3,rr} + (1/r)\,V_{3,r} + k_l{}^2 V_{3,yy} = -2\alpha k_l V_{1,yy}\,, \tag{36}$$

and

$$W_{3,rr} + (1/r)\,W_{3,r} + k_l{}^2 W_{3,yy} = -2\alpha k_l W_{1,yy}; \tag{37}$$

finally, on $r = 1$, we must have

$$\xi_{3,0} - \Phi_{3,r} = -\xi_{2,1} - k_l^2\xi_{1,y}\Phi_{2,y} + \xi_1\Phi_{2,rr} - k_l^2\xi_{2,y}\Phi_{1,y} \\ -k_l^2\xi_1\xi_{1,y}\Phi_{1,ry} + \xi_2\Phi_{1,rr} + \tfrac{1}{2}\xi_1^2\Phi_{1,rrr}, \tag{38}$$

$$\Phi_{3,0} + V_3 + \tfrac{1}{2}\xi_3 = -\Phi_{2,1} - \xi_1\Phi_{2,0r} - \xi_1 V_{2,r} - \xi_2 V_{1,r} - \tfrac{1}{2}\xi_1^2 V_{1,rr} \\ - \tfrac{1}{2}\xi_1\xi_2 - \xi_1\Phi_{1,1r} - \xi_2\Phi_{1,0r} - \tfrac{1}{2}\xi_1^2\Phi_{1,0rr} - \Phi_{1,r}\Phi_{2,r} \\ - \xi_1\Phi_{1,r}\Phi_{1,rr} - k_l^2\Phi_{1,y}\Phi_{2,y} - k_l^2\xi_1\Phi_{1,y}\Phi_{1,ry}, \tag{39}$$

$$V_{3,y} - W_{3,y} = -\xi_1 V_{2,ry} + \xi_1 W_{2,ry} - \xi_2 V_{1,ry} + \xi_2 W_{1,ry} \\ - \tfrac{1}{2}\xi_1^2 V_{1,rry} + \tfrac{1}{2}\xi_1^2 W_{1,rry}, \tag{40}$$

and

$$V_{3,r} - W_{3,r} + \xi_3 = -\xi_1 V_{2,rr} + \xi_1 W_{2,rr} - \xi_2 V_{1,rr} + \xi_2 W_{1,rr} \\ - \tfrac{1}{2}\xi_1^2 V_{1,rrr} + \tfrac{1}{2}\xi_1^2 W_{1,rrr} + \xi_1\xi_2 - \tfrac{1}{2}\xi_1^3. \tag{41}$$

Now, given the results derived in Appendix B of Paper I (cf. also Nayfeh [8]), we know that the second-order asymptotic expansions should be of the form

$$\xi \simeq \epsilon\theta_1(t_1)\cos y + \epsilon^2\{[\theta_{20}(t_0) + \theta_{21}(t_1)]\cos 2y + \theta_\lambda(t_1)\}, \tag{42}$$

$$\Phi \simeq -\tfrac{1}{4}t_0 + \epsilon\Psi_1(t_1) + \epsilon^2[\Phi_{20}(t_0)\cos 2y(I_0(2k_lr)/2k_lI_1(2k_l)) \\ + \Phi_{21}(t_1)\cos y(I_0(k_lr)/k_lI_1(k_l)) + \Psi_2(t_0)], \tag{43}$$

$$V \simeq \tfrac{1}{4}r^2 + \epsilon\mathscr{V}_1(t_1)\cos y(I_0(k_lr)/I_0(k_l)) \\ + \epsilon^2[\mathscr{V}_{20}(t_0) + \mathscr{V}_{21}(t_1)]\cos 2y(I_0(2k_lr)/I_0(2k_l)), \tag{44}$$

and

$$W \simeq \tfrac{1}{2}\log r + \epsilon\mathscr{W}_1(t_1)\cos y(K_0(k_lr)/K_0(k_l)) \\ + \epsilon^2[\mathscr{W}_{20}(t_0) + \mathscr{W}_{21}(t_1)]\cos 2y(K_0(2k_lr)/K_0(2k_l)). \tag{45}$$

Let us first consider Eqs. (23)–(27). In making use of Eq. (12), we obtain

$$\mathscr{V}_1(t_1) = \mathscr{W}_1(t_1) = -\tfrac{1}{2}\theta_1, \tag{46}$$

in which θ_1 remains, for the time being, an arbitrary function of t_1. By virtue of Eqs. (22) and (28), we must also have

$$\theta_1(0) = 1, \qquad \text{and} \qquad \theta_{1,1}(0) = 0. \tag{47}$$

Equations (28)–(33) imply next that

$$\theta_{20}(t_0) = -\beta \cos \omega t_0 , \tag{48}$$

$$\theta_{21}(t_1) = \beta \theta_1^2, \tag{49}$$

$$\theta_2(t_1) = -\tfrac{1}{4}\, \theta_1^2, \tag{50}$$

$$\Phi_{20}(t_0) = \beta\omega \sin \omega t_0 , \tag{51}$$

and

$$\Phi_{21}(t_1) = \theta_{1,1} , \tag{52}$$

where

$$\beta = (1/4\omega^2)\{2k_l(I_1(2k_l)/I_0(2k_l))\,[k_l(I_1(k_l)/I_0(k_l)) - 2k_l I_0(2k_l)\,K_1(2k_l)] - \omega^2\}, \tag{53}$$

and

$$\omega^2 = 2k_l(I_1(2k_l)/I_0(2k_l))\,[\tfrac{1}{2} - I_0(2k_l)\,K_0(2k_l)] \tag{54}$$

(viz., $\beta = -0.37367$ and $|\,\omega\,| = 0.62838$). We also obtain

$$\Psi_{2,0} = -\Psi_{1,1} + \tfrac{1}{4}\, k_l(I_1(k_l)/I_0(k_l))\, \theta_1^2; \tag{55}$$

note, however, that the functions Ψ_2 and Ψ_1 play no important role in the present analysis, for Φ is a velocity potential. Finally, we can write

$$\mathscr{V}_{20}(t_0) = \mathscr{W}_{20}(t_0) = \beta I_0(2k_l)\, K_0(2k_l) \cos \omega t_0 , \tag{56}$$

$$\begin{aligned} \mathscr{V}_{21}(t_1) &= \mathscr{W}_{21}(t_1) + \tfrac{1}{4}\, \theta_1^2 \\ &= \tfrac{1}{4}\, I_0(2k_l)\, K_0(2k_l)\, [2k_l(K_1(2k_l)/K_0(2k_l)) - (4\beta + 1)]\, \theta_1^2. \end{aligned} \tag{57}$$

If we now insert Eqs. (46)–(57) into expansions (42)–(45), we see that all solutions can be related to the single unknown function $\theta_1(t_1)$. The latter can be obtained by requiring that expansions (42)–(45) be *genuine* second-order asymptotic series. Therefore, in order to eliminate secular terms in neglected quantities, we must eventually resort to the third-order Eqs. (34)–(41).

A close inspection of Eqs. (42)–(45) shows that ξ_3, Φ_3, V_3, and W_3 (cf. Eqs. (18)–(21)) contain terms which are proportional to $\cos y$ and $\cos 3y$ (Φ_3 also includes a term which is proportional to $\cos 2y$). However, for the purpose of eliminating secular terms, it is only necessary to consider those expressions which are proportional to $\cos y$. Thus, we can write

$$\xi_3 = \theta_3(t_0 , t_1) \cos y + \cdots, \tag{58}$$

$$\Phi_3 = \Psi_3(t_0 , t_1) \cos y(I_0(k_l r)/k_l I_1(k_l)) + \cdots, \tag{59}$$

$$V_3 = [\mathscr{V}_3(t_0 , t_1)\,(I_0(k_l r)/I_0(k_l)) - (\alpha/2)\, \theta_1(t_1)\, r(I_1(k_l r)/I_0(k_l))] \cos y + \cdots, \tag{60}$$

and

$$W_3 = [\mathscr{W}_3(t_0, t_1)(K_0(k_l r)/K_0(k_l)) + (\alpha/2)\,\theta_1(t_1)\,r(K_1(k_l r)/K_0(k_l))] \cos y + \cdots, \tag{61}$$

(cf., in particular, Eqs. (36) and (37)). Equations (34)–(41) will now be used to express the fact that the foregoing unknown functions are free of terms proportional to t_0. (It can be shown that ignored quantities in Eqs. (58)–(61) do *not* contain secular terms.)

Equation (40) implies that

$$\mathscr{V}_3 - \mathscr{W}_3 = \tfrac{1}{8}(4\beta - 1)\,\theta_1^3 + (\alpha/k_l)\,\theta_1 + f(t_0), \tag{62}$$

where $f(t_0)$ includes all terms which do not solely depend on t_1 (cf. Eqs. (48) and (51)). From Eq. (41), we obtain the relation

$$\begin{aligned} &k_l(I_1(k_l)/I_0(k_l))\,\mathscr{V}_3 + k_l(K_1(k_l)/K_0(k_l))\,\mathscr{W}_3 + \theta_3 \\ &\quad = -\tfrac{1}{8}(4\beta - 2 + k_l^2)\,\theta_1^3 + g(t_0); \end{aligned} \tag{63}$$

similarly, the function $g(t_0)$ contains all quantities depending on t_0, and which are irrelevant to the present discussion. If we now combine Eqs. (62) and (63) to eliminate $\mathscr{W}_3$, we immediately find

$$\begin{aligned} \mathscr{V}_3 + \tfrac{1}{2}\theta_3 &= \tfrac{1}{2} k_l(K_1(k_l)/K_0(k_l))\,[\tfrac{1}{8}(4\beta - 1)\,\theta_1^3 + (\alpha/k_l)\,\theta_1] \\ &\quad - \tfrac{1}{16}(4\beta - 2 + k_l^2)\,\theta_1^3 + p(t_0), \end{aligned} \tag{64}$$

where $p(t_0)$ is a combination of $f(t_0)$ and $g(t_0)$. Finally, by virtue of Eqs. (39) and (64), we obtain

$$\Psi_{3,0} = -(\theta_{1,11} + \alpha a\theta_1 - 2b^2\theta_1^3) + q(t_0), \tag{65}$$

in which all circular functions of t_0 are contained in $q(t_0)$. The constant a was already defined in Eq. (14); we also have that $b = 0.20726$, i.e.,

$$\begin{aligned} b^2 = &-\tfrac{1}{16} k_l(4\beta - 1)\,a + \tfrac{1}{8} k_l^2(I_1(k_l)/I_0(k_l)) \\ &\times [k_l - (I_1(k_l)/I_0(k_l)) - 2k_l I_1(2k_l)\,K_1(2k_l) + (4\beta + 1)\,I_1(2k_l)\,K_0(2k_l)]. \end{aligned} \tag{66}$$

It can now readily be seen from Eq. (65) that, in order to remove secular terms from Ψ_3 (as from θ_3, $\mathscr{V}_3$, and $\mathscr{W}_3$), we must impose the condition

$$\theta_{1,11} = 2b^2(\theta_1^3 - u\theta_1), \tag{67}$$

where

$$u = \alpha/\alpha_c, \qquad \text{and} \qquad \alpha_c = 2b^2/a, \tag{68}$$

viz., $\alpha_c = 0.36663$. In view of conditions (47), a first integral of Eq. (67) can be written down at once. We obtain

$$\theta_{1,1}^2 = b^2(\theta_1{}^2 - 1)(\theta_1{}^2 + 1 - 2u), \tag{69}$$

which must be solved with the condition that $\theta_1(0) = 1$. Also, on the outside of the domain $| k - k_l | \lesssim O(\epsilon^2)$, $\theta_1(\epsilon t)$ must tend toward the function $\cos \sigma_0 t$ (cf. Eq. (13), and Paper I).

IV. Discussion

To sum up, near the linear cutoff, the distortion of the cylindrical surface can be brought to the form

$$\xi(z, t) \simeq \epsilon\theta_1(\epsilon t) \cos kz + \epsilon^2\{\beta[\theta_1{}^2(\epsilon t) - \cos \omega t] \cos 2kz - \tfrac{1}{4}\, \theta_1{}^2(\epsilon t)\}, \tag{70}$$

with an error of $O(\epsilon^3)$; $\theta_1(\epsilon t)$ designates *one* particular solution of Eq. (69).

Let us first note that, in view of Eq. (69), no small amplitude motion exists when $u < 1$ ($\alpha < \alpha_c$). Indeed, in the latter case, θ_1 always grows from its initial value one. On the contrary, when $u \geqslant 1$ ($\alpha \geqslant \alpha_c$), $\theta^2 \leqslant 1$; and, then, the cylindrical system merely oscillates about its position of equilibrium. Therefore, the value $u = 1$ separates stable from unstable flows. Hence, if we return to definitions (15) and (68), we see that the wave number

$$k_{nl} = k_l(1 + 0.34368\epsilon^2) \tag{71}$$

defines the nonlinear cutoff, when allowance is made for small—but finite—amplitude motions. Nonlinearity thus slightly *reduces* the critical wavelength beyond which no stable flow pattern can be maintained. For growing motions, however, expansion (70) is only valid for a finite period of time.

Equation (69) can easily be solved by means of Jacobian elliptic functions (cf., e.g., Jeffreys and Jeffreys [5]). By making use of the usual compact notation, we can express the function $\theta_1(\epsilon t)$ as follows: in the range $u \geqslant 1$, we get

$$\theta_1(\epsilon t) = \mathrm{cd}(\tau_1, \kappa_1), \tag{72}$$

where

$$\tau_1 = \epsilon b(2u - 1)^{1/2}\, t, \qquad \text{and} \qquad \kappa_1 = (2u - 1)^{-1/2}; \tag{73}$$

in the domain $\frac{1}{2} \leqslant u \leqslant 1$, we have

$$\theta_1(\epsilon t) = \mathrm{dc}(\tau_2, \kappa_2), \tag{74}$$

in which

$$\tau_2 = \epsilon bt, \quad \text{and} \quad \kappa_2 = (2u - 1)^{1/2}; \tag{75}$$

finally, when $u \leqslant \frac{1}{2}$,

$$\theta_1(\epsilon t) = \mathrm{nc}(\tau_3, \kappa_3), \tag{76}$$

where

$$\tau_3 = \epsilon b(2 - 2u)^{1/2}\, t, \quad \text{and} \quad \kappa_3 = ((1 - 2u)/(2 - 2u))^{1/2}. \tag{77}$$

Evidently, solutions (72) and (74) both reduce to $\theta_1(\epsilon t) \equiv 1$ when $u = 1$. Moreover, solutions (74) and (76) become equal (as they should) when $u = \frac{1}{2}$. Finally, in the limits $u \to +\infty$ and $u \to -\infty$, Eqs. (72) and (76) reduce to $\cos \sigma_0 t$ and $\cosh |\sigma_0|\, t$, respectively (cf. Eqs. (13), (15), and (68)).

In Fig. 1, we illustrate the behavior of $\theta_1(\epsilon t)$ for different values of k. Note in

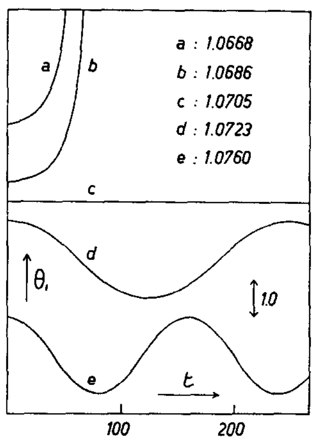

FIG. 1. The function $\theta_1(\epsilon t)$ for some values of k, when $\epsilon = 0.1$. The origin of the vertical scale is defined by the fact that, for each solution, $\theta_1(0) = 1$. The curves correspond to the different values of k listed on the figure. Unstable solutions labeled "a" and "b" become invalid when $t \gtrsim 50$ and $t \gtrsim 60$, respectively.

particular that, in view of Eqs. (70) and (71), the value $k = k_l$ already corresponds to a time-dependent, unstable flow (in the linear approximation, at $k = k_l$, the motion would then not depend on time; cf. Eq. (10) with $\sigma_0 = 0$). Also, when $k = k_{nl}$, solution (70) becomes

$$\xi(z, t) = \xi(z, 0) + \epsilon^2\beta(1 - \cos \omega t) \cos 2kz + O(\epsilon^3). \tag{78}$$

Thus, the nonlinear marginal state $k = k_{nl}$ depends on time. This is in sharp contrast with the results of the linearized theory.

To conclude, it is also worth mentioning that the configuration exhibits a quasiperiodical behavior, when $k \geqslant k_{nl}(u \geqslant 1)$. This typical nonlinear feature is due to the existence of *two* distinct periods (i.e., $P_\omega = 2\pi/\omega$, and $P_\theta = 4K(\kappa_1)/\epsilon b(2u - 1)^{1/2}$, where K denotes the complete elliptic integral of the first kind) which are *not* commensurable. The same conclusion holds true for wave numbers well above the nonlinear cutoff.

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