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Algebraic series and valuation rings over nonclosed fields

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Abstract

Suppose that k is an arbitrary field. Let $k[[x_1, \ldots, x_n]]$ be the ring of formal power series in n variables with coefficients in k. Let $\overline{k}$ be the algebraic closure of k and $\sigma \in \overline{k}[[x_1, \ldots, x_n]]$. We give a simple necessary and sufficient condition for σ to be algebraic over the quotient field of $k[[x_1, \ldots, x_n]]$. We also characterize valuation rings V dominating an excellent Noetherian local domain R of dimension 2, and such that the rank increases after passing to the completion of a birational extension of R. This is a generalization of the characterization given by M. Spivakovsky [Valuations in function fields of surfaces, Amer. J. Math. 112 (1990) 107–156] in the case when the residue field of R is algebraically closed.

MSC: 13F25; 13F30

1. Introduction

Suppose that k is an arbitrary field. Consider the field $k((x_1, \ldots, x_n))$, which is the quotient field of the ring $k[[x_1, \ldots, x_n]]$ of formal power series in the variables $x_1, \ldots, x_n$, with coefficients in k. Suppose that $\overline{k}$ is an algebraic closure of k, and $\sigma \in \overline{k}[[x_1, \ldots, x_n]]$ is a formal power series. In this paper, we give a very simple necessary and sufficient condition for σ to be algebraic over $k((x_1, \ldots, x_n))$. We prove the following theorem, which is restated in an equivalent formulation in Theorem 3.2.

Theorem 1.1. *Suppose that k is a field of characteristic $p \geq 0$, with algebraic closure $\overline{k}$. Suppose that*

$$\sigma(x_1, \ldots, x_n) = \sum_{i_1, \ldots, i_n \in \mathbb{N}} \alpha_{i_1, \ldots, i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \in \overline{k}[[x_1, \ldots, x_n]]$$

where $\alpha_{i_1, \ldots, i_n} \in \overline{k}$ for all i. Let

$$L = k(\{\alpha_{i_1, \ldots, i_n} \mid i_1, \ldots, i_n \in \mathbb{N}\})$$

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be the extension field of k generated by the coefficients of $\sigma(x_1,\ldots,x_n)$. Then $\sigma(x_1,\ldots,x_n)$ is algebraic over $k((x_1,\ldots,x_n))$ if and only if there exists $r\in\mathbb{N}$ such that $[kL^{p^r}:k]<\infty$, where kL^{p^r} is the compositum of k and L^{p^r} in $\overline{k}$.

In the case that L is separable over k (Corollary 3.4), or that k is a finitely generated extension field of a perfect field (Corollary 3.3), we have a stronger condition. In these cases, σ is algebraic if and only if $[L:k]<\infty$. The finiteness condition $[L:k]<\infty$ does not characterize algebraic series over arbitrary base fields k of positive characteristic. To illustrate this, we give a simple example, in Example 2.3, of an algebraic series in one variable for which $[L:k]=\infty$.

In Section 2, we prove Theorem 1.1 in the case $n=1$. The most difficult part of the proof arises when k is not perfect. Our proof uses the theorem of resolution of singularities of a germ of a plane curve singularity over an arbitrary field (cf. [1,20,8]). In Section 3, we prove Theorem 1.1 for any number of variables n. The proof involves induction on the number of variables, and uses the result for one variable proven in Section 2.

In the case when k has characteristic zero and $n=1$, the conclusions of Theorem 1.1 are classical. We recall the very strong known results, under the assumption that k has characteristic zero, and there is only one variable ($n=1$). The algebraic closure of the field of formal meromorphic power series $k((x))$ in the variable x is

$$\overline{k((x))}=\cup_F\bigcup_{n=1}^{\infty}F((x^{\frac{1}{n}})) \tag{1}$$

where F is any finite field extension of k contained in the algebraic closure $\overline{k}$ of k. The equality (1) is stated and proven in Ribenboim and Van den Dries' article [24]. A proof can also be deduced from Abhyankar's Theorem ([3] or Section 2.3 of [11]). The equality (1) already follows for an algebraically closed field k of characteristic zero from a classical algorithm of Newton [6,8].

If k has characteristic $p>0$, then the algebraic closure of $k((x))$ is much more complicated, even when k is algebraically closed, because of the existence of Artin Schreier extensions, as is shown in Chevalley's book [7]. In fact, the series

$$\sigma(x)=\sum_{i=1}^{\infty}x^{1-\frac{1}{p^i}}, \tag{2}$$

considered by Abhyankar in [4], is algebraic over $k((x))$, as it satisfies the relation

$$\sigma^p-x^{p-1}\sigma-x^{p-1}=0.$$

When k is an algebraically closed field of arbitrary characteristic, the "generalized power series" field $k((x^{\mathbb{Q}}))$ is algebraically closed, as is shown by Ribenboim in [23]. The approach of studying the algebraic closure of $k((x))$ through generalized power series is developed by Benhessi [5], Hahn [12], Huang, [15], Poonen [21], Rayner [22], Stefanescu [26] and Vaidya [28]. A complete solution when k is a perfect field is given by Kedlaya in [16]. He shows that the algebraic closure of $k((x))$ consists of all "twist recurrent series" $u=\sum\alpha_ix^i$ in $\overline{k}((x^{\mathbb{Q}}))$ such that all α_i lie in a common finite extension of k.

When $n>1$, the algebraic closure of $k((x_1,\ldots,x_n))$ is known to be extremely complicated, even when k is algebraically closed of characteristic 0. In this case, difficulties occur when the ramification locus of a finite extension is very singular. There is a good understanding in some important cases, such as when the ramification locus is a simple normal crossings divisor and the characteristic of k is 0 or the ramification is tame (Abhyankar [3], Grothendieck and Murre [11]) and for quasi-ordinary singularities (Lipman [18], González-Pérez [10]).

More generally, subrings of a power series ring can be very complex, and are a source of many extraordinary examples, such as [19,25,14].

As an application of our methods, we give a characterization of valuation rings V which dominate an excellent, Noetherian local domain R of dimension two, and such that the rank increases after passing to the completion of a birational extension of R. The characterization is known when the residue field of R is algebraically closed (Spivakovsky [27]). In this case (R/m_R algebraically closed) the rank increases under completion if and only if $\dim_R(V)=0$ (V/m_V is algebraic over R/m_R) and V is discrete of rank 1.

However, the characterization is more subtle over nonclosed fields. In Theorem 4.2, we show that the condition that the rank increases under completion is characterized by the two conditions that the residue field of V is finite over the residue field of R, and that V is discrete of rank 1. The case when the residue field of V is infinite algebraic over the residue field of R and the value group is discrete of rank 1 can occur, and the rank of such a valuation does not increase when passing to completion. In Corollary 4.3, we show that there exists a valuation ring V dominating R whose value group is discrete of rank 1 with $\dim_R(V) = 0$ such that the rank of V does not increase under completion if and only if the algebraic closure of R/m_R has infinite degree over R/m_R.

We point out the contrast of the conclusions of Theorem 1.1 with the results of Section 4. The finiteness condition $[L : k] < \infty$ of the coefficient field of a series over a base field k does not characterize algebraicity of a series in a positive characteristic, while the corresponding finiteness condition on residue field extensions does characterize algebraicity (the increase of rank) in the case of valuations dominating a local ring of Theorem 4.2. We illustrate this distinction in Example 4.4 by constructing the valuation ring determined by the series of Example 2.3. We conclude by showing a simple standard power series representation of the valuation associated to the algebraic series of (2), whose exponents do not have bounded denominators.

The concept of the rank increasing when passing to the completion already appears implicitly in Zariski's paper [29]. Some papers where the concept is developed are [27,13,9].

If R is a local (or quasi local) ring, we will denote its maximal ideal by m_R.

2. Series in one variable

Lemma 2.1. *Suppose that R is a two-dimensional regular local ring, and $x \in m_R$ is part of a regular system of parameters.*

Suppose that k_0 is a coefficient field of $\hat{R}$ and $y \in \hat{R}$ is such that x, y are regular parameters in $\hat{R}$. This determines an isomorphism

$$\hat{R} \overset{\lambda_0}{\to} k_0[[x, y]]$$

of $\hat{R}$ with a power series ring. Suppose that α is separably algebraic over k_0. Let $y_1 = \frac{y}{x} - \alpha$. Then there exists a maximal ideal $n \subset R[\frac{m_R}{x}]$ and an isomorphism

$$\widehat{R\left[\frac{m_R}{x}\right]_n} \overset{\lambda_1}{\to} k_0(\alpha)[[x, y_1]]$$

which makes the diagram

$$\begin{array}{ccc} \hat{R} & \overset{\lambda_0}{\to} & k_0[[x, y]] \\ \downarrow & & \downarrow \\ \widehat{R\left[\frac{m_R}{x}\right]_n} & \overset{\lambda_1}{\to} & k_0(\alpha)[[x, y_1]] \end{array}$$

commute, where the vertical arrows are the natural maps.

Proof. There exists $\tilde{y} \in R$ such that $\tilde{y} = y + h$ where $h \in m_R^3\hat{R}$. We have

$$\frac{\tilde{y}}{x} - \alpha = y_1 + \frac{h(x, x(y_1 + \alpha))}{x} = y_1 + h_1(x_1, y_1)$$

where $h_1 \in k_0(\alpha)[[x, y_1]]$ is a series of order ≥ 2. Thus we have natural change of variables $k_0[[x, y]] = k_0[[x, \tilde{y}]]$ and $k_0(\alpha)[[x, y_1]] = k_0(\alpha)[[x, \frac{\tilde{y}}{x} - \alpha]]$. We may thus assume that $y \in R$.

We have a natural inclusion induced by λ_0,

$$R\left[\frac{y}{x}\right] \subset \hat{R}\left[\frac{y}{x}\right] \subset k_0(\alpha)[[x, y_1]].$$

Let $n = (x, y_1) \cap R[\frac{y}{x}]$.

Let $h(t)$ be the minimal polynomial of α over k_0, and $f \in R[\frac{y}{x}]$ be a lift of

$$h\left(\frac{y}{x}\right) \in k_0\left[\frac{y}{x}\right] \cong R\left[\frac{y}{x}\right]/xR\left[\frac{y}{x}\right].$$

Then $n = (x, f)$ and we see that $R[\frac{y}{x}]/n \cong k_0(\alpha)$. Now the conclusions of the lemma follow from Hensel's Lemma (cf. Lemma 3.5 [8]). $\square$

Theorem 2.2. *Suppose that k is a field, with algebraic closure $\overline{k}$. Let $\overline{k}((x))$ be the field of formal Laurent series in a variable x with coefficients in $\overline{k}$. Suppose that*

$$\sigma(x) = \sum_{i=d}^{\infty} \alpha_i x^i \in \overline{k}((x))$$

where $d \in \mathbb{Z}$ and $\alpha_i \in \overline{k}$ for all i. Let $L = k(\{\alpha_i \mid i \in \mathbb{N}\})$, and suppose that L is separable over k. Then $\sigma(x)$ is algebraic over $k((x))$ if and only if

$$[L : k] < \infty.$$

Proof. We reduce to the case where $d \geq 1$, by observing that σ is algebraic over $k((x))$ if and only if $x^{1-d}\sigma$ is.

First suppose that $[L : k] < \infty$. Let M be a finite Galois extension of k which contains L. Let G be the Galois group of M over k. G acts naturally by k algebra isomorphisms on $M[[x]]$, and the invariant ring of the action is $k[[x]]$. Let $f(y) = \prod_{\tau \in G}(y - \tau(\sigma)) \in M[[x]][y]$. Since f is invariant under the action of G, $f(y) \in k[[x]][y]$. Since $f(\sigma) = 0$, we have that σ is algebraic over $k((x))$.

Now suppose that $\sigma(x) = \sum_{i=1}^{\infty} \alpha_i x^i$ is algebraic over $k((x))$. Then there exists

$$g(x, y) = a_0(x)y^n + a_1(x)y^{n-1} + \cdots + a_n(x) \in k[[x]][y]$$

such that $a_0(x) \neq 0$, $n \geq 1$, g is irreducible and $g(x, \sigma(x)) = 0$.

Let

$$y_0 = y,\ y_1 = \frac{y}{x} - \alpha_1,\ y_2 = \frac{y_1}{x} - \alpha_2, \ldots, y_i = \frac{y_{i-1}}{x} - \alpha_i, \ldots$$

and define

$$S_0 = k[[x, y]],\ S_1 = k(\alpha_1)[[x, y_1]], \ldots S_i = k(\alpha_1, \ldots, \alpha_i)[[x, y_i]], \ldots.$$

We have natural inclusions

$$S_0 \to S_1 \to \cdots \to S_i \to \cdots.$$

By Lemma 2.1, there exists a sequence of inclusions

$$R_0 \to R_1 \to \cdots \to R_i \to \cdots \tag{3}$$

where $R_0 = k[[x]][y]_{(x,y)}$ and each R_i is a localization at a maximal ideal of the blow up of the maximal ideal $m_{R_{i-1}}$ of R_{i-1}, and we have a commutative diagram of homomorphisms

$$\begin{array}{ccccccccc} S_0 & \to & S_1 & \to & \cdots & \to & S_i & \to & \cdots \\ \uparrow & & \uparrow & & & & \uparrow & & \\ R_0 & \to & R_1 & \to & \cdots & \to & R_i & \to & \cdots \end{array}$$

where the vertical arrows induce isomorphisms of the m_{R_i}-adic completions $\hat{R}_i$ of R_i with S_i. We further have that x is part of a regular system of parameters in R_i for all i, and $m_{R_{i-1}}R_i = xR_i$ for all i.

By our construction, we have that

$$R_i/m_{R_i} \cong k(\alpha_1, \ldots, \alpha_i) \tag{4}$$

for all i.

For all i, write $g = x^{b_i} g_i$ where $g_i \in R_i$ and x does not divide g_i in R_i.

In $\overline{k}[[x, y_i]]$, we have a factorization

$$y - \sigma = x^i \left(y_i - \sum_{j=i+1}^{\infty} \alpha_j x^{j-i} \right).$$

Since $y - \sigma$ divides g in $\overline{k}[[x, y]]$, we have that $y_i - \sum_{j=i+1}^{\infty} \alpha_j x^{j-i}$ divides g_i in $\overline{k}[[x, y_i]]$. Thus g_i is not a unit in $\overline{k}[[x, y_i]]$, and is thus not a unit in R_i.

Let C be the curve germ $g = 0$ in the germ $\mathrm{Spec}(R_0)$ of a nonsingular surface. The sequence (3) is obtained by blowing up the closed point in $\mathrm{Spec}(R_i)$, and localizing at a point which is on the strict transform of C. $g_i = 0$ is a local equation of the strict transform of C in $\mathrm{Spec}(R_i)$. By embedded resolution of plane curve singularities ([1,20] or a simple generalization of Theorem 3.15 and Exercise 3.13 of [8]) we obtain that there exists i_0 such that the total transform of C in $\mathrm{Spec}(R_i)$ is a simple normal crossings divisor for all $i \geq i_0$. Since $x^{b_i} g_i = g = 0$ is a local equation of the total transform of C in $\mathrm{Spec}(R_i)$, we have that x, g_i are regular parameters in R_i for all $i \geq i_0$. Thus $g_{i_0} = x^{i-i_0} g_i$ for all $i \geq i_0$, and $R_i = R_{i-1}[\frac{g_{i-1}}{x}]_{(x,g_i)}$ for all $i \geq i_0 + 1$.

We thus have that $R_i/m_{R_i} \cong R_{i_0}/m_{R_{i_0}}$ for all $i \geq i_0$, and we see that

$$L = \bigcup_{i\geq 0} R_i/m_{R_i} = R_{i_0}/m_{R_{i_0}} = k(\alpha_1, \ldots, \alpha_{i_0}).$$

Thus $[L : k] < \infty$. □

Example 2.3. The conclusions of Theorem 2.2 may fail if L is not separable over k.

Proof. Let p be a prime and $\{t_i \mid i \in \mathbb{N}\}$ be algebraically independent over the finite field $\mathbb{Z}_p$. Let $k = \mathbb{Z}_p(\{t_i \mid i \in \mathbb{N}\})$. Define

$$\sigma(x) = \sum_{i=1}^{\infty} t_i^{\frac{1}{p}} x^i \in \overline{k}[[x]].$$

Let

$$f(y) = y^p - \sum_{i=1}^{\infty} t_i x^{ip} \in k[[x]][y].$$

$\sigma(x)$ is algebraic over $k[[x]]$ since

$$f(\sigma(x)) = (\sigma(x))^p - \sum_{i=1}^{\infty} t_i x^{ip} = 0.$$

However,

$$[k(\{t_i^{\frac{1}{p}} \mid i \in \mathbb{N}\}) : k] = \infty. \quad \square$$

Suppose that k is a field of characteristic $p > 0$ and L is an extension field of k. For $n \in \mathbb{N}$, let

$$L^{p^n} = \{f^{p^n} \mid f \in L\}.$$

If k has characteristic $p = 0$, we take $L^{p^n} = L$ for all n.

Theorem 2.4. *Suppose that k is a field of characteristic $p > 0$, with algebraic closure $\overline{k}$. Let $\overline{k}((x))$ be the field of formal Laurent series in the variable x with coefficients in $\overline{k}$. Suppose that*

$$\sigma(x) = \sum_{i=d}^{\infty} \alpha_i x^i \in \overline{k}((x))$$

where $d \in \mathbb{Z}$ and $\alpha_i \in \overline{k}$ for all i. Let $L = k(\{\alpha_i \mid i \in \mathbb{N}\})$, and assume that L is purely inseparable over k. Then $\sigma(x)$ is algebraic over $k((x))$ if and only if there exists $n \in \mathbb{N}$ such that $L^{p^n} \subset k$.

Proof. As in the proof of Theorem 2.2, we may assume that $d \geq 1$.

First suppose that $L^{p^n} \subset k$ for some n. Then $\tau(x) = \sigma(x)^{p^n} \in k[[x]]$, and $\sigma(x)$ is the root of $y^{p^n} - \tau(x) = 0$. Thus σ is algebraic over $k((x))$.

Now suppose that $\sigma(x) = \sum_{i=1}^{\infty} \alpha_i x^i \in \overline{k}[[x]]$ is algebraic over $k((x))$. Then there exists

$$g(x, y) = a_0(x)y^n + a_1(x)y^{n-1} + \cdots + a_n(x) \in k[[x]][y]$$

such that $a_0(x) \neq 0$, $n \geq 1$, g is irreducible and $g(x, \sigma(x)) = 0$.

Let K be the quotient field of $\overline{k}[[x]][y]$, and let $R_0 := S_0 := k[[x]][y]_{(x,y)}$. We will first construct a series of subrings S_i of K.

Define a local k-algebra homomorphism $\pi_0 : S_0 \to \overline{k}[[x]]$ by prescribing that $\pi_0(x) = x$ and $\pi_0(y) = \sigma(x)$. The kernel of π_0 is the prime ideal gS_0.

$$\frac{y}{x} = \sum_{i=0}^{\infty} \alpha_{i+1} x^i \in \overline{k}[[x]]$$

defines a k-algebra homomorphism $S_0[\frac{y}{x}] \to \overline{k}[[x]]$ which extends π_0. Let $\lambda(1) \in \mathbb{N}$ be the smallest natural number such that $\alpha_1^{p^{\lambda(1)}} \in k$. Then the maximal ideal $x\overline{k}[[x]]$ of $\overline{k}[[x]]$ contracts to

$$x\overline{k}[[x]] \cap S_0\left[\frac{y}{x}\right] = \left(x, \left(\frac{y}{x}\right)^{p^{\lambda(1)}} - \alpha_1^{p^{\lambda(1)}}\right).$$

Set $y_1 = \left(\frac{y}{x}\right)^{p^{\lambda(1)}} - \alpha_1^{p^{\lambda(1)}}$. Let

$$S_1 = S_0\left[\frac{y}{x}\right]_{(x,y_1)}.$$

Let $\pi_1 : S_1 \to \overline{k}[[x]]$ be the local k-algebra homomorphism induced by π_0.

We have that x, y_1 is a regular system of parameters in S_1, with

$$y_1 = \sum_{i=1}^{\infty} \alpha_{i+1}^{p^{\lambda(1)}} x^{ip^{\lambda(1)}}.$$

$S_1/m_{S_1} \cong k(\alpha_1)$ and

$$[S_1/m_{S_1} : S_0/m_{S_0}] = [k(\alpha_1) : k] = p^{\lambda(1)}.$$

Let $\lambda(2) \in \mathbb{N}$ be the smallest natural number such that $\alpha_2^{p^{\lambda(1)+\lambda(2)}} \in k(\alpha_1)$. Let

$$y_2 = \left(\frac{y_1}{x^{p^{\lambda(1)}}}\right)^{p^{\lambda(2)}} - \alpha_2^{p^{\lambda(1)+\lambda(2)}}.$$

Then there is an expansion in $\overline{k}[[x]]$

$$y_2 = \sum_{i=1}^{\infty} \alpha_{i+2}^{p^{\lambda(1)+\lambda(2)}} x^{ip^{\lambda(1)+\lambda(2)}}.$$

Let $S_2 = S_1[\frac{y_1}{x^{p^{\lambda(1)}}}, \alpha_1]_{(x,y_2)} \subset K$. We have a local k-algebra homomorphism $\pi_2 : S_2 \to \overline{k}[[x]]$ which extends π_1. We have $S_2/m_{S_2} \cong k(\alpha_1, \alpha_2^{p^{\lambda(1)}})$, so that

$$[S_2/m_{S_2} : S_1/m_{S_1}] = [k(\alpha_1, \alpha_2^{p^{\lambda(1)}}) : k(\alpha_1)] = p^{\lambda(2)}.$$

We iterate the above construction, defining for $i \geq 2$,

$$\begin{aligned} y_i &= \left(\frac{y_{i-1}}{x^{p^{\lambda(1)+\cdots+\lambda(i-1)}}}\right)^{p^{\lambda(i)}} - \alpha_i^{p^{\lambda(1)+\cdots+\lambda(i)}} \\ &= \sum_{j=1}^{\infty} \alpha_{j+i}^{p^{\lambda(1)+\cdots+\lambda(i)}} x^{jp^{\lambda(1)+\cdots+\lambda(i)}} \end{aligned}$$

where $p^{\lambda(i)} \in \mathbb{N}$ is the smallest natural number such that

$$\alpha_i^{p^{\lambda(1)+\cdots+\lambda(i)}} \in k(\alpha_1, \alpha_2^{p^{\lambda(1)}}, \ldots, \alpha_{i-1}^{p^{\lambda(1)+\cdots+\lambda(i-2)}}).$$

Define

$$S_i = S_{i-1}\left[\frac{y_{i-1}}{x^{p^{\lambda(1)+\cdots+\lambda(i-1)}}}, \alpha_{i-1}^{p^{\lambda(1)+\cdots+\lambda(i-2)}}\right]_{(x,y_i)},$$

to construct an infinite commutative diagram of regular local rings, which are contained in K,

$$\begin{array}{ccccccccc} S_0 & \to & S_1 & \to & \cdots & \to & S_i & \to & \cdots \\ \pi_0\downarrow & & \pi_1\downarrow & & & & \pi_i\downarrow & & \\ \overline{k}[[x]] & = & \overline{k}[[x]] & = & \cdots & = & \overline{k}[[x]] & = & \cdots \end{array}$$

We have

$$S_i/m_{S_i} \cong S_{i-1}/m_{S_{i-1}}[\alpha_i^{p^{\lambda(1)+\cdots+\lambda(i-1)}}] \tag{5}$$

and

$$[S_i/m_{S_i} : S_{i-1}/m_{S_{i-1}}] = p^{\lambda(i)}.$$

For all i, the field

$$k_i := k(\alpha_1, \alpha_2^{p^{\lambda(1)}}, \ldots, \alpha_{i-1}^{p^{\lambda(1)+\cdots+\lambda(i-2)}}) \subset S_i,$$

and

$$S_i/m_{S_i} \cong k_i[\alpha_i^{p^{\lambda(1)+\cdots+\lambda(i-1)}}].$$

We now construct a sequence

$$R_0 \to R_1 \to \cdots \to R_i \to \cdots$$

of birationally equivalent regular local rings such that there is a commutative diagram of local k-algebra homomorphisms

$$\begin{array}{ccccccccc} R_0 & \to & R_1 & \to & \cdots & \to & R_i & \to & \cdots \\ \downarrow & & \downarrow & & & & \downarrow & & \\ S_0 & \to & S_1 & \to & \cdots & \to & S_i & \to & \cdots \end{array}$$

satisfying

$$m_{R_i}S_i = m_{S_i} \quad \text{and} \quad S_i/m_{S_i} \cong R_i/m_{R_i}$$

for all i. The vertical arrows are inclusions.

This is certainly the case for $R_0 = S_0$, so we suppose that we have constructed the sequence out to $R_i \to S_i$, and show that we may extend it to $R_{i+1} \to S_{i+1}$.

We have

$$\alpha_{i+1}^{p^{\lambda(1)+\cdots+\lambda(i+1)}} \in k(\alpha_1, \alpha_2^{p^{\lambda(1)}}, \ldots, \alpha_i^{p^{\lambda(1)+\cdots+\lambda(i-1)}}) \cong R_i/m_{R_i}.$$

Thus there exists $\varphi \in R_i$ such that the class of φ in R_i/m_{R_i} is

$$[\varphi] = \alpha_{i+1}^{p^{\lambda(1)+\cdots+\lambda(i+1)}}.$$

Our assumptions $m_{R_i}S_i = m_{S_i}$ and $S_i/m_{S_i} \cong R_i/m_{R_i}$ imply that

$$m_{R_i}^n/m_{R_i}^{n+1} \cong m_{S_i}^n/m_{S_i}^{n+1} \tag{6}$$

as R_i/m_{R_i} vector spaces for all $n \in \mathbb{N}$.

By (6), there exists $z_i \in R_i$ such that

$$z_i = y_i + h$$

with $h \in m_{S_i}^{2+p^{\lambda(1)+\cdots+\lambda(i)}}$. We then have that $m_{R_i} = (x, z_i)$, since $m_{R_i}/m_{R_i}^2 \cong m_{S_i}/m_{S_i}^2$ as R_i/m_{R_i} vector spaces, and by Nakayama's Lemma. Now

$$\begin{aligned}\left(\frac{z_i}{x^{p^{\lambda(1)+\cdots+\lambda(i)}}}\right)^{p^{\lambda(i+1)}} &= \left(\frac{y_i}{x^{p^{\lambda(1)+\cdots+\lambda(i)}}}\right)^{p^{\lambda(i+1)}} + \left(\frac{h}{x^{p^{\lambda(1)+\cdots+\lambda(i)}}}\right)^{p^{\lambda(i+1)}}\\ &= \left(\frac{y_i}{x^{p^{\lambda(1)+\cdots+\lambda(i)}}}\right)^{p^{\lambda(i+1)}} + xh'\end{aligned}$$

for some $h' \in S_i[\frac{y_i}{x^{p^{\lambda(1)+\cdots+\lambda(i)}}}]$.

$$\left(\frac{z_i}{x^{p^{\lambda(1)+\cdots+\lambda(i)}}}\right)^{p^{\lambda(i+1)}} - \varphi \in R_i\left[\frac{z_i}{x^{p^{\lambda(1)+\cdots+\lambda(i)}}}\right] \subset S_i\left[\frac{y_i}{x^{p^{\lambda(1)+\cdots+\lambda(i)}}}\right]$$

has residue

$$\left(\frac{y_i}{x^{p^{\lambda(1)+\cdots+\lambda(i)}}}\right)^{p^{\lambda(i+1)}} - \alpha_{i+1}^{p^{\lambda(1)+\cdots+\lambda(i+1)}}$$

in $S_{i+1}/xS_{i+1} \cong S_i/m_{S_i}\left[\frac{y_i}{x^{p^{\lambda(1)+\cdots+\lambda(i)}}}\right]$. Thus

$$m_{S_{i+1}} \cap R_i\left[\frac{z_i}{x^{p^{\lambda(1)+\cdots+\lambda(i)}}}\right] = (x, \left(\frac{z_i}{x^{p^{\lambda(1)+\cdots+\lambda(i)}}}\right)^{p^{\lambda(i+1)}} - \varphi).$$

Let

$$R_{i+1} = R_i\left[\frac{z_i}{x^{p^{\lambda(1)+\cdots+\lambda(i)}}}\right]_{\left(x, \left(\frac{z_i}{x^{p^{\lambda(1)+\cdots+\lambda(i)}}}\right)^{p^{\lambda(i+1)}} - \varphi\right)}.$$

We have $m_{R_{i+1}}S_{i+1} = m_{S_{i+1}}$ (by Nakayama's Lemma) and $R_{i+1}/m_{R_{i+1}} \cong S_{i+1}/m_{S_{i+1}}$.

We have factorizations $g(x, y) = x^{\beta_i} g_i$ where $\beta_i \in \mathbb{N}$ and $g_i \in R_i$ is either irreducible or a unit. g_i is a strict transform of g in R_i. Since $\pi_i(x) \neq 0$, we have that g_i is contained in the kernel of the map $R_i \to S_i \xrightarrow{\pi} \overline{k}[[x]]$, and thus the ideal (g_i) is the (nontrivial) kernel of $R_i \to \overline{k}[[x]]$. In particular, $g_i \in M_{R_i}$ for all i.

Each extension $R_i \to R_{i+1}$ can be factored as a sequence of $p^{\lambda(1)+\cdots+\lambda(i)}$ birationally equivalent regular local rings, each of which is a quadratic transform (the blow up of the maximal ideal followed by localization). The j-th local ring with $j < p^{\lambda(1)+\cdots+\lambda(i)}$, has the maximal ideal $(x, \frac{z_i}{x^j})$.

By embedded resolution of plane curve singularities [1,8,20], we obtain that there exists i_0 such that $g = 0$ is a simple normal crossings divisor in Spec(R_i) for all $i \geq i_0$, so that x, g_i is a regular system of parameters in R_i for all $i \geq i_0$. Thus

$$R_{i+1} = R_i\left[\frac{z_i}{x^{p^{\lambda(1)+\cdots+\lambda(i)}}}\right]_{(x, \frac{g_i}{x^{p^{\lambda(1)+\cdots+\lambda(i)}}})}$$

for all $i \geq i_0$, and

$$S_{i+1}/m_{S_{i+1}} \cong R_{i+1}/m_{R_{i+1}} \cong R_i/m_{R_i} \cong S_i/m_{S_i}$$

for all $i \geq i_0$. Thus $\lambda(i) = 0$ for all $i \geq i_0 + 1$.

Let

$$M = k(\alpha_1, \alpha_2^{p^{\lambda(1)}}, \ldots, \alpha_{i_0}^{p^{\lambda(1)+\cdots+\lambda(i_0-1)}}) \cong S_{i_0}/m_{S_{i_0}}.$$

From (5), we see that $L^{p^{\lambda(1)+\cdots+\lambda(i_0)}} \subset M$. Since M is a finitely generated purely inseparable extension of k, there exists $r \in \mathbb{N}$ such that $M^{p^r} \subset k$. Thus $L^{p^{\lambda(1)+\cdots+\lambda(i_0)+r}} \subset k$. □

Theorem 2.5. *Suppose that k is a field of characteristic $p \geq 0$, with algebraic closure $\overline{k}$. Let $\overline{k}((x))$ be the field of formal Laurent series with coefficients in $\overline{k}$. Suppose that*

$$\sigma(x) = \sum_{i=d}^{\infty} \alpha_i x^i \in \overline{k}((x))$$

where $d \in \mathbb{Z}$ and $\alpha_i \in \overline{k}$ for all i. Let $L = k(\{\alpha_i \mid i \in \mathbb{N}\})$. Then $\sigma(x)$ is algebraic over $k((x))$ if and only if there exists $n \in \mathbb{N}$ such that $[kL^{p^n} : k] < \infty$, where kL^{p^n} is the compositum of k and L^{p^n} in $\overline{k}$.

Proof. First suppose that $[kL^{p^n} : k] < \infty$ for some n. After possibly replacing n with a larger value of n, we may assume that kL^{p^n} is separable over k. Then $\sigma(x)^{p^n}$ is algebraic over $k((x))$ by Theorem 2.2, and thus $\sigma(x)$ is algebraic over $k((x))$.

Now suppose that $\sigma(x)$ is algebraic over $k((x))$. Let M be the separable closure of k in L. Then $\sigma(x)$ is algebraic over $M((x))$. Since L is a purely inseparable extension of M, it follows from Theorem 2.4 that $\tau(x) = \sigma(x)^{p^n} \in M[[x]]$ for some $n \in \mathbb{N}$. Since $\tau(x)$ is algebraic over $k((x))$, we have that $[kL^{p^n} : k] < \infty$ by Theorem 2.2. $\square$

Corollary 2.6. *Suppose that k is a field of characteristic $p \geq 0$ such that k is a finitely generated extension of a perfect field, with algebraic closure $\overline{k}$. Let $\overline{k}((x))$ be the field of formal Laurent series with coefficients in $\overline{k}$. Suppose that*

$$\sigma(x) = \sum_{i=d}^{\infty} \alpha_i x^i \in \overline{k}((x))$$

where $d \in \mathbb{Z}$ and $\alpha_i \in \overline{k}$ for all i. Let $L = k(\{\alpha_i \mid i \in \mathbb{N}\})$. Then $\sigma(x)$ is algebraic over $k((x))$ if and only if

$$[L : k] < \infty.$$

Proof. If $[L : k] < \infty$, then $\sigma(x)$ is algebraic over $k((x))$ by Theorem 2.5.

Suppose that $\sigma(x)$ is algebraic over $k((x))$. By assumption, there exists a perfect field F and $s_1, \ldots, s_r \in k$ such that $k = F(s_1, \ldots, s_r)$. By Theorem 2.5, there exists n such that $[kL^{p^n} : k] < \infty$. Thus $kL^{p^n} = F(s_1, \ldots, s_r, \beta_1, \ldots, \beta_s)$ where $\beta_1, \ldots, \beta_s \in kL^{p^n}$ are algebraic over k. Thus

$$L \subset F(s_1^{\frac{1}{p^n}}, \ldots, s_r^{\frac{1}{p^n}}, \beta_1^{\frac{1}{p^n}}, \ldots, \beta_s^{\frac{1}{p^n}}).$$

Now

$$[F(s_1^{\frac{1}{p^n}}, \ldots, s_r^{\frac{1}{p^n}}) : F(s_1, \ldots, s_r)] < \infty$$

and since $\beta_1, \ldots, \beta_s$ are algebraic over $F(s_1, \ldots, s_r)$,

$$[F(s_1^{\frac{1}{p^n}}, \ldots, s_r^{\frac{1}{p^n}}, \beta_1^{\frac{1}{p^n}}, \ldots, \beta_s^{\frac{1}{p^n}}) : F(s_1^{\frac{1}{p^n}}, \ldots, s_r^{\frac{1}{p^n}})] < \infty.$$

Thus

$$[L : k] \leq [F(s_1^{\frac{1}{p^n}}, \ldots, s_r^{\frac{1}{p^n}}, \beta_1^{\frac{1}{p^n}}, \ldots, \beta_s^{\frac{1}{p^n}}) : k] < \infty. \quad \square$$

3. Series in several variables

We will now generalize Theorem 2.5 to higher dimensions.

Denote by X an n-dimensional indeterminate vector $(x_1, x_2, \ldots, x_n)$ and by I an n-dimensional exponent vector $(i_1, i_2, \ldots, i_n) \in \mathbb{N}^n$. Then for $1 \leq l \leq n$ write $X_l = (x_1, x_2, \ldots, x_l)$, $I_l = (i_1, i_2, \ldots, i_l)$ and $X_l^I = X_l^{I_l} = x_1^{i_1} x_2^{i_2} \cdots x_l^{i_l}$. If E is a field denote by $E[[X]]$ the formal power series ring in n variables with coefficients in E and by $E((X))$ the quotient field of $E[[X]]$. Also denote by E^c the perfect closure of E and by $\overline{E}$ the algebraic closure of E.

Lemma 3.1. *Suppose that E is a field and F is a field extension of E. Let*

$$\sigma = \sum_{I \in \mathbb{N}^n} \alpha_I X^I \in F[[X]], \quad \text{with } \alpha_I \in F,$$

be a formal power series in n variables with coefficients in F. For any $1 \le l \le n$ and $I \in \mathbb{N}^n$ define the following power series in 1 variable with coefficients in F

$$a_{I,l} = \sum_{j=0}^{\infty} \alpha_J x_l^j, \quad \text{where } J = (i_1, i_2, \ldots, i_{l-1}, j, i_{l+1}, \ldots, i_n).$$

Then σ is algebraic over $E((X))$ implies $a_{I,l}$ is algebraic over $E((x_l))$.

Proof. We use induction on the number of variables. If $n = 1$ the statement is trivial. Suppose that $n > 1$. After possibly permuting the variables we may assume that $l = 1$. Write $X_{n-1} = (x_1, \ldots, x_{n-1})$ and for all $m \in \mathbb{N}$ consider the power series in $n - 1$ variables

$$\delta_m = \sum_{R \in \mathbb{N}^n,\, r_n = m} \alpha_R X_{n-1}^R = \sum_{R \in \mathbb{N}^n,\, r_n = m} \alpha_R x_1^{r_1} x_2^{r_2} \cdots x_{n-1}^{r_{n-1}}.$$

If δ_{i_n} is algebraic over $E((X_{n-1}))$ it will follow from the inductive hypothesis that $a_{I,1}$ is algebraic over $E((x_1))$. We will show that δ_m is algebraic over $E((X_{n-1}))$ for all $m \in \mathbb{N}$.

Consider the algebraic dependency relation for σ over $E((X))$

$$c_t(X)\sigma^t + c_{t-1}(X)\sigma^{t-1} + \cdots + c_1(X)\sigma + c_0(X) = 0.$$

By clearing the denominators we may assume that $c_j \in E[[X]]$ for all $0 \le j \le t$. Let g be the highest power of x_n that divides c_j for all j. Set $c'_j = (x_n^{-g} c_j)(x_1, x_2, \ldots, x_{n-1}, 0)$. Then $c'_j \in E[[X_{n-1}]]$ and the following equation holds:

$$c'_t(X_{n-1})\delta_0^t + c'_{t-1}(X_{n-1})\delta_0^{t-1} + \cdots + c'_1(X_{n-1})\delta_0 + c'_0(X_{n-1}) = 0,$$

where $c'_j \ne 0$ for some $0 \le j \le t$. Thus δ_0 is algebraic over $E((X_{n-1}))$.

Set $\sigma_1 = x_n^{-1}(\sigma - \delta_0)$. Then $\sigma_1 \in F[[X]]$ and it is algebraic over $E((X))$. Arguing as above we get that δ_1 is algebraic over $E((X_{n-1}))$. In general we define $\sigma_m = x_n^{-1}(\sigma_{m-1} - \delta_{m-1})$ recursively for all $m \in \mathbb{N}$ and use σ_m to prove that δ_m is algebraic over $E((X_{n-1}))$. □

Theorem 3.2. *Suppose that k is a field of characteristic $p \ge 0$. Suppose that*

$$\sigma = \sum_{I \in \mathbb{N}^n} \alpha_I X^I \in \overline{k}[[X]], \quad \text{with } \alpha_I \in \overline{k}$$

is a formal power series in n variables with coefficients in $\overline{k}$. Let $L = k(\{\alpha_I \mid I \in \mathbb{N}^n\})$ be the extension field of k generated by the coefficients of σ. Then σ is algebraic over $k((X))$ if and only if there exists $r \in \mathbb{N}$ such that $[kL^{p^r} : k] < \infty$, where kL^{p^r} is the compositum of k and L^{p^r} in $\overline{k}$.

Proof. First suppose that there exists $r \in \mathbb{N}$ such that $[kL^{p^r} : k] < \infty$. After possibly increasing r we may assume that kL^{p^r} is a separable extension of k. Let M be a finite Galois extension of k which contains kL^{p^r}. Notice that $kL^{p^r} = k(\{\alpha_I^{p^r} \mid I \in \mathbb{N}^n\})$ and, therefore $\sigma^{p^r} \in M[[X]]$. Let G be the Galois group of M over k. G acts naturally by k algebra isomorphisms on $M[[X]]$, and the invariant ring of the action is $k[[X]]$. Let $f(y) = \prod_{\tau \in G}(y - \tau(\sigma^{p^r})) \in M[[X]][y]$. Since f is invariant under the action of G, $f(y) \in k[[X]][y]$. Since $f(\sigma^{p^r}) = 0$, we have that σ is algebraic over $k[[X]]$.

To prove the other implication we use induction on the number of variables. When $n = 1$ the statement follows from Theorem 2.5. Assume that $n > 1$.

For all $I \in \mathbb{N}^n$ let

$$a_I = \sum_{j=0}^{\infty} \alpha_J x_n^j, \quad \text{with } J = (i_1, i_2, \ldots, i_{n-1}, j),$$

be a power series in 1 variable with coefficients in $\overline{k}$. If $K = k((x_n))$ then by Lemma 3.1 a_I is algebraic over K for all $I \in \mathbb{N}^n$. Then

$$\sigma = \sum_{\{I\in\mathbb{N}^n | i_n=0\}} a_I X_{n-1}^I$$

is a series in $n-1$ variables with coefficients in $\overline{K}$. By the inductive hypothesis there exists $N \in \mathbb{N}$ and $r \in \mathbb{N}$ such that $K(\{a_I^{p^r} \mid I \in \mathbb{N}^n\}) = K(a_{I_1}^{p^r}, a_{I_2}^{p^r}, \ldots, a_{I_N}^{p^r})$. Thus, for all $I \in \mathbb{N}^n$ we have $a_I^{p^r}$ is a polynomial in $a_{I_1}^{p^r}, a_{I_2}^{p^r}, \ldots, a_{I_N}^{p^r}$ with coefficients in K.

Fix $I \in \mathbb{N}$, if $j \in \mathbb{N}$ set $J = (i_1, i_2, \ldots, i_{n-1}, j)$ and write

$$\sum_{j=0}^{\infty} \alpha_J^{p_r} x_n^{jp^r} = a_I^{p^r} = \sum_{S\in\{0,1,\ldots,T\}^N} \left(\sum_{m=-M_S}^{\infty} \gamma_{S,m} x_n^m \right) (a_{I_1}^{p^r})^{s_1} (a_{I_2}^{p^r})^{s_2} \cdots (a_{I_N}^{p^r})^{s_N},$$

where $T \in \mathbb{N}$, $S = (s_1, s_2, \ldots, s_N)$ is an index vector, $M_S \in \mathbb{N}$ and $\gamma_{S,m} \in k$ for all S and m. This implies that for all $I \in \mathbb{N}$ and $j \in \mathbb{N}$, $\alpha_J^{p^r}$ is a polynomial in the coefficients of power series $a_{I_1}^{p^r}, a_{I_2}^{p^r}, \ldots, a_{I_N}^{p^r}$ over k. Moreover, for all $r' \geq r$ we also have $\alpha_J^{p^{r'}}$ is a polynomial in the coefficients of power series $a_{I_1}^{p^{r'}}, a_{I_2}^{p^{r'}}, \ldots, a_{I_N}^{p^{r'}}$ over k. Thus $kL^{p^{r'}}$ is the field extension of k generated by the coefficients of power series $a_{I_1}^{p^{r'}}, a_{I_2}^{p^{r'}}, \ldots, a_{I_N}^{p^{r'}}$.

Applying Theorem 2.5 to each of the series $a_{I_1}, a_{I_2}, \ldots, a_{I_N}$ we see that there exists $R \in \mathbb{N}$ such that kL^{p^R} is finitely generated over k. $\square$

Similarly to the case of one variable we deduce the following corollary.

Corollary 3.3. *Suppose that k is a field of characteristic $p \geq 0$ such that k is a finitely generated extension of a perfect field. Suppose that*

$$\sigma = \sum_{I\in\mathbb{N}^n} \alpha_I X^I \in \overline{k}[[X]], \quad \text{with } \alpha_I \in \overline{k}$$

is a formal power series in n variables with coefficient in $\overline{k}$. Let $L = k(\{\alpha_I \mid I \in \mathbb{N}^n\})$ be the extension field of k generated by the coefficients of σ. Then σ is algebraic over $k((X))$ if and only if $[L : k] < \infty$.

Also notice that if E is a field of characteristic $p \geq 0$ and a is separable algebraic over E; then for all $r \in \mathbb{N}$ we have $E[a^{p^r}] = E[a]$. Thus if F is a separable extension of E, $EF^{p^r} = F$ for all $r \in \mathbb{N}$. So we have the following statement in case of separable extensions.

Corollary 3.4. *Suppose that k is a field of characteristic $p \geq 0$. Suppose that*

$$\sigma = \sum_{I\in\mathbb{N}^n} \alpha_I X^I \in \overline{k}[[X]], \quad \text{with } \alpha_I \in \overline{k}$$

is a formal power series in n variables with coefficient in $\overline{k}$. Let $L = k(\{\alpha_I \mid I \in \mathbb{N}^n\})$ be the extension field of k generated by the coefficients of σ. Suppose that L is separable over k. Then σ is algebraic over $k((X))$ if and only if $[L : k] < \infty$.

4. Valuations whose rank increases under completion

Suppose that K is a field and V is a valuation ring of K. We will say that *the rank of V increases under completion* if there exists an analytically normal local domain T with quotient field K such that V dominates T and there exists an extension of V to a valuation ring of the quotient field of $\hat{T}$ which dominates $\hat{T}$ which has higher rank than the rank of V.

Suppose that V dominates an excellent local ring R of dimension 2. Then by resolution of surface singularities [17], there exists a regular local ring R_0 and a birational extension $R \to R_0$ such that V dominates R_0. Let

$$R_0 \to R_1 \to \cdots \to R_n \to \cdots \tag{7}$$

be the infinite sequence of regular local rings obtained by blowing up the maximal ideal of R_i and localizing at the center of V. Since R has dimension 2, we have that $V = \cup_{i=0}^{\infty} R_i$ (as is shown in [2]), and thus $V/m_V = \cup_{i=0}^{\infty} R_i/m_{R_i}$. We see that V/m_V is countably generated over R/m_R.

Suppose that the rank of V increases under completion. Then there exists n such that for all $i \geq n$, there exists a valuation ring V_1 of the quotient field of the regular local ring $\hat{R}_i$ which extends V, dominates $\hat{R}_i$, and has rank larger than 1. By the Abhyankar inequality ([2] or Proposition 3 of Appendix 2 [30]), we have that R_i has dimension 2, V_1 is discrete of rank 2, and V_1/m_{V_1} is algebraic over $\hat{R}_i/m_{\hat{R}_i}$. Thus V/m_V is algebraic over R/m_R and V is discrete of rank 1.

It was shown by Spivakovsky [27] in the case that R/m is algebraically closed that the converse holds, giving the following simple characterization.

Theorem 4.1 (*Spivakovsky [27]*). *Suppose that V dominates an excellent two-dimensional local ring R who residue field R/m_R is algebraically closed. Then the rank of V increases under completion if and only if* $\dim_R(V) = 0$ *and V is discrete of rank* 1.

The condition that the transcendence degree $\dim_R(V)$ of V/m_V over R/m_R is zero is just the statement that V/m_V is algebraic over R/m_R. In the case that R/m_R is algebraically closed, $\dim_R(V) = 0$ if and only if $V/m_V = R/m_R$.

Using a similar method to that used in the proof of our algebraicity theorem on power series, Theorem 2.2, we prove the following extension of Theorem 4.1.

Theorem 4.2. *Suppose that V is a valuation ring of a field K, and V dominates an excellent two-dimensional local domain R whose quotient field is K. Then the rank of V increases under completion if and only if V/m_V is finite over R/m_R and V is discrete of rank* 1.

Proof. First assume that the rank of V increases under completion. Consider the sequence (7). We observed above after (7) that V/m_V is algebraic over R/m_R and V is discrete of rank 1. Further, there exists R_i and a valuation V_1 of the quotient field of $\hat{R}_i$ which dominates $\hat{R}_i$ whose intersection with the quotient field K of R is V, and the rank of V_1 is 2. Without loss of generality, we may assume that $R_i = R_0$.

For $i \geq 0$, let $p(R_i)_\infty$ be the (nontrivial) prime ideal in $\hat{R}_i$ of Cauchy sequences whose value is greater than n for any $n \in \mathbb{N}$ (Section 5 of [9]). Since $\hat{R}_i$ is a two-dimensional regular local ring, $p(R_i)_\infty$ is generated by an irreducible element in $\hat{R}_i$ for all i. Let f be a generator of $p(R_0)_\infty$. By resolution of plane curve singularities [1,8,20], there exists i in the sequence (7) such that $f = h_i f_i$, where $h_i \in R_i$ is such that $h_i = 0$ is supported on the exceptional locus of $\mathrm{Spec}(R_i) \to \mathrm{Spec}(R)$, and $f_i \in \hat{R}_i$ is such that $\hat{R}_i/f_i\hat{R}_i$ is a regular local ring. We necessarily have that $p(R_i)_\infty = f_i\hat{R}_i$. Again, without loss of generality, we may assume that $i = 0$. Let $T_0 = \hat{R}_0$, and let

$$T_0 \to T_1 \to \cdots \to T_n \to \cdots$$

be the infinite sequence of regular local rings obtained by blowing up the maximal ideal of the regular local ring T_i and localizing at the center of V_1. We then have a commutative diagram

$$\begin{array}{ccccccccc} R_0 & \to & R_1 & \to & \cdots & \to & R_i & \to & \cdots \\ \downarrow & & \downarrow & & & & \downarrow & & \\ T_0 & \to & T_1 & \to & \cdots & \to & T_i & \to & \cdots \\ \downarrow & & \downarrow & & & & \downarrow & & \\ \hat{R}_0 & \to & \hat{R}_1 & \to & \cdots & \to & \hat{R}_i & \to & \cdots. \end{array} \tag{8}$$

There exists $x \in R_0$ such that x, f_0 is a regular system of parameters in T_0. Thus $T_1 = T_0[\frac{f_0}{x}]_{(x, \frac{f_0}{x})}$. Define $f_i = \frac{f_0}{x^i}$ for $i \geq 1$. Then $T_i = T_0[f_i]_{(x, f_i)}$ and $p(R_i)_\infty = f_i\hat{R}_i$ for all $i \geq 0$. Thus $R_i/m_{R_i} \cong T_i/m_{T_i} \cong T_0/(x, f_0) \cong R_0/m_{R_0}$ for all i. Since $V/m_V = \cup_{i\geq 0} R_i/m_{R_i} = R_0/m_{R_0}$ and R_0/m_{R_0} is finite over R/m_R, we have the conclusions of the theorem.

Now assume that V/m_V is finite over R/m_R and V is discrete of rank 1. Consider the sequence (7). There exists i such that $R_i/m_{R_i} = V/m_V$. Without loss of generality, we may assume that $R = R_i$. Let ν be a valuation of K such that V is the valuation ring of ν. We may also assume that there are regular parameters x, y in R such that $\nu(x) = 1$ generates the value group $\mathbb{Z}$ of ν. Let $\pi : R \to R/m_R = V/m_V$ be the residue map. Let $y_0 = y$. There exists $n_0 \in \mathbb{N}$

such that $\nu(y) = n_0$. Let $\alpha_0 \in R$ be such that $\pi(\alpha_0) = [\frac{y}{x^{n_0}}] \in V/m_V$. Let $y_1 = y - \alpha_0 x^{n_0}$, and let $n_1 = \nu(y_1)$. We have $n_1 > n_0$. Iterate, to construct $y_i \in R$ and $n_i \in \mathbb{N}$ with $\nu(y_i) = n_i$ for $i \in \mathbb{N}$ by choosing $\alpha_i \in R$ such that $y_{i+1} = y_i - \alpha_i x^{n_i}$ satisfies $n_{i+1} > n_i$. Thus $\{y_i\}$ is a Cauchy sequence in R. Let σ be the limit of $\{y_i\}$ in $\hat{R}$. Let $\hat{\nu}$ be an extension of ν to the quotient field of $\hat{R}$ which dominates $\hat{R}$. Then $\hat{\nu}(\sigma) > n$ for all $n \in \mathbb{N}$, so that $\hat{\nu}$ has rank $2 > 1$, and we see that the rank of V increases under completion. □

We see that the condition that V/m_V is finite over R/m_R thus divides the class of discrete rank 1 valuation rings with $\dim_R(V) = 0$ into two subclasses, those whose rank increases under completion ($[V/m_V : R/mR] < \infty$), and those whose rank does not increase ($[V/m_V : R/mR] = \infty$). We have the following precise characterization of when this division into subclasses is nontrivial.

Corollary 4.3. *Suppose that R is an excellent two-dimensional local ring. Then there exists a rank 1 discrete valuation ring V of the quotient field of R which dominates R such that $\dim_R(V) = 0$ and the rank of V does not increase under completion if and only if $[\overline{k} : k] = \infty$, where $\overline{k}$ is the algebraic closure of $k = R/m_R$.*

Proof. Suppose that $[\overline{k} : k] < \infty$, and V is a rank 1 discrete valuation ring of the quotient field of R which dominates R such that $\dim_R(V) = 0$. Then

$$[V/m_V : k] \le [\overline{k} : k] < \infty.$$

Thus the rank of V must increase under completion by Theorem 4.2.

Now suppose that $[\overline{k} : k] = \infty$. We will construct a rank 1 discrete valuation ring V of the quotient field of R which dominates R such that $\dim_R(V) = 0$ and the rank of V does not increase under completion.

There exists a two-dimensional regular local ring R_0 which birationally dominates R. We have $[\overline{k} : R_0/m_{R_0}] = \infty$. Let x, y_0 be a regular system of parameters in R_0. We will inductively construct an infinite birational sequence of regular local rings

$$R_0 \to R_1 \to \cdots \to R_i \to \cdots$$

such that R_i has a regular system of parameters x, y_i and $[R_i/m_{R_i} : R_{i-1}/m_{R_{i-1}}] > 1$ for all i. Suppose that we have defined the sequence out to R_i. Choose $\alpha_{i+1} \in \overline{k} - R_i/m_{R_i}$. Let $h_{i+1}(t)$ be the minimal polynomial of α_{i+1} in the polynomial ring $R_i/m_{R_i}[t]$. We have an isomorphism

$$R_i\left[\frac{m_{R_i}}{x}\right]/xR_i\left[\frac{m_{R_i}}{x}\right] \cong R_i/m_{R_i}\left[\frac{y_i}{x}\right].$$

Let y_{i+1} be a lift of $h_{i+1}(\frac{y_i}{x})$ to $R_i\left[\frac{m_{R_i}}{x}\right]$. Let

$$R_{i+1} = R_i\left[\frac{m_{R_i}}{x}\right]_{(x,y_{i+1})}.$$

We have that $R_{i+1}/m_{R_{i+1}} \cong R_i/m_{R_i}(\alpha_{i+1})$.

Let $V = \cup_{i=0}^{\infty} R_i$. V is a valuation ring which dominates R (as is shown in [2]). $V/m_V = \cup_{i=0}^{\infty} R_i/m_{R_i}$ so that $\dim_R(V) = 0$ and $[V/m_V : k] = \infty$.

V must have rank 1 since $[V/m_V : k] = \infty$ (for instance by the Abhyankar inequality, [2] or Proposition 3 [30]). By our construction, $\nu(x) \le \nu(f)$ for any $f \in m_V = \cup_{i=1}^{\infty} m_{R_i}$. Thus the value group of V is discrete. Since $[V/m_V : k] = \infty$, by Theorem 4.2 the rank of V does not increase under completion. □

When a valuation ring V with quotient field K is equicharacteristic and discrete of rank 1, it can be explicitly described by a representation in a power series ring in one variable over the residue field of V. In fact, since V is discrete of rank 1, it is Noetherian (Theorem 16, Section 10, Chapter VI [30]). As V is equicharacteristic, the m_V-adic completion $\hat{V}$ of V has a coefficient field L by Cohen's theorem, and thus $\hat{V} \cong L[[t]]$ is a power series ring in one variable over $L \cong V/m_V$. We have $V = K \cap \hat{V}$. The subtlety of this statement is that if k is a subfield of K contained in V such that V/m_V is not separably generated over k, then there may not exist a coefficient field L of $\hat{V}$ which contains k.

Although the completion of a rank 1 valuation ring is a power series ring, in positive characteristic, the valuation determined by associating to a system of parameters specific power series may not be easily recognizable from a series representation of the valuation ring. This can be seen from the contrast of the conclusions of Theorem 1.1 with the results of this section. The finiteness condition $[L : k] < \infty$ of the coefficient field of a series over a base field k does not characterize algebraicity of a series in positive characteristic, while the corresponding finiteness condition on residue field extensions does characterize algebraicity in the case of valuations dominating a local ring of Theorem 4.2. We illustrate this distinction in the following example.

Example 4.4. The valuation induced by the series of Example 2.3, whose coefficient field is infinitely algebraic over the base field k, has a residue field which is finite over k.

Proof. With notation of Example 2.3, we have a k-algebra homomorphism

$$R = k[u, v]_{(u,v)} \xrightarrow{\pi} \overline{k}[[x]]$$

defined by the substitutions

$$u = x, v = \sigma(x) = \sum_{i=1}^{\infty} t_i^{\frac{1}{p}} x^i.$$

π is 1–1 since x, y and the $t_i^{\frac{1}{p}}$ are algebraically independent over k. The order valuation on $\overline{k}[[x]]$ induces a rank 1 valuation ν on the quotient field of R. Let $v_1 = (\frac{v}{u})^p - t_1$.

$R_1 = R[\frac{v}{u}]_{(u,v_1)}$ is dominated by ν. From the expansion

$$v_1 = \sum_{i=1}^{\infty} t_{i+1} x^{ip},$$

we inductively define

$$v_{j+1} = \frac{v_j}{u^p} - t_{j+1} = \sum_{i=1}^{\infty} t_{i+j} x^{ip}$$

and

$$R_{j+1} = R_i \left[\frac{v_j}{u^p}\right]_{(u,v_{j+1})}$$

for $j \geq 1$. The R_j are dominated by ν for all j, so that $V = \cup_{j\geq 1} R_j$ is the valuation ring of ν. We have that the residue field of V is $V/m_V = R_1/m_{R_1} = k(t_1^{\frac{1}{p}})$. This is a finite extension of k, in contrast to the fact that the field of coefficients $L = k(\{t_i^{\frac{1}{p}} \mid i \in \mathbb{N}\})$ of $\sigma(x)$ has infinite degree over k. □

An especially strange representation of a rank 2 discrete valuation is given by the example (2) of a power series whose exponents have unbounded denominators.

Let k be a field of characteristic $p > 0$, and consider the series

$$\sigma = \sum_{i=1}^{\infty} x^{1-\frac{1}{p^i}} \tag{9}$$

of (2). σ is algebraic over $k(x)$, with irreducible relation $\sigma^p - x^{p-1}\sigma - x^{p-1} = 0$.

Consider the two-dimensional regular local ring $R_0 = k[x, y]_{(x,y)}$. x and y are regular parameters in R_0. Let $y = \sigma(x)$. We see from (9) that y does not have a fractional power series representation in terms of x. However, by expanding x in terms of y, we have an expansion

$$x = y^{\frac{p}{p-1}}(1+y)^{-\frac{1}{p-1}} \tag{10}$$

which represents x as a fractional power series in y with bounded denominators.

Let $g = y^p - x^{p-1}y - x^{p-1} \in R_0$. $g = 0$ has a singularity of order $p - 1$ in R. Let

$$R_1 = R\left[\frac{x}{y}, y\right]_{(\frac{x}{y}, y)}.$$

$x_1 = \frac{x}{y}$ and y are regular parameters in R_1. $g = y^{p-1} g_1$, where

$$g_1 = y - x_1^{p-1} y - x_1^{p-1}$$

is a strict transform of g in R_1. $g_1 = 0$ is nonsingular. From the equation $g_1 = 0$ we deduce that

$$\begin{aligned} y &= x_1^{p-1}(1 - x_1^{p-1})^{-1} \\ &= x_1^{p-1}(1 + x_1^{p-1} + x_1^{2(p-1)} + \cdots) \\ &= \sum_{i=1}^{\infty} x_1^{i(p-1)}, \end{aligned}$$

obtaining a standard power series expansion of y in terms of x.

We obtain a fractional power series of x_1 in terms of y with bounded denominators either from the equation $g_1 = 0$, or by substitution in (10).

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