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Supersymmetry and the formal loop space

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Abstract

For any algebraic super-manifold M we define the super-ind-scheme LM of formal loops and study the transgression map (Radon transform) on differential forms in this context. Applying this to the super-manifold $M = \mathcal{S}X$, the spectrum of the de Rham complex of a manifold X, we obtain, in particular, that the transgression map for X is a quasi-isomorphism between the $[2,3)$-truncated de Rham complex of X and the additive part of the $[1,2)$-truncated de Rham complex of $\mathcal{L}X$. The proof uses the super-manifold $\mathcal{S}\mathcal{S}X$ and the action of the Lie super-algebra $\mathfrak{sl}(1|2)$ on this manifold. This quasi-isomorphism result provides a crucial step in the classification of sheaves of chiral differential operators in terms of geometry of the formal loop space.
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Contents

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0. Introduction

Let X be a smooth complex algebraic variety. The *de Rham spectrum* $\mathcal{S}X = \mathrm{Spec}(\Omega^\bullet_X)$, is a super-manifold which can be seen as the configuration space of a supersymmetric particle moving in X, see, e.g., [34]. The particle itself can be understood as the super-manifold $\mathbb{A}^{0|1} = \mathrm{Spec}\,\Lambda[\eta]$. It was pointed out by M. Kontsevich that the de Rham differential comes from the internal symmetry of the particle, i.e., from the action of the super-group of automorphisms of $\mathbb{A}^{0|1}$. In fact, representations of this super-group are the same as cochain complexes, see [27] and Section 2.2 below.

The functor $\mathcal{S}$ can be applied to any super-manifold, in particular, we can form $\mathcal{S}\mathcal{S}X = \mathrm{Spec}(\Omega^\bullet_{\mathcal{S}X})$. It has a similar interpretation to the above, but in terms of $\mathbb{A}^{0|2} = \mathrm{Spec}\,\Lambda[\eta_1, \eta_2]$ which can be seen as an "$N = 2$ supersymmetric particle", moving in X. Mathematically, the most immediate part of $N = 2$ supersymmetry is the super-group $\underline{\mathrm{Aut}}(\mathbb{A}^{0|2})$, acting on $\mathcal{S}\mathcal{S}X$. Its Lie algebra includes the two natural differentials on $\Omega^\bullet_{\mathcal{S}X}$. A remarkable feature of the $N = 2$ case, lost if we pass to $\mathbb{A}^{0|N}$ for $N > 2$, is that $\underline{\mathrm{Aut}}(\mathbb{A}^{0|2})$ is isomorphic to the special linear super-group $SL_{1|2}$. Therefore $SL_{1|2}$ acts on the double complex $\Omega^\bullet_{\mathcal{S}X}$, and this action gives a detailed information about the cohomology of the rows and columns.

The goal of the present paper is to apply these ideas to the study of $\mathcal{L}X$, the ind-scheme of formal loops in X, introduced by us in [23]. We showed that $\mathcal{L}X$ possesses a nonlinear analog of a vertex algebra structure, called the structure of a *factorization semigroup*. Therefore, natural linear objects on $\mathcal{L}X$ give rise to vertex algebras. In particular, we showed how to obtain Ω^{ch}_X, the chiral de Rham complex of X, see [30], from geometry of $\mathcal{L}X$. Our point of view suggests a similar interpretation of the sheaves of chiral differential operators (CDO) on X studied in [13,14]: a CDO can be obtained from an object of the determinantal gerbe of $\mathcal{L}X$ which is *factorizing* (compatible with the factorization semigroup structure). In fact, the factorization structure on $\mathcal{L}X$ leads naturally to the factorization conditions for all sorts of geometric objects on $\mathcal{L}X$: functions, forms, line bundles, gerbes, etc. For functions and forms the factorization is understood in the additive sense, so we will speak about *additive forms* on $\mathcal{L}X$, see Definition 6.1.6. In [14], the CDO's were classified in terms of the complex $\Omega_X^{[2,3)}$ which is the second of the two truncated de Rham complexes below (on the Zariski topology of X):

$$\Omega_X^{[1,2)} = \{\Omega_X^1 \xrightarrow{d} \Omega_X^{2,\mathrm{cl}}\}, \qquad \Omega_X^{[2,3)} = \{\Omega_X^2 \xrightarrow{d} \Omega_X^{3,\mathrm{cl}}\}.$$

Here $\Omega_X^{i,\mathrm{cl}}$ is the sheaf of closed differential i-forms on X. Note that $\Omega_X^{[1,2)}$ governs rings of *twisted differential operators* on X, see [2]; such rings form a stack of Picard categories, of which the gerbe of CDO is often said to be a "higher" analog. On the other hand, our approach with the determinantal gerbe also leads to the complex $\Omega^{[1,2)}$, but on $\mathcal{L}X$.

Our main result, Corollary 6.3.5, says that $\Omega_X^{[2,3)}$ is quasi-isomorphic (although not isomorphic) to a subcomplex in $\Omega_{\mathcal{L}X}^{[1,2)}$ consisting of additive forms. The quasi-isomorphism is given by the transgression (Radon transform)

$$\tau : \Omega^p(X) \longrightarrow \Omega^{p-1}(\mathcal{L}X).$$

In fact, we prove a general statement about the full de Rham complexes of X and $\mathcal{L}X$ (Theorem 6.3.1), of which Corollary 6.3.5 is a consequence. The proof uses $N = 2$ supersymmetry.

In [25] we proved that additive functions f on $\mathcal{L}X$ are identified with closed 2-forms ω on X via a version of the *symplectic action functional*

$$\omega \longmapsto S(\omega) = d^{-1}\big(\tau(\omega)\big).$$

The argument in [25] used vertex algebras and the result of [30] on realization of $\Omega_X^{2,\mathrm{cl}}$ as the sheaf of vertex automorphisms of the chiral de Rham complex. Here we give a direct proof of this fact from first principles (Theorem 6.2.3), by expanding f around constant loops. Here X can be any super-manifold. Now, viewing additive forms on $\mathcal{L}X$ as additive functions on $\mathcal{SL}X = \mathcal{LS}X$, we identify their space with $\Omega^{2,\mathrm{cl}}_{\Omega_X^\bullet}$ which, by the $N = 2$ supersymmetry analysis, is quasi-isomorphic to the cohomological truncation

$$\Omega_X^{2,\mathrm{cl}} \longrightarrow \Omega_X^2 \longrightarrow \Omega_X^3 \longrightarrow \cdots,$$

giving Theorem 6.3.1. Note that $\mathcal{LS}X$ can be seen as the space of "super-loops" in X, i.e., of maps from a super-thickening of the punctured formal disk. In Section 5.3 we show how $\mathcal{LS}^N X$ gives rise to a factorization semigroup on any $(1|N)$-dimensional super-curve.

This paper was originally intended as an appendix to [26] but it seemed better to us to write it separately, collecting together the aspects of the theory with less emphasis on categorical issues. These categorical issues, i.e., complete yoga of factorization as applied to not just functions and forms on $\mathcal{L}X$, but $\mathcal{D}$-modules, line bundles, gerbes, etc., form the natural subject of [26], whose place in the logical order is after the present paper. In fact, factorizing gerbes can be given a de Rham-type description, much in the spirit of the book [8] by Brylinski. This description leads to a direct identification of the gerbe of CDO with the gerbe corresponding to the additive part of the complex $\Omega_{\mathcal{L}X}^{[1,2)}$. The results of the present paper, identifying this additive part with $\Omega_X^{[2,3)}$, provide then a clear explanation of the classification of [14] from the first principles.

Here is a brief outline of the paper. In the first section we provide the necessary background for scheme-theoretic algebraic geometry in the super-setting. As there seems to be no systematic reference in the required generality, we had to give a somewhat longer treatment. Section 2 is devoted to the discussion of extended supersymmetry from the point of view of super-version of schemes of infinitesimally near points in the spirit of A. Weil [33]. Here we analyze representations of $\underline{\mathrm{Aut}}(\mathbb{A}^{0|2}) = SL_{1|2}$ as double complexes with appropriate partial contracting homotopies. In Section 3, we discuss the formalism of super-ind-schemes, quite parallel to that of usual ind-schemes. In Section 4 we define formal loop spaces in the super-setting while in Section 5 we discuss their factorization structure. The formalism of factorization data which we discuss differs slightly from that of [3]; it is better adapted to studying coherence conditions needed for factorizing line bundles, gerbes, etc. Finally, in Section 6 we prove our main results: first about additive functions, then about additive forms.

1. Super-schemes

1.1. Basic definitions

We start by discussing basic concepts of algebraic geometry in the super situation, following [28,31,32]. See also [10] for a general background in a more differential-geometric context.

First of all, recall that a *ringed space* is a pair $X = (\underline{X}, \mathcal{O}_X)$ where $\underline{X}$ is a topological space, and $\mathcal{O}_X$ is a sheaf of rings, not necessarily commutative, on $\underline{X}$. A morphism $f : X = (\underline{X}, \mathcal{O}_X) \to Y = (\underline{Y}, \mathcal{O}_Y)$ of ringed spaces consists of a continuous map of spaces $f_\sharp : \underline{X} \to \underline{Y}$, and a morphism of sheaves of rings $f^\flat : f_\sharp^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$ on $\underline{X}$.

An *open embedding* of ringed spaces is a morphism f such that $f_\sharp$ is an open embedding of topological spaces, while $f^\flat$ is an isomorphism of sheaves of rings.

A *locally ringed space* is a ringed space $X = (\underline{X}, \mathcal{O}_X)$ such that each stalk $\mathcal{O}_{X,x}$, $x \in \underline{X}$, is a local ring. A *local morphism* of locally ringed spaces is a morphism f as above such that each morphism of stalks $f_x^\flat : \mathcal{O}_{Y, f_\sharp(x)} \to \mathcal{O}_{X,x}$ is a local homomorphism of local rings, i.e., takes the maximal ideal of one ring into the maximal ideal of the other. For example, an open embedding of locally ringed spaces is always a local morphism. We denote by **Lrs** the category of locally ringed spaces and their local morphisms.

We also denote by **Sch** the category of schemes. Recall that **Sch** is a full subcategory in **Lrs**. In particular, for any commutative ring R we have the scheme $\mathrm{Spec}(R)$ whose underlying topological space (i.e., the set of prime ideals in R with the Zariski topology) will be denoted $\underline{\mathrm{Spec}(R)}$.

We denote by **SAb** the symmetric monoidal category of $\mathbb{Z}/2$-graded abelian groups $A = A_{\bar{0}} \oplus A_{\bar{1}}$. The symmetry transformation $A \otimes B \to B \otimes A$ in this category is given by the Koszul sign rule:

$$a \otimes b \longmapsto (-1)^{\deg(a)\deg(b)} b \otimes a$$

on homogeneous elements. We denote by $\Pi : \mathbf{SAb} \to \mathbf{SAb}$ the functor of change of parity: $(\Pi A)_{\bar{0}} = A_{\bar{1}}$ and vice versa.

Recall that a *super-commutative ring* is a commutative ring object in the symmetric monoidal category **SAb**. Explicitly, it is a $\mathbb{Z}/2$-graded ring $R = R_{\bar{0}} \oplus R_{\bar{1}}$ such that $ab = (-1)^{\deg(a)\deg(b)} ba$ for homogeneous elements. The following is then clear.

Proposition 1.1.1. *Let R be a super-commutative ring. Then:*

(a) *A $\mathbb{Z}/2$-graded ideal $\mathfrak{p} = \mathfrak{p}_{\bar{0}} \oplus \mathfrak{p}_{\bar{1}} \subset R$ is prime (in the sense that $R/\mathfrak{p}$ has no zero-divisors), if and only if $\mathfrak{p}_{\bar{0}}$ is a prime ideal in $R_{\bar{0}}$. In this case $\mathfrak{p}_{\bar{1}} = R_{\bar{1}}$.*
(b) *The Jacobson radical of R is equal to the sum $\sqrt{R_{\bar{0}}} \oplus R_{\bar{1}}$.*
(c) *R is local with maximal ideal $\mathfrak{m}$ if and only if $R_{\bar{0}}$ is local with maximal ideal $\mathfrak{m}_{\bar{0}} = \mathfrak{m} \cap R_{\bar{0}}$.*

A *super-space* is a locally ringed space $(\underline{X}, \mathcal{O}_X)$ where $\mathcal{O}_X$ is equipped with a $\mathbb{Z}/2$-grading $\mathcal{O}_X = \mathcal{O}_{X,\bar{0}} \oplus \mathcal{O}_{X,\bar{1}}$ making it into a sheaf of super-commutative rings. A *morphism of super-spaces* is a local morphism of locally ringed spaces $f = (f_\sharp, f^\flat)$ such that $f^\flat$ preserves the $\mathbb{Z}/2$-grading. We denote by **Ssp** the category of super-spaces.

A super-space $X = (\underline{X}, \mathcal{O}_X)$ is called a *super-scheme* if $(\underline{X}, \mathcal{O}_{X,\bar{0}})$ is a scheme and $\mathcal{O}_{X,\bar{1}}$ is a quasi-coherent of $\mathcal{O}_{X,\bar{0}}$-module. We denote by **Ssch** $\subset$ **Ssp** the full subcategory formed by super-schemes.

In particular for any super-commutative ring R we have a super-scheme $\mathrm{Spec}(R)$. Its underlying space is $\underline{\mathrm{Spec}}(R) = \underline{\mathrm{Spec}}(R_{\bar{0}})$, with $\mathcal{O}_{\mathrm{Spec}(R)} = \mathcal{O}_{\mathrm{Spec}(R),\bar{0}} \oplus \mathcal{O}_{\mathrm{Spec}(R),\bar{1}}$ where $\mathcal{O}_{\mathrm{Spec}(R),\bar{0}}$ is the structure sheaf of $\mathrm{Spec}(R_{\bar{0}})$, while $\mathcal{O}_{\mathrm{Spec}(R),\bar{1}}$ is the quasi-coherent sheaf of $\mathcal{O}_{\mathrm{Spec}(R),\bar{0}}$-modules corresponding to the $R_{\bar{0}}$-module $R_{\bar{1}}$. Super-schemes of the form $\mathrm{Spec}(R)$ will be called *affine*. It is clear that every super-scheme is locally isomorphic to an affine super-scheme.

Given a super-scheme X, a *quasi-coherent sheaf* of $\mathcal{O}_X$-modules is a $\mathbb{Z}/2$-graded sheaf $\mathcal{F} = \mathcal{F}_{\bar{0}} \oplus \mathcal{F}_{\bar{1}}$ which is quasi-coherent as a sheaf of $\mathcal{O}_{X,\bar{0}}$-modules. For any super-commutative ring R quasi-coherent sheaves on $\mathrm{Spec}(R)$ are in bijection with $\mathbb{Z}/2$-graded R-modules.

For a morphism of super-commutative algebras $A \to B$ we denote by $\Omega^1(B/A)$ the $\mathbb{Z}/2$-module of Kähler differentials of B over A understood in the super-sense, so that $d: B \to \Omega^1(B/A)$ preserves the $\mathbb{Z}/2$-grading, annihilates the image of A, and satisfies the super-Leibniz rule. Alternatively,

$$\Omega^1(B/A) = I/I^2, \quad I = \mathrm{Ker}\{B \otimes_A B \xrightarrow{\text{mult.}} B\}. \tag{1.1.2}$$

For a morphism of super-schemes $X \to Y$ we have then the quasi-coherent sheaf $\Omega^1_{X/Y}$ on X.

Given a super-scheme X and a quasi-coherent sheaf $\mathcal{A}$ of super-commutative $\mathcal{O}_X$-algebras, we have a super-scheme $\mathrm{Spec}_X(\mathcal{A}) \to X$ obtained by gluing affine schemes $\mathrm{Spec}(\mathcal{A}(U))$ for open affine subschemes $U \subset X$. A morphism $Y \to X$ of super-schemes is called *affine*, if it is isomorphic to one of the form $\mathrm{Spec}_X(\mathcal{A}) \to X$. Note the particular case when $\mathcal{A} = \mathcal{O}_X/\mathcal{I}$ is the quotient of $\mathcal{O}_X$ by a sheaf of ideals. In this case $Y = \mathrm{Spec}_X(\mathcal{A})$ is called a *closed sub-super-scheme* in X, and any morphism isomorphic to $Y \to X$ of this type is called a *closed embedding* of super-schemes. An *immersion* is a morphism of super-schemes which can be represented as the composition of an open embedding followed by a closed embedding.

A super-scheme X will be called *quasi-compact*, if the topological space $\underline{X}$ is quasi-compact, i.e., each open covering of $\underline{X}$ has a finite sub-covering. For example every affine super-scheme is quasi-compact.

As in the case of ordinary schemes, we have the following fact.

Proposition 1.1.3.

(a) *The category* **Ssch** *has finite projective limits, in particular, finite products and fiber products.*
(b) *Let I be a filtering poset and $(X_i)_{i\in I}$ be a projective system of super-schemes with structure morphisms $u_{ij} : X_j \to X_i$, given for $i \leqslant j$. If all u_{ij} are affine morphisms, then the limit*

$\varprojlim^{\mathbf{Ssch}}_{i\in I} X_i$ *exists. Denoting this limit by* X, *we have that the natural projection* $p_i : X \to X_i$ *is affine for any* i, *in fact*

$$X = \operatorname{Spec}_{X_i}\Big(\varinjlim_{j\geqslant i} u_{ij*}\mathcal{O}_{X_j}\Big).$$

Moreover, we have $\underline{X} = \varprojlim^{\mathbf{Top}}_{i\in I} \underline{X}_i$.

Proof. (a) In any category, existence of finite projective limits is equivalent to the existence of finite products and fiber products. Now, for affine super-schemes, the fiber product of

$$\operatorname{Spec}(A) \longrightarrow \operatorname{Spec}(C) \longleftarrow \operatorname{Spec}(B)$$

is found as $\operatorname{Spec}(A \otimes_C B)$, like for ordinary schemes. After that, fiber products of arbitrary super-schemes are defined by gluing affine charts of the kind described.

(b) The argument is identical to [17] (8.2.3) (existence of the limit and its realization as a relative spectrum) and (8.2.10) (description of $\underline{X}$). □

Given a super-scheme $X = (\underline{X}, \mathcal{O}_X)$ its *even part* is defined to be the scheme

$$X_{\text{even}} = \big(\underline{X}, \mathcal{O}_{X,\bar{0}}/\big(\mathcal{O}^2_{X,\bar{1}}\big)\big), \tag{1.1.4}$$

while the corresponding *reduced scheme* is

$$X_{\text{red}} = \big(\underline{X}, \mathcal{O}_{X,\bar{0}}/\sqrt{\mathcal{O}_{X,\bar{0}}}\big), \tag{1.1.5}$$

similarly to the case of ordinary schemes.

From now on we work over the field $\mathbb{C}$ of complex numbers. All rings will be assumed to contain $\mathbb{C}$ and all super-schemes will be super-schemes over $\mathbb{C}$. We denote by **SVect** the symmetric monoidal category of $\mathbb{Z}/2$-graded $\mathbb{C}$-vector spaces, and by **Alg** the category of super-commutative $\mathbb{C}$-algebras. We also denote **Aff** the category of affine super-schemes, i.e., the dual category of **Alg**.

An affine super-scheme (over $\mathbb{C}$) is said to be of *finite type* if it is isomorphic to $\operatorname{Spec}(R)$ where R is a finitely generated super-commutative $\mathbb{C}$-algebra. More generally a *super-scheme of finite type* is a super-scheme which can be covered by finitely many affine super-schemes of finite type. Let **Fsch** be the full category of super-schemes of finite type.

Example 1.1.6. (a) For $d_1, d_2 \geqslant 0$ we denote by $\mathbb{C}^{d_1|d_2} \in \mathbf{SVect}$ the coordinate $\mathbb{Z}/2$-graded space with d_1 even dimensions and d_2 odd dimensions. For $R \in \mathbf{Alg}$ we denote by $R^{d_1|d_2}$ the $\mathbb{Z}/2$-graded R-module $R \otimes \mathbb{C}^{d_1|d_2}$. If R is local, then any finitely generated projective R-module M is free, i.e., isomorphic to $R^{d_1|d_2}$ for a unique pair $(d_1|d_2)$ which is called the *rank* of M. If R is finitely generated, then a finitely generated projective R-module is free, locally on the Zariski topology of $\underline{\operatorname{Spec}}(R)$, so its rank is a locally constant function $\underline{\operatorname{Spec}}(R) \to \mathbb{Z}_+ \times \mathbb{Z}_+$.

We define $\mathbb{A}^{d_1|d_2}$, the *affine super-space* of dimension $d_1|d_2$ to be the super-scheme

$$\mathbb{A}^{d_1|d_2} = \operatorname{Spec}\big(\mathbb{C}[x_1, \ldots, x_{d_1}] \otimes \Lambda[\xi_1, \ldots, \xi_{d_2}]\big).$$

(b) It will be convenient to use the following unified notation. Given $N = d_1 + d_2$ generators $a_1, \ldots, a_N$ of which d_1 are even and d_2 are odd, we will simply write $\mathbb{C}[a_1, \ldots, a_N]$ for the tensor product of the polynomial algebra on the even generators and the exterior algebra on the odd generators. We also write $\mathbb{C}[[a_1, \ldots, a_N]]$ for the completion of $\mathbb{C}[a_1, \ldots, a_N]$ with respect to the ideal $(a_1, \ldots, a_N)$, which is the tensor product of the formal power series algebra on the even generators and the exterior algebra on the odd generators.

(c) For any $d_1, d_2 \geqslant 0$ we have the group super-scheme $GL_{d_1|d_2}$ such that for a super-commutative algebra R, the group $GL_{d_1|d_2}(R)$ consists of $\mathbb{Z}/2$-homogeneous automorphisms of the R-module $R \otimes \mathbb{C}^{d_1|d_2}$. Such automorphisms can be represented by block matrices over R of format $(d_1 + d_2) \times (d_1 + d_2)$

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with entries of A, D belonging to $R_{\bar{0}}$, entries of B, C to $R_{\bar{1}}$, and A, D invertible. The group super-scheme $GL_{d_1|d_2}$ is called the *general linear group* of format $d_1|d_2$. In particular, $GL_{1|0} = \mathbb{G}_m$ is the multiplicative group. The *Berezin determinant* is a morphism of group super-schemes $\mathrm{ber}: GL_{d_1|d_2} \to \mathbb{G}_m$ which on R-points sends a matrix g as above to

$$\mathrm{ber}(g) = \det\bigl(A - BD^{-1}C\bigr) / \det(D) \in R_{\bar{0}}^{*} = \mathbb{G}_m(R).$$

Its kernel is denoted by $SL_{d_1|d_2}$ and called the *special linear group* of format $d_1|d_2$.

1.2. Smooth and étale morphisms

Let $f : X \to Y$ be a morphism of super-schemes. As in the classical (even) case, we say that f is *locally of finite presentation*, if $\mathcal{O}_X$ is, locally on the Zariski topology of X, finitely presented as an $\mathcal{O}_Y$-algebra, i.e., given by finitely many generators and relations.

A morphism of super-commutative algebras $u : R \to R'$ is called a *simple extension*, if u is surjective, and $I = \mathrm{Ker}(u)$ satisfies $I^2 = 0$. Recall that each super-scheme X gives a covariant functor $h^X : \mathbf{Alg} \to \mathbf{Set}$, sending R to $\mathrm{Hom}_{\mathbf{Ssch}}(\mathrm{Spec}(R), X)$.

Definition 1.2.1. (a) Let $f : h \to h'$ be a morphism of covariant functors $\mathbf{Alg} \to \mathbf{Set}$. We say that f is *formally smooth* (resp. *formally étale*) if, for any simple extension $R \to R'$ the natural morphism

$$h(A) \longrightarrow h'(A) \times_{h'(A')} h\bigl(A'\bigr)$$

is surjective (resp. bijective).

(b) A morphism $f : X \to Y$ of super-schemes is called *formally smooth* (resp. *formally étale*), if the corresponding morphism of functors $h^X \to h^Y$ is formally smooth (resp. formally étale).

A morphism of super-schemes is called *smooth*, if it is formally smooth and locally of finite presentation.

Proposition 1.2.2.

(a) *Let $\psi : A \to B$ be a morphism of super-commutative algebras such that $\psi^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a formally smooth morphism of super-schemes. Then the B-module $\Omega^1(B/A)$ is projective.*
(b) *Let $f : X \to Y$ be a smooth morphism of super-schemes. Then $\Omega^1_{X/Y}$ is locally free, as a sheaf of $\mathcal{O}_X$-modules.*

In particular, the rank of $\Omega^1_{X/Y}$ is a locally constant function on $\underline{X}$ with values in $\mathbb{Z}_+ \times \mathbb{Z}_+$ denoted by $\dim(X/Y)$ and called the *relative dimension* of X over Y. If f is étale, then $\dim(X/Y)$ is identically equal to 0.

Proof of Proposition 1.2.2. (a) The classical argument (contained in a more general form in [16, (19.5.4.1)]), is completely formal and goes in our case as follows.

Any $\mathbb{Z}/2$-graded B-module Q gives a super-commutative B-algebra $B \oplus Q$ with $Q^2 = 0$ and with the multiplication of B and Q given by the module structure. A B-module homomorphism $u : \Omega^1(B/A) \to Q$ is the same as a Q-valued derivation $\delta : B \to Q$ vanishing on A, and this gives a homomorphism of B-algebras

$$(\mathrm{Id}, \delta) : B \longrightarrow B \oplus Q, \quad b \longmapsto \big(b, \delta(b)\big).$$

To prove that $\Omega^1(B/A)$ is projective, let $s : P \to Q$ be a surjective morphism of B-modules, and $u : \Omega^1(B/A) \to Q$ be any morphism of B-modules. We prove that u can be lifted to a $v : \Omega^1(B/A) \to P$. Indeed, $(\mathrm{Id} \oplus s) : B \oplus P \to B \oplus Q$ is a simple extension of super-commutative algebras, and we have a commutative square

$$\begin{tikzcd} A \arrow[r, "(\psi,0)"] \arrow[d, "\psi"'] & B \oplus P \arrow[d, "\mathrm{Id}\oplus s"] \\ B \arrow[r, "(\mathrm{Id},\delta)"] & B \oplus Q. \end{tikzcd}$$

So by the condition that $\psi^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is formally smooth, we find that there is an algebra homomorphism $w : B \to B \oplus P$ splitting the square into two commutative triangles. The second component of w gives a derivation $B \to P$ lifting δ, i.e., a homomorphism $v : \Omega^1(B/A) \to P$ as claimed.

(b) This follows from the fact (proved in the same way as in the commutative case) that a finitely presented projective module over any super-commutative algebra is locally free. □

A *smooth algebraic super-variety* is a super-scheme X of finite type (over $\mathbb{C}$) such that $X \to \mathrm{Spec}(\mathbb{C})$ is a smooth morphism. In this case X_{even} is a smooth algebraic variety over $\mathbb{C}$ in the usual sense. We write $\dim(X)$ for $\dim(X/\mathbb{C})$. If X is irreducible (i.e., $\underline{X}$ is an irreducible topological space), then this function is constant, so $\dim(X) = (d_1|d_2)$ for some $d_1, d_2 \in \mathbb{Z}_+$, and $\dim(X_{\mathrm{even}}) = d_1$. If $\dim(X) = (1|N)$, we say that X is a *super-curve*.

Proposition 1.2.3. *Let $X \to Y$ be a smooth morphism, $x \in X(\mathbb{C})$ be such that* $\dim(X/Y) = (d_1|d_2)$ *at x. Then there are Zariski open sets $U \subset X$ containing x such that there is a morphism of Y-schemes $\phi : U \to Y \times \mathbb{A}^{d_1|d_2}$ which is étale.*

Proof. Proof analogous to the purely even case which is proved in [18, (17.11.4)]. As f is locally of finite presentation, we find U' and V' with $x \in U'$, $f(x) \in V'$, $f(U') \subset V'$ so that there a morphism of V'-schemes $i : U' \to V' \times \mathbb{A}^{D_1|D_2}$ which is a closed embedding. Then, we can choose a subset of the coordinates x_i, ξ_j on $\mathbb{A}^{D_1|D_2}$ such that dx_i, $d\xi_j$ from that subset form a set of free generators of $\Omega^1_{U'/V'}$ in some $U \subset U'$ containing x. The projection on the coordinate affine subspace $\mathbb{A}^{d_1|d_2}$ corresponding to this subset, is the étale morphism required. $\square$

2. Infinitesimally near points and supersymmetry

2.1. Infinitesimally near points

Definition 2.1.1. Let $u, u' : S \to X$ be morphisms of super-schemes. We say that u and u' are *infinitesimally near*, if $u = u'$ on S_{red}.

In this section, we want to study super-schemes which classify such morphisms for a particular class of super-schemes S. We start with general categorical remarks.

Let $\mathbf{C}$ be any category with finite products. Given two objects B, C of $\mathbf{C}$, we have the contravariant functor

$$\mathbf{C} \longrightarrow \mathbf{Set}, \quad T \longmapsto \mathrm{Hom}_{\mathbf{C}}(T \times B, C).$$

If this functor is representable, then the representing object of $\mathbf{C}$ is denoted by $\underline{\mathrm{Hom}}(B, C)$ and is called the *internal Hom* from B to C. Note that if $B = C$, then $\underline{\mathrm{Hom}}(B, B)$ is a semigroup object in $\mathbf{C}$. Indeed, for every T, the set $\mathrm{Hom}_{\mathbf{C}}(S \times B, B)$ is a semigroup with unit being the canonical projection $S \times B \to B$. Further, consider the functor associating to T the set of invertible elements in the semigroup $\mathrm{Hom}_{\mathbf{C}}(T \times B, B)$. If this functor is representable, then the representing object in $\mathbf{C}$ is denoted by $\underline{\mathrm{Aut}}(B)$ and is called the *internal automorphism group* of B. It is a group object of $\mathbf{C}$.

We now specialize to $\mathbf{C} = \mathbf{Ssch}$. Let $\mathfrak{o}$ be a finite dimensional local super-commutative $\mathbb{C}$-algebra.

Proposition 2.1.2.

(a) *For any super-scheme S we have an identification of super-spaces* $T \times \mathrm{Spec}(\mathfrak{o}) = (\underline{T}, \mathcal{O}_T \otimes \mathfrak{o})$.
(b) *Let X be any super-scheme. Then there exists the internal Hom super-scheme* $X^{\mathfrak{o}} = \underline{\mathrm{Hom}}(\mathrm{Spec}(\mathfrak{o}), X)$ *representing the functor*

$$T \longmapsto \mathrm{Hom}\big(T \times \mathrm{Spec}(\mathfrak{o}), X\big).$$

(c) *If $U \subset X$ is open then $U^{\mathfrak{o}} = X^{\mathfrak{o}} \times_X U$. In particular, $U^{\mathfrak{o}}$ is open in $X^{\mathfrak{o}}$.*
(d) *We have $(X^{\mathfrak{o}_1})^{\mathfrak{o}_2} = X^{\mathfrak{o}_1 \otimes \mathfrak{o}_2}$.*
(e) *The functor $X \mapsto X^{\mathfrak{o}}$ takes closed embeddings to closed embeddings.*

The super-scheme $X^{\mathfrak{o}}$ will be called the *super-scheme of* $\mathfrak{o}$*-infinitesimally near points* of X. This terminology and notation is borrowed from A. Weil [33].

Proof of Proposition 2.1.2. (a) Clear since $\mathfrak{o}$ is a finite dimensional local super-commutative algebra, and so its maximal ideal consists of nilpotent elements.

(b) Assume that $X = \mathrm{Spec}(A)$ is affine. Choose a basis $(e_i)_{i\in I}$ of homogeneous elements of $\mathfrak{o}$ with the following properties. First, we assume that I has a distinguished element 0, and $e_0 = 1$. Second, we assume that all e_i, $i \neq 0$, lie in the maximal ideal of $\mathfrak{o}$. After this, write the multiplication law in $\mathfrak{o}$ as

$$e_i e_j = \sum_k c_{ij}^k e_k.$$

Define a super-commutative algebra $A^{\mathfrak{o}}$ containing A generated by symbols $a[i]$, with $a \in A$, $i \in I$, subjects to the relations

$$\begin{aligned} (ab)[k] &= \sum_{i,j} c_{ij}^k a[i] b[j], \\ (a + \lambda b)[i] &= a[i] + \lambda\big(b[i]\big), \quad a, b \in A,\ i \in I,\ \lambda \in \mathbb{C}. \end{aligned} \tag{2.1.3}$$

Here the degree of $a[i]$ is the sum of the degrees of a and e_i. Notice that the correspondence $a \mapsto a[0]$ defines an algebra embedding $A \subset A^{\mathfrak{o}}$, because $(ab)[0] = a[0]b[0]$ for each a, b. We claim that

$$\mathrm{Hom}_{\mathbf{Alg}}\big(A^{\mathfrak{o}}, R\big) = \mathrm{Hom}(A, R \otimes \mathfrak{o}),$$

for each super-commutative $\mathbb{C}$-algebra R. Indeed, given $f : A \to R \otimes \mathfrak{o}$, we expand it in the form

$$f(a) = \sum_i f_i(a) \otimes e_i, \quad a \in A.$$

Then we form the map

$$\phi : A^{\mathfrak{o}} \longrightarrow R, \quad a[i] \longmapsto f_i(a).$$

Note that the relations in $A^{\mathfrak{o}}$ insure that ϕ is a well-defined homomorphism. This proves (b) for $X = \mathrm{Spec}(A)$, and an affine super-scheme $T = \mathrm{Spec}(R)$. This implies the equality for any T in virtue of part (a), because the two functors are sheaves on the Zariski topology of T.

We next prove (c) in the particular case where $X = \mathrm{Spec}(A)$ is affine and $U = \mathrm{Spec}(A[s^{-1}])$ is a principal open subset. We identify the functors represented by $U^{\mathfrak{o}}$ and $X^{\mathfrak{o}} \times_X U$ on an affine super-scheme $T = \mathrm{Spec}(R)$. First, $\mathrm{Hom}(T, U^{\mathfrak{o}})$ consists of algebra homomorphisms $f : A \to R \otimes \mathfrak{o}$ such that $f(s)$ is invertible in $R \otimes \mathfrak{o}$.

Next, $\mathrm{Hom}(T, X^{\mathfrak{o}} \times_X U)$ consists of f's as before such that $f_0(s)$ is invertible in R. They coincide by Nakayama's lemma. Having proved (c), and (b) for an affine X, we deduce (b), (c) for any X by gluing along open parts.

Part (d) is clear because the two super-schemes represent the same functor.

Finally, let us prove (e). It is enough to observe that a surjective algebra homomorphism $A \to B$ yields a surjective algebra homomorphism $A^{\mathfrak{o}} \to B^{\mathfrak{o}}$. $\square$

Consider the particular case where $X = \mathrm{Spec}(\mathfrak{o})$. Then

$$\mathrm{Spec}(\mathfrak{o})^{\mathfrak{o}} = \underline{\mathrm{Hom}}\big(\mathrm{Spec}(\mathfrak{o}), \mathrm{Spec}(\mathfrak{o})\big)$$

is a semigroup super-scheme.

Proposition 2.1.4. *The object* $\underline{\mathrm{Aut}}(\mathrm{Spec}(\mathfrak{o}))$ *exists. It is an open subgroup-super-scheme in the semigroup super-scheme* $\mathrm{Spec}(\mathfrak{o})^{\mathfrak{o}}$.

We abbreviate $G_{\mathfrak{o}} = \underline{\mathrm{Aut}}(\mathrm{Spec}(\mathfrak{o}))$. Its Lie algebra $\mathfrak{g}_{\mathfrak{o}} = \mathrm{Der}_{\mathbb{C}}(\mathfrak{o}, \mathfrak{o})$ is just the Lie superalgebra of derivations of the super-commutative $\mathbb{C}$-algebra $\mathfrak{o}$. By construction $G_{\mathfrak{o}}$ acts on $X^{\mathfrak{o}}$ for any X.

Proof of Proposition 2.1.4. By construction the algebra $\mathfrak{o}^{\mathfrak{o}}$ is generated by the elements $u_{ij} = e_i[j]$ of degree equal to the sum of the degrees of e_i and e_j. We have therefore a matrix $U = (u_{ij})_{i,j\in I}$ over $\mathfrak{o}^{\mathfrak{o}}$. Let $\mathfrak{o}^{\mathfrak{o}}[U^{-1}]$ be the localization of $\mathfrak{o}^{\mathfrak{o}}$ obtained by adjoining the matrix elements of U^{-1}. More precisely, we have a decomposition $I = I_{\bar{0}} \sqcup I_{\bar{1}}$ according to the parities of the e_i's. The matrix U has the corresponding block decomposition

$$\begin{pmatrix} U_{\bar{0}\bar{0}} & U_{\bar{0}\bar{1}} \\ U_{\bar{1}\bar{0}} & U_{\bar{1}\bar{1}} \end{pmatrix},$$

and elements of U_{pq} have the $\mathbb{Z}/2$-degree $p+q$. Therefore the algebra $\mathfrak{o}^{\mathfrak{o}}[U^{-1}]$ is obtained by inverting the determinants of the even matrices $U_{\bar{0}\bar{0}}$ and $U_{\bar{1}\bar{1}}$. Our proposition is implied by the following.

Lemma 2.1.5. *The functor assigning to a given super-scheme T the set of invertible elements in* $\mathrm{Hom}_{\mathbf{Sch}}(T \times \mathrm{Spec}(\mathfrak{o}), \mathrm{Spec}(\mathfrak{o}))$ *is represented by* $\mathrm{Spec}(\mathfrak{o}^{\mathfrak{o}}[U^{-1}])$.

Proof. Assume that $T = \mathrm{Spec}(R)$. Then

$$\mathrm{Hom}_{\mathbf{Sch}}\big(T \times \mathrm{Spec}(\mathfrak{o}), \mathrm{Spec}(\mathfrak{o})\big) = \mathrm{Hom}_{\mathbf{Alg}}(\mathfrak{o}, R \otimes \mathfrak{o}).$$

To every homomorphism $f : \mathfrak{o} \to R \otimes \mathfrak{o}$ we associate the matrix (f_{ij}) over R such that

$$f(e_i) = \sum_j f_{ij} \otimes e_j.$$

Then the composition in the semigroup $\mathrm{Hom}_{\mathbf{Alg}}(\mathfrak{o}, R \otimes \mathfrak{o})$ corresponds to the multiplication of matrices. Next, we have an identification

$$\mathrm{Hom}_{\mathbf{Alg}}(\mathfrak{o}, R \otimes \mathfrak{o}) = \mathrm{Hom}_{\mathbf{Alg}}\big(\mathfrak{o}^{\mathfrak{o}}, R\big)$$

which takes f to the map $u_{ij} \mapsto f_{ij}$. Therefore f is invertible if and only if the matrix (f_{ij}) is invertible over R, which means that the matrix U is mapped to an invertible matrix. $\square$

2.2. $N=1$ supersymmetry

Let $\mathfrak{o}=\Lambda[\eta]$ be the exterior algebra in one variable, so that $\mathrm{Spec}(\mathfrak{o})=\mathbb{A}^{0|1}$.

Let us describe the group super-scheme

$$G_{\Lambda[\eta]}=\underline{\mathrm{Aut}}\big(\mathbb{A}^{0|1}\big)$$

and its Lie algebra. For any super-commutative algebra R the group

$$\mathrm{Hom}\big(\mathrm{Spec}(R),G_{\Lambda[\eta]}\big)$$

consists of changes of variables of the form $\eta\mapsto a+b\eta$, where $a\in R_{\bar 1}$ is arbitrary and $b\in R_{\bar 0}$ is invertible. The even part of the super-group is $\mathbb{G}_m$. The Lie super-algebra $\mathrm{Der}\,\Lambda[\eta]$ consists of the derivations

$$(a+b\eta)\frac{d}{d\eta},\quad a,b\in\mathbb{C}.$$

So its basis is formed by

$$D=\frac{d}{d\eta},\qquad \Theta=\eta\frac{d}{d\eta},$$

with D odd and Θ even, subject to the relations

$$[D,D]=[\Theta,\Theta]=0,\qquad [\Theta,D]=D. \tag{2.2.1}$$

The following fact was pointed out by M. Kontsevich [27].

Proposition 2.2.2. *Let $V=V_{\bar 0}\oplus V_{\bar 1}$ be a super-vector space. Then an action of $G_{\Lambda[\eta]}$ on V is the same as a structure of a cochain complex on V, i.e., a choice of a $\mathbb{Z}$-grading $V=\bigoplus_{n\in\mathbb{Z}}V^n$ such that*

$$V_{\bar 0}=\bigoplus_{n\in 2\mathbb{Z}}V^n,\qquad V_{\bar 1}=\bigoplus_{n\in 1+2\mathbb{Z}}V^n,$$

and a differential $d:V\to V$ of degree 1 with $d^2=0$.

Proof. The action of $\mathbb{G}_m\subset G_{\Lambda[\eta]}$ gives the grading, so that the action of Θ is given by $\Theta=n$ on V^n. The action of $D\in\mathrm{Der}\,\Lambda[\eta]$ gives d. The fact that d is of degree 1 follows from the relation $[\Theta,D]=D$. $\square$

Given a super-scheme X, we denote $X^{\Lambda[\eta]}=\underline{\mathrm{Hom}}(\mathbb{A}^{0|1},X)$ by $\mathcal{S}X$ and call it the *de Rham spectrum* of X. The super-scheme $\mathbb{A}^{0|1}$ can be called the $N=1$ *supersymmetric particle* in the same sense as $\mathrm{Spec}(\mathbb{C})$ can be thought as representing a point particle. The super-scheme $\mathcal{S}X$ is therefore the configuration space of an $N=1$ supersymmetric particle moving in X.

Denote by Ω^1_X the sheaf of Kähler differentials on X, and $\Omega^\bullet_X=S^\bullet(\Pi\Omega^1_X)$ be the sheaf of differential forms on X. Here Π is the change of parity functor. The derivation $d:\mathcal{O}_X\to\Omega^1_X$

gives rise to a derivation d on $\Omega_X^\bullet$ of degree one and square zero. Let $\varpi : \mathcal{S}X \to X$ be the projection.

Proposition 2.2.3. *We have $\varpi_* \mathcal{O}_{\mathcal{S}X} = \Omega_X^\bullet$, with the structure of a complex on the right-hand side corresponding to the $G_{\Lambda[\eta]}$-action on the left-hand side.*

Proof. Let $X = \mathrm{Spec}(A)$. A basis of the algebra $\Lambda[\eta]$ consists of two elements $e_0 = 1$ and $e_1 = \eta$. Therefore $A^{\Lambda[\eta]}$ is the algebra generated by $a[0] = a$, $a[1]$, given for $a \in A$ and subject to the relations

$$(ab)[1] = a\big(b[1]\big) + a[1]b, \quad a, b \in A.$$

These relations are identical to those defining Ω_X^1, with $a[1]$ corresponding to da. Further, $\deg(a[1]) = \deg(a) = 1$. So taking the super-commutative algebra $A^{\Lambda[\eta]}$ amounts to forming the symmetric algebra of $\Pi\Omega_X^1$. $\square$

Example 2.2.4. In particular, the de Rham differential in $\Omega_X^\bullet$ corresponds to a vector field D on $\mathcal{S}X$. Assume that X is a smooth super-manifold with local coordinates $x_1, \ldots, x_n$. Then on $\mathcal{S}X$ we have local coordinates $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$ where $\xi = dx_i$. The vector fields D and Θ have the form

$$D = \sum \xi_i \frac{\partial}{\partial x_i}, \qquad \Theta = \sum \xi_i \frac{\partial}{\partial \xi_i}.$$

2.3. $N = 2$ *supersymmetry*

Let $\mathfrak{o} = \Lambda[\eta_1, \eta_2]$ be the exterior algebra in two variables. The super-scheme $\mathrm{Spec}(\mathfrak{o}) = \mathbb{A}^{0|2}$ can be called the $N = 2$ *supersymmetric particle*. The group super-scheme

$$G_{\Lambda[\eta_1,\eta_2]} = \underline{\mathrm{Aut}}\big(\mathbb{A}^{0|2}\big)$$

and its Lie algebra $\mathrm{Der}(\Lambda[\eta_1, \eta_2])$ possess remarkable symmetry properties. By definition, for any super-commutative algebra R the group

$$\mathrm{Hom}\big(\mathrm{Spec}(R), G_{\Lambda[\eta_1,\eta_2]}\big)$$

consists of change of variables of the form

$$\begin{aligned}
\eta_1 &\longmapsto a^1 + b_1^1\eta_1 + b_2^1\eta_2 + c^1\eta_1\eta_2, \\
\eta_2 &\longmapsto a^2 + b_1^2\eta_1 + b_2^2\eta_2 + c^2\eta_1\eta_2
\end{aligned}$$

where $a^i, c^i \in R_{\bar{1}}$ are arbitrary and $b_j^i \in R_{\bar{0}}$ are such that the matrix (b_j^i) is invertible. The even part of the group is GL_2. Let us introduce special notations for the elements of the obvious basis of the Lie super-algebra $\mathrm{Der}(\Lambda[\eta_1, \eta_2])$:

$$D_i = \frac{\partial}{\partial \eta_i}, \qquad D_1^* = \eta_1\eta_2 \frac{\partial}{\partial \eta_1}, \qquad D_2^* = \eta_2\eta_1 \frac{\partial}{\partial \eta_2}, \tag{2.3.1}$$

$$\Theta_i = \eta_i \frac{\partial}{\partial \eta_i}, \qquad E = \eta_i \frac{\partial}{\partial \eta_2}, \qquad F = \eta_2 \frac{\partial}{\partial \eta_1}, \quad i = 1, 2. \tag{2.3.2}$$

We will call the D_i the *differentials*, the D_i^* the *homotopies*. They exhaust the odd basis elements. The even elements Θ_i will be called the *grading operators*, while E, F will be called the $\mathfrak{sl}_2$*-operators*. Note that

$$\big[D_1, D_1^*\big] = \Theta_2, \qquad \big[D_2, D_2^*\big] = \Theta_1. \tag{2.3.3}$$

The following classical fact (known to V.G. Kac in 1970's, see [22, (3.3.3)]), explains the special role of the $N = 2$ case.

Proposition 2.3.4. *The group super-scheme $G_{\Lambda[\eta_1,\eta_2]}$ is isomorphic to $SL_{1|2}$.*

Proof *(sketch).* We have the $1|2$-dimensional super-space $\Lambda[\eta_1, \eta_2]/\mathbb{C} \cdot 1 \simeq \mathbb{C}^{1|2}$. The group super-scheme $G_{\Lambda[\eta_1,\eta_2]}$ acts on this space by linear transformations, so we have a morphism of group super-schemes $G_{\Lambda[\eta_1,\eta_2]} \to GL_{1|2}$. We then verify directly that this morphism factors through $SL_{1|2}$. To see that the resulting morphism $\varphi : G_{\Lambda[\eta_1,\eta_2]} \to SL_{1|2}$ is an isomorphism, we first verify this on the level of the underlying even schemes, which are identified with GL_2 for both the source and the target of φ. After this it remains to verify that φ induces an isomorphism on the level of Lie super-algebras. This is checked directly, using the above basis in $\mathrm{Der}(\Lambda[\eta_1, \eta_2])$ and a standard basis in $\mathfrak{sl}_{1|2}$. □

Motivated by Proposition 2.2.2, we give the following

Definition 2.3.5. An $N = 2$ *supersymmetric complex* is a super-vector space V with an action of the group super-scheme $G_{\Lambda[\eta_1,\eta_2]} = SL_{1|2}$.

Our goal is now to analyze the structures on an $N = 2$ supersymmetric complex V in more detail. First of all, the action on V of the torus $\mathbb{G}_m \times \mathbb{G}_m \subset GL_2$ in the even part of $G_{\Lambda[\eta_1,\eta_2]}$, gives a bigrading $V = \bigoplus_{i,j} V^{ij}$. In other words, the operator Θ_1 (resp. Θ_2) is equal to i (resp. j) on V^{ij}. This bigrading is compatible with the $\mathbb{Z}/2$-grading by parity, i.e.,

$$V_{\bar{0}} = \bigoplus_{i+j \in 2\mathbb{Z}} V^{ij}, \qquad V_{\bar{1}} = \bigoplus_{i+j \in 1+2\mathbb{Z}} V^{ij}.$$

This just expresses the fact that Θ_1 and Θ_2 are even operators. Next, the operators D_1 and D_2 define anticommuting differentials on $V^{\bullet\bullet}$ of square 0 and degrees $(1, 0)$ and $(0, 1)$ respectively. This follows from the commutation relations

$$[D_\mu, D_\nu] = 0, \qquad [\Theta_\mu, D_\nu] = \delta_{\mu\nu} D_\nu, \quad \mu, \nu = 1, 2,$$

which are verified at once from (2.3.1). So $V^{\bullet\bullet}$ is, in particular, a double complex. Further, the permutation matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in GL_2(\mathbb{C}) \tag{2.3.6}$$

identifies V^{ij} with V^{ji} and interchanges D_1 and D_2. Finally, and most importantly, we have:

Proposition 2.3.7. *Let $V^{\bullet\bullet}$ be an $N = 2$ supersymmetric complex. Then every row of $V^{\bullet\bullet}$ except, possibly, the* 0*th row, is exact with respect to D_1. Similarly, every column except, possibly, the* 0*th column, is exact with respect to D_2.*

Proof. This follows from (2.3.3), which means that D_ν^* provides a contracting homotopy for D_ν outside of the 0th row (for $\nu = 1$) or the 0th column (for $\nu = 2$). □

For a double complex $(C^{\bullet\bullet}, d_1, d_2)$ we denote by $\mathrm{Tot}(C^{\bullet\bullet})$ its total complex, with differential $d_1 + d_2$.

Corollary 2.3.8. *Suppose that the bigrading on V is such that $V^{ij} = 0$ for $i \ll 0$ or $j \ll 0$. Then the complex* $\mathrm{Tot}(V^{\bullet\bullet})$ *is quasi-isomorphic to the* 0*th row* $(V^{\bullet,0}, D_1)$*, as well as to the* 0*th column* $(V^{0\bullet}, D_2)$.

Proof. Consider the obvious morphisms of double complexes

$$V^{\bullet,0} \xleftarrow{\varphi} V^{\bullet,\geqslant 0} \xrightarrow{\psi} V^{\bullet\bullet},$$

with φ being surjective and ψ injective. As all the rows of $V^{\bullet,\bullet}$ other than the 0th row are exact, we see that both $\mathrm{Tot}(\mathrm{Ker}(\varphi))$ and $\mathrm{Tot}(\mathrm{Coker}(\psi))$ have increasing filtrations with acyclic quotients, whence the statement. □

We now consider the particular case where $V^{\bullet\bullet}$ is concentrated in the first quadrant, i.e., in the range $i, j \geqslant 0$. Fix $p > 0$ and let $V_{\mathrm{cl}}^{pq} \subset V^{pq}$ be the kernel of D_1. We have then the complex

$$V_{\mathrm{cl}}^{p,\bullet} = \left\{V_{\mathrm{cl}}^{p0} \overset{D_2}{\hookrightarrow} V_{\mathrm{cl}}^{p1} \xrightarrow{D_2} V_{\mathrm{cl}}^{p2} \xrightarrow{D_2} \cdots\right\}. \tag{2.3.9}$$

Note that the very first differential is an injective map as $[D_2, D_2^*] = \Theta_1 = p$ on V^{pq}, and we assumed $p > 0$.

Proposition 2.3.10. *Let $V^{\bullet\bullet}$ be an $N = 2$ supersymmetric complex concentrated in the first quadrant, and $p > 0$. Then the complex $V_{\mathrm{cl}}^{p,\bullet}$ is isomorphic, in the derived category, to the cohomological truncation*

$$\left(\mathrm{t}_{\geqslant p+1}\left(V^{0\bullet}\right), D_2\right) = \left\{\mathrm{Ker}(D_2) \hookrightarrow V^{0,p} \xrightarrow{D_2} V^{0,p+1} \xrightarrow{D_2} V^{0,p+2} \xrightarrow{D_2} \cdots\right\}$$

with the grading normalized so that $\mathrm{Ker}(D_2)$ *is in degree* 0.

Proof. Since the action of the permutation matrix (2.3.6) interchanges the two differentials, it suffices to identify $V_{\mathrm{cl}}^{p,\bullet}$ (up to quasi-isomorphism) with

$$\big(\mathfrak{t}_{\geqslant p+1}\big(V^{\bullet,0}\big), D_1\big) = \big\{\mathrm{Ker}(D_1) \hookrightarrow V^{p,0} \xrightarrow{D_1} V^{p+1,0} \xrightarrow{D_1} V^{p+2,0} \xrightarrow{D_1} \cdots\big\},$$

where $\mathrm{Ker}(D_1) = V_{\mathrm{cl}}^{p,0}$. To achieve this, for each $j \geqslant 0$ consider a similar complex:

$$\mathfrak{t}_{\geqslant p+1}\big(V^{\bullet,j}\big) = \big\{V_{\mathrm{cl}}^{p,j} \longrightarrow V^{p,j} \longrightarrow V^{p+1,j} \longrightarrow \cdots\big\}.$$

By Proposition 2.3.7, $\tau_{\geqslant p+1}(V^{\bullet,j})$ is exact for $j \geqslant 0$. So we consider the double complex

$$W^{\bullet\bullet} = \big\{\mathfrak{t}_{\geqslant p+1}\big(V^{\bullet,0}\big) \longrightarrow \mathfrak{t}_{\geqslant p+1}\big(V^{\bullet,1}\big) \longrightarrow \mathfrak{t}_{\geqslant p+1}\big(V^{\bullet,2}\big) \longrightarrow \cdots\big\} \tag{2.3.11}$$

and denote its total complex by $W^\bullet$. Then one edge of $W^{\bullet\bullet}$ is $\mathfrak{t}_{\geqslant p+1}(V^{\bullet,0})$, the other edge is $V_{\mathrm{cl}}^{p,\bullet}$ and all the rows and columns other than these edges are exact. Therefore the projections

$$\mathfrak{t}_{\geqslant p+1}\big(V^{\bullet,0}\big) \longleftarrow W^\bullet \longrightarrow V_{\mathrm{cl}}^{p,\bullet} \tag{2.3.12}$$

of the total complex onto the two edges are quasi-isomorphisms. □

Remark 2.3.13. Representations of the Lie super-algebra $\mathrm{Der}\,\Lambda[\eta_1,\eta_2] = \mathfrak{sl}(1|2)$ have attracted a lot of attention. In particular, there is a complete classification of finite dimensional irreducible [6] and even indecomposable [12,29] representations. In this paper we do not need any more information about these representations than what is given by Proposition 2.3.10.

2.4. *The double de Rham complex*

Let X be a super-scheme. The super-scheme

$$\mathcal{S}^2X = X^{\Lambda[\eta_1,\eta_2]} = \underline{\mathrm{Hom}}\big(\mathbb{A}^{0|2}, X\big)$$

can be seen as the configuration space of an $N=2$ supersymmetric particle moving in X. The group super-scheme $G_{\Lambda[\eta_1,\eta_2]}$ acts on $\mathcal{S}^2X$. Denoting by $\varpi^2 : \mathcal{S}^2X \to X$ and $\varpi : \mathcal{S}X \to X$ the projections, we see that $\varpi^2_*\mathcal{O}_{\mathcal{S}^2X}$ is a sheaf of $N=2$ supersymmetric complexes on X. These complexes are concentrated in the first quadrant. Viewing $\mathcal{S}^2X$ as $\mathcal{S}\mathcal{S}X$, we can view $\varpi^2_*\mathcal{O}_{\mathcal{S}^2X}$ as $\varpi_*\Omega^\bullet_{\mathcal{S}X}$, i.e., the de Rham complex of the de Rham complex of X. It has two differentials: $D_1 = d_{\mathrm{DR}}^{\mathcal{S}X}$, the de Rham differential of $\mathcal{S}X$, and $D_2 = \mathrm{Lie}_D$, where D is the vector field on $\mathcal{S}X$ corresponding to the de Rham differential d_{DR}^X. These differentials are just a part of the structure of an $N=2$ supersymmetric complex.

Example 2.4.1. Suppose that X is a purely even smooth algebraic variety with an étale coordinate system $\phi : X \to \mathbb{A}^n$, so we have the regular functions $x_1,\dots,x_n$ on X. Then $\mathcal{S}X$ has étale coordinates $x_1,\dots,x_n,\xi_1,\dots,\xi_n$ with $\xi_i = d_{\mathrm{DR}}^X(x_i)$ odd. Accordingly, $\mathcal{S}^2X$ has an étale coordinate system consisting of $2n$ even coordinates x_i and $d_{\mathrm{DR}}^{\mathcal{S}X}(\xi_i)$ and $2n$ odd coordinates ξ_i and $d_{\mathrm{DR}}^{\mathcal{S}X}(x_i)$, $i=1,\dots,n$.

Let $\Omega_{\mathcal{S}X}^{p,\mathrm{cl}}$ be the sheaf of $d_{\mathrm{DR}}^{\mathcal{S}X}$-closed p-forms on $\mathcal{S}X$. The direct image $\varpi_*\Omega_{\mathcal{S}X}^{p,\mathrm{cl}}$ onto X has the residual grading and differential coming from the action of $G_{\Lambda[\eta_2]}$ on $\mathcal{S}X$. In local étale coordinates as above this is the grading assigning degree 1 to ξ_i and to $d_{\mathrm{DR}}^{\mathcal{S}X}(\xi_i)$ and degree 0 to the other generators, while the differential is $D = D_2$. Proposition 2.3.10 implies the following.

Corollary 2.4.2. *The complex*

$$\varpi_*\big(\Omega_{\mathcal{S}X}^{p,\mathrm{cl}}\big)^0 \xrightarrow{D} \varpi_*\big(\Omega_{\mathcal{S}X}^{p,\mathrm{cl}}\big)^1 \xrightarrow{D} \varpi_*\big(\Omega_{\mathcal{S}X}^{p,\mathrm{cl}}\big)^2 \xrightarrow{D} \cdots$$

of sheaves on X is quasi-isomorphic to

$$\Omega_X^{p,\mathrm{cl}} \hookrightarrow \Omega_X^p \xrightarrow{d_{\mathrm{DR}}^X} \Omega_X^{p+1} \xrightarrow{d_{\mathrm{DR}}^X} \Omega_X^{p+2} \xrightarrow{d_{\mathrm{DR}}^X} \cdots$$

where $\Omega_X^{p,\mathrm{cl}}$ is the sheaf of closed differential p-forms on X.

3. Super-ind-schemes

3.1. Basic definitions

We refer to [1,19] for general background on ind- and pro-objects. By a *super-ind-scheme* in this paper we mean an ind-object in **Sch** represented as a filtering inductive limit of quasi-compact super-schemes and their immersions

$$Y = \text{“}\varinjlim_{\alpha\in A}\text{”}\, Y^\alpha. \tag{3.1.1}$$

Alternatively, Y can be identified with the corresponding (ind-)representable functor

$$h_Y : \mathbf{Sch} \longrightarrow \mathbf{Set}, \quad S \longmapsto \varinjlim_{\alpha\in A}{}^{\mathbf{Set}}\, \mathrm{Hom}_{\mathbf{Sch}}\big(S, Y^\alpha\big). \tag{3.1.2}$$

We denote by h^Y the covariant functor $\mathbf{Alg} \to \mathbf{Set}$ given by $h^Y(R) = h_Y(\mathrm{Spec}(R))$.

Let **Isch** be the category of super-ind-schemes. It is a general property of ind-objects that for a quasi-compact super-scheme S we have $h_Y(S) = \mathrm{Hom}_{\mathbf{Isch}}(S, Y)$.

Proposition 3.1.3. *Let X be a super-scheme X. Consider the object*

$$\text{“}X\text{”} = \text{“}\varinjlim_{U\subset X \text{ quasi-comp.}}\text{”}\; U \in \mathbf{Isch}$$

where U runs over quasi-compact open sub-super-schemes in X. Associating $X \to$ “X” defines an embedding of **Ssch** *into* **Isch** *as a full subcategory.*

Our requirement that X_i be quasi-compact follows [4, §7.11]. Note that if we defined a super-ind-scheme as simply an ind-object in the category of all super-schemes, then a non-quasi-compact super-scheme X would be represented by two ind-objects: X itself and “X”, which are not isomorphic in general.

Let $f : X \to Y$ be a morphism of super-ind-schemes. We say that f is *formally smooth*, if the induced morphism $h^X \to h^Y$ of contravariant functors $\mathbf{Aff} \to \mathbf{Set}$ is formally smooth in the sense of Definition 1.2.1(a). The even and reduced parts of a super-ind-scheme Y as in (3.1.1) are defined by

$$Y_{\text{even}} = \text{“}\varinjlim_{\alpha \in A}\text{”}\, Y^{\alpha}_{\text{even}}, \qquad Y_{\text{red}} = \text{“}\varinjlim_{\alpha \in A}\text{”}\, Y^{\alpha}_{\text{red}}. \tag{3.1.4}$$

Similarly, let $\mathfrak{o}$ be a finite dimensional local $\mathbb{C}$-super-algebra. Using Proposition 2.1.2, we extends the functor $X \to X^{\mathfrak{o}}$ to ind-schemes by

$$Y^{\mathfrak{o}} = \text{“}\varinjlim_{\alpha \in A}\text{”}\big(Y^{\alpha}\big)^{\mathfrak{o}}. \tag{3.1.5}$$

Example 3.1.6. Let B be a super-scheme, I a finite set, and B^I the Ith Cartesian power of B. A morphism $u : S \to B^I$ is thus the same as an I-tuple of morphisms $u_i : S \to B$, $i \in I$. Denoting $\Delta \subset B^I$ the small diagonal $\{(b, b, \ldots, b)\}$ and by $\mathcal{I}_\Delta \subset \mathcal{O}_{B^I}$ its sheaf of ideals, we can view the formal neighborhood of Δ in B^I as an ind-scheme

$$B^{[I]} = \text{“}\varinjlim_{n \geqslant 0}\text{”}\, B^{[I]}_n, \quad B^{[I]}_n = \operatorname{Spec}_{B^I}\big(\mathcal{O}_{B^I}/\mathcal{I}^{n+1}_\Delta\big).$$

A morphism from a super-scheme S into $B^{[I]}$ is the same as an I-tuple of morphisms $u_i : S \to B$ as above but with the condition that any two u_i, u_j are infinitesimally near, in the sense of Definition 2.1.1. Note that for a 1-element set I we have $B^I = B^{[I]} = B$.

Further, any map $p : J \to I$ of finite sets induces a morphism of schemes $p^* : B^I \to B^J$ and a morphism of ind-schemes $[p]^* : B^{[I]} \to B^{[J]}$. If p is injective, then p^* and $[p]^*$ are coordinate projections; if p is surjective, then p^* and $[p]^*$ are diagonal embeddings.

We now discuss the concept of an integrable connection, following the approach of Grothendieck [15], see also [3, (3.4.7)].

Let B be a super-scheme, and $E \to B$ be a super-ind-scheme over B. For a morphism of super-schemes $u : S \to B$ we denote by $u^*E = E \times_B S \to S$ the pullback of E.

Proposition 3.1.7. *For a given $E \to B$ as above, the following systems of data* (1) *and* (2) *are in a bijection:*

(1) *For each super-scheme S and each pair of infinitesimally near morphisms $u, u' : S \to B$, an isomorphism $M_{S,u,u'} : u^*E \to u'^*E$ of super-ind-schemes over S, satisfying the following conditions:*
   (1a) *Transitivity: for each three infinitesimally near morphisms $u, u', u'' : S \to B$, we have*

$$M_{S,u,u''} = M_{S,u',u''} \circ M_{S,u,u'}.$$

   (1b) *Compatibility with restrictions: for any u, u' as above and any morphism $v : S' \to S$, we have*

$$M_{S',uv,u'v} = v^* M_{S,u,u'}.$$

(2) *For each nonempty finite set I, a super-ind-scheme $E_I \to B^{[I]}$ such that $E_I = E$ for any 1-element I, and for any map $p : J \to I$ we have an isomorphism $\alpha_p : E_I \to [p]^* E_J$, these isomorphisms compatible with compositions of maps.*

We will call a datum of either type an *integrable connection* on E along B.

Proof of Proposition 3.1.7. Given a datum of type (2), any infinitesimally near $u, u' : S \to B$ give a morphism $(u, u') : S \to B^{[\{1,2\}]}$. On the other hand, $B^{[\{1,2\}]}$ is the formal neighborhood of the diagonal in $B \times B$, and the isomorphisms α_{i_1}, α_{i_2} corresponding to the maps $i_1 : \{1\} \hookrightarrow \{1,2\}$, $i_2 : \{2\} \hookrightarrow \{1,2\}$ identify $E_{[\{1,2\}]}$ with the pullback of E via the two projections $[i_1]^*, [i_2]^* : B^{[\{1,2\}]} \to B$, whence the isomorphism $M_{S,u,u'}$. Transitivity follows from considering the morphism $(u, u', u'') : S \to B^{[\{1,2,3\}]}$. Compatibility with restrictions follows because the morphism $(uv, u'v) : S' \to B^{[\{1,2\}]}$ is the composition of (u, u') and v.

Conversely, let a datum of type (1) be given. To construct E_I, we fix $n \geqslant 0$ and take $S = B_n^{[I]}$. The coordinate projections $p_i : S \to B$, $i \in I$, are infinitesimally close to each other so the ind-schemes $p_i^* E \to B_n^{[I]}$ are canonically identified with each other via the M-isomorphisms. We can say that we have one ind-scheme $E_{I,n}$, identified with them all. When n increases, these $E_{I,n}$ form a filtering inductive system of super-ind-schemes and closed embeddings, so their limit E_I is a well-defined object of **Isch**. The remaining verifications are left to the reader. $\square$

3.2. Functions and forms on super-ind-schemes

As in [20] and [24, §2], to any super-ind-scheme Y as in (3.1.1), we associate a topological space

$$\underline{Y} = \varinjlim_{\alpha \in A}{}^{\mathbf{Top}}\, \underline{Y}^{\alpha}. \tag{3.2.1}$$

Let $i_\alpha : \underline{Y}^\alpha \to \underline{Y}$ be the canonical embedding. We then have a sheaf of super-commutative pro-algebras $\mathcal{O}_Y$ on $\underline{Y}$

$$\mathcal{O}_Y = \varprojlim_{\alpha \in A} (i_\alpha)_* \mathcal{O}_{Y^\alpha}, \tag{3.2.2}$$

which we can consider as a sheaf of topological algebras. We define the sheaves of differential forms in a similar way:

$$\Omega_Y^p = \varprojlim_{\alpha \in A} (i_\alpha)_* \Omega_{Y^\alpha}^p. \tag{3.2.3}$$

Note that

$$\Omega_Y^\bullet = \varpi_* \mathcal{O}_{\mathcal{S}Y}, \tag{3.2.4}$$

where $\pi : \mathcal{S}Y \to Y$ is the natural projection.

4. The formal loop space of a super-manifold

4.1. Nil-Laurent series

For a super-commutative ring R we denote by $R((t))^{\sqrt{}}$ the subring of $R((t))$ consisting of Laurent series $\sum_{i\gg-\infty}^{\infty} a_i t^i$ such that a_i is nilpotent for $i<0$. We proved in [23, Prop. 1.3.1], that if R is a commutative local ring, then so is $R((t))^{\sqrt{}}$. We need the following version of this.

Lemma 4.1.1. *Let S be a super-scheme. Then $\mathcal{O}_S[[t]]$ and $\mathcal{O}_S((t))^{\sqrt{}}$ are sheaves of super-commutative local rings.*

Proof. It is enough to assume that $S=\operatorname{Spec}(R)$. Let $\mathfrak{p}\in\underline{\operatorname{Spec}}(R)$, i.e., $\mathfrak{p}\subset R_0$ is a prime ideal. We first treat the case of $\mathcal{O}_S[[t]]$. The stalk of this sheaf at $\mathfrak{p}$ is the ring

$$\mathcal{O}_S[[t]]_{\mathfrak{p}}=\varinjlim_{U\ni\mathfrak{p}}\mathcal{O}(U)[[t]]=R[[t]]\big[(R-\mathfrak{p})^{-1}\big].$$

We claim that

$$\mathfrak{p}'=\left\{b^{-1}\sum_{n=0}^{\infty}a_nt^n;\ a_n\in R,\ a_0\in\mathfrak{p},\ b\in R-\mathfrak{p}\right\}$$

is the maximal ideal in $R[[t]][(R-\mathfrak{p})^{-1}]$, i.e., any element not in $\mathfrak{p}'$ is invertible. This is obvious by using the geometric series and inverting $a_0\notin\mathfrak{p}$.

Consider now the case of $\mathcal{O}_S((t))^{\sqrt{}}$. As before, we have

$$\mathcal{O}_S((t))^{\sqrt{}}_{\mathfrak{p}}=R((t))^{\sqrt{}}\big[(R-\mathfrak{p})^{-1}\big].$$

We define

$$\tilde{\mathfrak{p}}=\left\{b^{-1}\sum_{n\gg-\infty}^{\infty}a_nt^n;\ a_n\in R,\ a_{<0}\in\sqrt{R},\ a_0\in\mathfrak{p},\ b\in R-\mathfrak{p}\right\}\tag{4.1.2}$$

and claim that it is the maximal ideal in $\mathcal{O}_S((t))^{\sqrt{}}_{\mathfrak{p}}$. Indeed, the fact that $\tilde{\mathfrak{p}}$ is an ideal is obvious. On the other hand, if $u(t)\in\mathcal{O}_S((t))^{\sqrt{}}_{\mathfrak{p}}-\tilde{\mathfrak{p}}$, then we write $u(t)$ as the sum

$$u(t)=u_-(t)+a_0b^{-1}+u_+(t)$$

where $u_\pm(t)$ is the sum of the terms with $\pm n>0$. Now, $u_-(t)$ is nilpotent, a_0b^{-1} is invertible in $R[(R-\mathfrak{p})^{-1}]$, and $u_+(t)$ is topologically nilpotent. So the invertibility follows in the same way as in [23, Prop. 1.3.1]. □

As in [23, (1.6)], denote by $\mathbf{E}$ the set of sequences

$$\epsilon=(\epsilon_{-1},\epsilon_{-2},\ldots),\quad \epsilon_j\in\mathbb{Z}_+,\ \epsilon_j=0,\ j\ll 0.\tag{4.1.3}$$

It is equipped with a natural partial order such that $\epsilon \leqslant \epsilon'$ if $\epsilon_j \leqslant \epsilon'_j$ for all j. For a super-commutative algebra R we define the subset

$$R((t))^{\sqrt{}}_{\epsilon} = \left\{ \sum_{n\in\mathbb{Z}} a_n t^n \mid a_n^{1+\epsilon_n} = 0,\ n < 0 \right\}. \tag{4.1.4}$$

Thus series from this set have both the number of negative coefficients and their order of nilpotency bounded.

Proposition 4.1.5. *Any finitely generated subalgebra A in $R((t))^{\sqrt{}}$ is contained in $R((t))^{\sqrt{}}_{\epsilon}$ for some ϵ.*

Proof. Let $f_1, \dots, f_r$ be generators of A, which we can assume to be homogeneous with respect to the $\mathbb{Z}/2$-grading. Write $f_i = f_{i,+} + f_{i,-}$, where $f_{i,+} \in R[[t]]$, while $f_{i,-}$ is the sum of the terms with negative powers of t. Then, each $f_{i,-}$ is nilpotent. This implies that among the infinite number of monomials

$$f_-^m := f_{1,-}^{m_1} f_{2,-}^{m_2} \cdots f_{r,-}^{m_r}, \quad m = (m_1, \dots, m_r),\ m_i \geqslant 0,$$

only finitely many are nonzero. Let $m^{(1)}, \dots, m^{(s)}$ be the exponents of all the nonzero ones. Look now at similar monomials $f^n = f_1^{n_1} \cdots f_r^{n_r}$ formed out of the f_i. They form a spanning set for A. On the other hand, expanding them using $f_i = f_{i,+} + f_{i,-}$ and the binomial formula, we find that each f^n can be expressed as

$$f^n = \sum_{\nu=1}^{s} F_\nu^n f_-^{m^{(\nu)}}, \quad F_\nu^n \in R[[t]].$$

The finitely many monomials $f_-^{m^{(\nu)}} \in R((t))^{\sqrt{}}$ clearly admit $N, d \geqslant 0$ with the following properties. First, all the $f_-^{m^{(\nu)}}$ have zero coefficients at t^j, $j < -N$. Second, all the coefficients of these $f_-^{m^{(\nu)}}$ at monomials with t^j, $-N \leqslant j \leqslant -1$, are nilpotent of degree $d+1$. Look now at elements of the form Ff_-^m with $F \in R[[t]]$. Each of them clearly satisfies the first property: the order of pole is still bounded by N. As for the second property, each coefficient of Ff_-^m at each negative power of t is a sum of at most $N-1$ summands, each nilpotent of degree $d+1$. This implies that there is d' depending only on d and N such that each coefficient of each Ff_-^m at each negative power of t, is nilpotent of degree $d'+1$. This means that $A \subset R((t))^{\sqrt{}}_{\epsilon}$, where ϵ is such that $\epsilon_{-1} = \cdots = \epsilon_{-N} = d'$, and $\epsilon_i = 0$ for $i < N$. $\square$

4.2. Basics on $\mathcal{L}^0 X$ and $\mathcal{L}X$

Let X be a super-scheme. We define the super-scheme

$$\mathcal{L}_n^0 X = X^{\mathbb{C}[t]/t^{n+1}}.$$

For different n the $\mathcal{L}_n^0 X$ form a projective system of affine morphisms of super-schemes. We define the super-scheme $\mathcal{L}^0 X$ to be the projective limit of this system, and call it the *super-scheme of formal arcs* in X. Compare with [11].

Proposition 4.2.1. *For any super-commutative ring R and, more generally, for any super-scheme S we have*

$$\operatorname{Hom}_{\mathbf{Ssch}}\big(\operatorname{Spec}(R), \mathcal{L}^0 X\big) = \operatorname{Hom}_{\mathbf{Ssch}}\big(\operatorname{Spec}\big(R[[t]]\big), X\big),$$
$$\operatorname{Hom}_{\mathbf{Ssch}}\big(S, \mathcal{L}^0 X\big) = \operatorname{Hom}_{\mathbf{Ssp}}\big(\big(\underline{S}, \mathcal{O}_S[[t]]\big), X\big).$$

This was asserted for schemes in [23, Prop. 1.2.1(b)] but with an incorrect proof (the first equality in Lemma 1.2.3 of [23] does not hold in general). Here we supply the proof.

Proof of Proposition 4.2.1. Note that if S is any super-scheme, then, by Proposition 2.1.2 applied to $\mathfrak{o} = \mathbb{C}[t]/t^{n+1}$ we have

$$\operatorname{Hom}_{\mathbf{Ssch}}\big(S, \mathcal{L}_n^0 X\big) = \operatorname{Hom}_{\mathbf{Ssp}}\big(\big(\underline{S}, \mathcal{O}_S[t]/t^{n+1}\big), X\big).$$

Next, we have

$$\mathcal{O}_S[[t]] = \varprojlim_{n \geqslant 0} \mathcal{O}_S[t]/t^{n+1}$$

in the category of sheaves of local rings on $\underline{S}$, so

$$\big(\underline{S}, \mathcal{O}_S[[t]]\big) = \varprojlim_{n \geqslant 0}{}^{\mathbf{Ssp}} \big(\underline{S}, \mathcal{O}_S[t]/t^{n+1}\big),$$

and therefore

$$\operatorname{Hom}_{\mathbf{Ssp}}\big(\big(\underline{S}, \mathcal{O}_S[[t]]\big), X\big) = \varprojlim_{n \geqslant 0} \operatorname{Hom}_{\mathbf{Ssp}}\big(\big(\underline{S}, \mathcal{O}_S[t]/t^{n+1}\big), X\big).$$

Note that **Ssch** is a full subcategory in **Ssp**, so Hom on the right-hand side can be taken in either category. Now the fact that

$$\mathcal{L}^0 X = \varprojlim_{n \geqslant 0}{}^{\mathbf{Ssch}} \mathcal{L}_n^0 X$$

implies that

$$\operatorname{Hom}_{\mathbf{Ssch}}\big(S, \mathcal{L}^0 X\big) = \varprojlim_{n \geqslant 0} \operatorname{Hom}_{\mathbf{Ssch}}\big(S, \mathcal{L}_n^0 X\big) = \operatorname{Hom}_{\mathbf{Ssp}}\big(\big(\underline{S}, \mathcal{O}_S[[t]]\big), X\big),$$

as claimed. □

As in [23] we define the functor $\lambda_X : \mathbf{Ssch} \to \mathbf{Set}$ as follows:

$$\lambda_X(S) = \operatorname{Hom}_{\mathbf{Ssp}}\big(\big(\underline{S}, \mathcal{O}_S((t))^{\surd}\big), X\big). \tag{4.2.2}$$

Proposition 4.2.3.

(a) *If* $X = \mathrm{Spec}(A)$, $S = \mathrm{Spec}(R)$ *are affine super-schemes, then*

$$\lambda_X(S) = \mathrm{Hom}_{\mathbf{Alg}}\big(A, R((t))^{\surd}\big).$$

(b) *For any super-scheme* X *of finite type the functor* λ_X *is representable by a super-ind-scheme* $\mathcal{L}X$, *and* $\mathcal{L}X = \varinjlim_{U \subset X \text{ affine}} \mathcal{L}U$ *in the category of super-ind-schemes.*

Remark 4.2.4. In [23, Prop. 1.4.5], we claimed (with an incorrect proof, based on erroneous Lemma 1.4.3(a)), that the analog of Proposition 4.2.3(a) holds for any X of finite type. In fact, this stronger statement is unnecessary, and Proposition 4.2.3(a) is sufficient to establish Proposition 4.2.3(b) and all the properties of $\mathcal{L}X$ claimed in [23].

Proof of Proposition 4.2.3(a). Let $f \in \lambda_X(S)$, i.e.,

$$f = \big(f_\flat, f^\sharp\big) : \big(\underline{\mathrm{Spec}}(R), \mathcal{O}_{\mathrm{Spec}(R)}((t))^{\surd}\big) \longrightarrow \big(\underline{\mathrm{Spec}}(A), \mathcal{O}_{\mathrm{Spec}(A)}\big)$$

is a morphism of super-spaces. Thus $f_\flat : \underline{\mathrm{Spec}}(R) \to \underline{\mathrm{Spec}}(A)$ is a morphism of topological spaces, and

$$f^\sharp : f_\flat^{-1}\mathcal{O}_{\mathrm{Spec}(A)} \longrightarrow \mathcal{O}_{\mathrm{Spec}(R)}((t))^{\surd}$$

is a morphism of sheaves of super-commutative local rings. It induces a morphism of rings

$$\varphi = \Gamma\big(f^\sharp\big) : A = \Gamma\big(\underline{\mathrm{Spec}}(A), \mathcal{O}_{\mathrm{Spec}(A)}\big) \longrightarrow R((t))^{\surd} = \Gamma\big(\underline{\mathrm{Spec}}(R), \mathcal{O}_{\mathrm{Spec}(R)}((t))^{\surd}\big)$$

and so a morphism of super-schemes

$$g : \mathrm{Spec}\, R((t))^{\surd} \longrightarrow \mathrm{Spec}(A).$$

So it is enough to prove:

Lemma 4.2.5. *The correspondence* $f \mapsto \varphi$ *gives a bijection*

$$\Phi : \lambda_X(S) \longrightarrow \mathrm{Hom}_{\mathbf{Alg}}\big(A, R((t))^{\surd}\big).$$

Proof. We construct the inverse map

$$\Psi : \mathrm{Hom}_{\mathbf{Alg}}\big(A, R((t))^{\surd}\big) \longrightarrow \lambda_X(S).$$

We have a morphism of super-spaces

$$h = \big(h_\flat, h^\sharp\big) : \big(\underline{\mathrm{Spec}}(R), \mathcal{O}_{\mathrm{Spec}(R)}((t))^{\surd}\big) \longrightarrow \mathrm{Spec}\big(R((t))^{\surd}\big)$$

with the map of topological spaces $h_\flat$ defined as the composition

$$\underline{\mathrm{Spec}}(R) = \underline{\mathrm{Spec}}(\overline{R}) \xrightarrow{u} \underline{\mathrm{Spec}}\big(\overline{R}[[t]]\big) \xrightarrow{v} \underline{\mathrm{Spec}}\big(R((t))^{\surd}\big).$$

Here we have denoted $\overline{R} = R/\sqrt{R}$, and u is induced by the evaluation homomorphism

$$\overline{R}[[t]] \longrightarrow \overline{R} = \overline{R}[[t]]/t\overline{R}[[t]],$$

while v is induced by the termwise factorization by $\sqrt{R}$:

$$R((t))^{\sqrt{\ }} \longrightarrow \overline{R}((t))^{\sqrt{\ }} = \overline{R}[[t]].$$

The morphism $h^\sharp$ is induced by the inclusions

$$R((t))^{\sqrt{\ }}[1/b] \subset \big(R[1/b]\big)((t))^{\sqrt{\ }}, \quad b \in R.$$

Given $\varphi : A \to R((t))^{\sqrt{\ }}$, it induces a morphism of super-schemes

$$g : \operatorname{Spec} R((t))^{\sqrt{\ }} \longrightarrow \operatorname{Spec}(A),$$

and we define $f = \Psi(\varphi)$ to be the composition

$$\Psi(\varphi) = gh : \big(\underline{\operatorname{Spec}}(R), \mathcal{O}_{\operatorname{Spec}(R)}((t))^{\sqrt{\ }}\big) \longrightarrow \operatorname{Spec}(A) = X.$$

We now claim that the maps Φ and Ψ are inverse to each other. Indeed, the equality $\Phi\Psi = \mathrm{Id}$ is obvious, it follows from the fact that $\Gamma(h^\sharp)$ is the identity of $R((t))^{\sqrt{\ }}$.

Let us prove that $\Psi\Phi = \mathrm{Id}$. The proof is analogous to the classical proof that a morphism of affine schemes is the same as a homomorphism of the corresponding rings. So let $f = (f_\flat, f^\sharp) \in \lambda_X(S)$, and $g = (g_\flat, g^\sharp) = \Phi(f)$. By construction

$$\Psi(g) = \big(g_\flat h_\flat, g_\flat^{-1}\big(h^\sharp\big)g^\sharp\big).$$

Let us prove the equality of maps $g_\flat h_\flat = f_\flat$, leaving the other equality to the reader.

Let $\mathfrak{p} \in \underline{\operatorname{Spec}}(R)$, so $\mathfrak{p} \subset R_0$ is a prime ideal. By definition of $g_\flat$ the equality $f_\flat(\mathfrak{p}) = g_\flat h_\flat(\mathfrak{p})$ is equivalent to

$$f_\flat(\mathfrak{p}) = \varphi^{-1} h_\flat(\mathfrak{p}), \tag{4.2.6}$$

where $\varphi = \Gamma(f^\sharp)$ fits into the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & R((t))^{\sqrt{\ }} \\ \downarrow & & \downarrow \\ A_{f_\flat(\mathfrak{p})} & \xrightarrow{f^\sharp_{\mathfrak{p}}} & (\mathcal{O}_{\operatorname{Spec} R}((t))^{\sqrt{\ }})_{\mathfrak{p}}. \end{array}$$

The vertical maps in this diagram are obtained by taking the stalks, and the map $f^\sharp_{\mathfrak{p}}$ is a local homomorphism of local rings. We use the notation (4.1.2) for the maximal ideal $\tilde{\mathfrak{p}}$ in $(\mathcal{O}_{\operatorname{Spec} R}((t))^{\sqrt{\ }})_{\mathfrak{p}}$. Let $\tilde{\tilde{\mathfrak{p}}}$ be its inverse image in $R((t))^{\sqrt{\ }}$. Explicitly, we have

$$\tilde{\tilde{\mathfrak{p}}} = \left\{ \sum_{n \gg -\infty}^{+\infty} a_n t^n \in R((t))^{\sqrt{\ }};\ a_0 \in \mathfrak{p} \right\}.$$

Since the diagram above commutes and $f_{\mathfrak{p}}^{\sharp}$ is a local homomorphism, we have $\varphi^{-1}(\tilde{\tilde{\mathfrak{p}}}) = f_\flat(\mathfrak{p})$. So to prove (4.2.6) we need to show that $h_\flat(\mathfrak{p}) = \tilde{\tilde{\mathfrak{p}}}$, which is obvious.

This ends the proof of Proposition 4.2.3, part (a). The proof of part (b) is then achieved as in [23]. Indeed, for X affine, part (a) implies that λ_X is represented by the formal neighborhood of $\mathcal{L}^0 X$ in $\tilde{\mathcal{L}}X$, see [23, p. 219]. For general X of finite type, $\mathcal{L}X$ is glued from $\mathcal{L}U$, $U \subset X$ affine, as in [23, Prop. 1.4.6]. $\square$

Recall the de Rham spectrum functor $\mathcal{S}$ from Section 2.2.

Proposition 4.2.7. *For any super-scheme X of finite type we have an isomorphism of super-ind-schemes $\mathcal{L}\mathcal{S}X = \mathcal{S}\mathcal{L}X$.*

Proof. Both super-ind-schemes represent the same functor

$$S \longmapsto \mathrm{Hom}_{\mathbf{Ssp}}\big(\big(\underline{S}, \mathcal{O}_S((t))^{\surd}[\eta]\big), (\underline{X}, \mathcal{O}_X)\big),$$

where η is an odd generator, so that $\eta^2 = 0$. Indeed, for any super-commutative ring R we have

$$\big(R[\eta]\big)((t))^{\surd} = R((t))^{\surd}[\eta]. \qquad \square$$

4.3. $\mathcal{L}X$ and loco-modules of Borisov

By construction that there are morphisms

$$X \xleftarrow{\pi} \mathcal{L}^0 X \xrightarrow{i} \mathcal{L}X, \tag{4.3.1}$$

where π is affine and i realizes $\mathcal{L}X$ as a formal thickening of $\mathcal{L}^0 X$. They are induced by the obvious morphisms of sheaves of local rings on any super-scheme S:

$$\mathcal{O}_S \longleftarrow \mathcal{O}_S[[t]] \hookrightarrow \mathcal{O}_S((t))^{\surd}.$$

We are going to describe explicitly $\pi_* \mathcal{O}_{\mathcal{L}^0 X}$, which is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras, and $\pi_* \mathcal{O}_{\mathcal{L}X}$, which is a sheaf of pro-$\mathcal{O}_X$-algebras.

Let A be a super-commutative algebra. Specializing (2.1.3) to the particular case of $\mathfrak{o} = \mathbb{C}[t]/t^{n+1}$ and of the basis of $\mathfrak{o}$ formed by $1, t, \ldots, t^n$, we find:

Corollary 4.3.2. *The super-scheme $\mathcal{L}_n^0(\operatorname{Spec} A)$ is identified with $\operatorname{Spec}(A^{\mathbb{C}[t]/t^{n+1}})$, where $A^{\mathbb{C}[t]/t^{n+1}}$ the super-commutative algebra generated by the symbols $a[m]$, $0 \leqslant m \leqslant n$ such that the $\mathbb{Z}/2$-degree of $a[m]$ is the same as that of a, and which are subject to the relations:*

$$(a+b)[m] = a[m] + b[m], \qquad (\lambda a)[n] = \lambda\big(a[n]\big), \quad \lambda \in \mathbb{C}; \tag{4.3.2(a)}$$

$$1[m] = 0, \quad m \neq 0; \tag{4.3.2(b)}$$

$$(ab)[m] = \sum_{i+j=m} a[i] \cdot b[j]. \tag{4.3.2(c)}$$

We denote

$$A^{[\![t]\!]} = \varinjlim_{n \geqslant 0} A^{\mathbb{C}[t]/t^{n+1}}. \tag{4.3.3}$$

This algebra can be defined by generators $a[m]$ given for all $m \geqslant 0$ subject to the same relations as in (4.3.2)(a)–(c). Note that we have an embedding of algebras

$$A \hookrightarrow A^{[\![t]\!]}, \quad a \longmapsto a[0]. \tag{4.3.4}$$

By applying the limit construction (inductive for algebras, projective for schemes) to the above corollary and to Proposition 2.1.2, we obtain:

Proposition 4.3.5.

(a) *If $X = \mathrm{Spec}(A)$, then $\mathcal{L}^0 X = \mathrm{Spec}(A^{[\![t]\!]})$, with the projection π induced by* (4.3.3).
(b) *If $S \subset A$ is a multiplicative subset, then $(A[S^{-1}])^{[\![t]\!]} = A^{[\![t]\!]}[S^{-1}]$. In particular, for any super-scheme S the sheaf $\mathcal{O}_X^{[\![t]\!]}$ is quasi-coherent.*
(c) *We have an identification*

$$\pi_* \mathcal{O}_{\mathcal{L}^0 X} = \mathcal{O}_X^{[\![t]\!]}.$$

To be precise, (a) follows from Corollary 4.3.2 since projective limits of affine super-schemes correspond to inductive limits of algebras. Part (b) follows from Proposition 2.1.2(c) since localization commutes with inductive limits. Finally, part (c) follows from part (a).

For each sequence $\epsilon \in \mathbf{E}$ as in (4.1.3) let $A_\epsilon^{((t))}$ be the algebra with generators $a[n]$ for $a \in A$ and $n \in \mathbb{Z}$ (arbitrary integers), subject to the relations

$$a[n]^{1+\epsilon_n} = 0, \quad a \in A,\ n < 0, \tag{4.3.6}$$

together with the relations identical to (4.3.2)(a)–(c) but with $n, i, j \in \mathbb{Z}$. Note that (4.3.6) implies that $a[n] = 0$ for any a and $n \ll 0$, so the sum in (4.3.2)(c) remains finite.

For $\epsilon \leqslant \epsilon'$ we have a surjection of algebras $A_{\epsilon'}^{((t))} \to A_\epsilon^{((t))}$, and we define the pro-algebra

$$A^{((t))} = \varprojlim_{\epsilon \in \mathbf{E}} A_\epsilon^{((t))}. \tag{4.3.7}$$

Recall, see Section 3.2, that every super-ind-scheme Y gives a topological space $\underline{Y}$ and a sheaf $\mathcal{O}_Y$ over $\underline{Y}$ of pro-super-commutative rings. In particular, if $Y = \mathcal{L}X$ then $\underline{Y} = \underline{\mathcal{L}^0 X}$, so that we have a sheaf $\mathcal{O}_{\mathcal{L}X}$ over $\underline{\mathcal{L}^0 X}$.

Proposition 4.3.8.

(a) *Let $X = \mathrm{Spec}(A)$ be an affine super-scheme of finite type (i.e., A is finitely generated as an algebra). Then*

$$\mathcal{L}X = \mathrm{Spf}\, A^{((t))} := \text{“}\varinjlim_{\epsilon \in \mathbf{E}}\text{”}\, \mathrm{Spec}\, A_\epsilon^{((t))}.$$

(b) *If X is any super-scheme of finite type, then we have an identification of sheaves of pro-algebras on X*

$$\pi_* \mathcal{O}_{\mathcal{L}X} = \mathcal{O}_X^{((t))}.$$

Proof. (a) By definition, for any super-commutative algebra R we have the first of the following two equalities:

$$\mathrm{Hom}(\mathrm{Spec}\, R, \mathcal{L}X) = \mathrm{Hom}_{\mathbf{Alg}}\big(A, R((t))^{\sqrt{\ }}\big) = \varinjlim_{\epsilon \in \mathbf{E}} \mathrm{Hom}_{\mathbf{Alg}}\big(A, R((t))_\epsilon^{\sqrt{\ }}\big).$$

The second equality is a consequence of Proposition 4.1.5, since A is assumed finitely generated. It remains to notice that

$$\mathrm{Hom}_{\mathbf{Alg}}\big(A, R((t))_\epsilon^{\sqrt{\ }}\big) = \mathrm{Hom}\big(\mathrm{Spec}\, R, \mathrm{Spec}\, A_\epsilon^{((t))}\big).$$

This proves (a) since a super-ind-scheme is uniquely determined by the functor it represents on affine super-schemes. Part (b), being a local statement, follows from (a). □

Remark 4.3.9. The above considerations are very similar to the work of Borisov [7]. In particular, his "loco-modules" can be understood as sheaves of discrete modules over the sheaf of topological (or pro-) algebras $\mathcal{O}_X^{((t))}$, i.e., as certain sheaves on the ind-scheme $\mathcal{L}X$.

Example 4.3.10. Let $X = \mathbb{A}^N$, so $A = \mathbb{C}[a_1, \ldots, a_N]$. For $i = 1, \ldots, N$ and $n \in \mathbb{Z}$ let $b_n^i = a_i[n] \in A^{((t))}$. Thus the b_n^i are the components of N indeterminate power series

$$a_i(t) = \sum_{n \gg -\infty} b_n^i t^n$$

forming a point of $\mathcal{L}\mathbb{A}^N$. We have then:

$$A^{[[t]]} = \mathbb{C}\big[b_n^i,\ i = 1, \ldots, N,\ n \geqslant 0\big];$$
$$A^{((t))} = \varprojlim_{m>0} \mathbb{C}\big[b_n^i,\ i = 1, \ldots, N,\ n \geqslant 0\big]\big[\big[b_n^i,\ i = 1, \ldots, N,\ n = -m, \ldots, -1\big]\big].$$

The case when $X = \mathbb{A}^{d_1|d_2}$ is a super-affine space, is considered similarly: we have even and odd coordinates $a_1, \ldots, a_N$, $N = d_1 + d_2$, and use the convention of Example 1.1.6(b) for super-polynomial rings.

Remark 4.3.11. Assume now that X is a smooth algebraic super-variety of dimension $d_1|d_2$. If $U \subset X$ is a Zariski open set admitting an étale map $\phi : U \to \mathbb{A}^{d_1|d_2}$, then $\mathcal{L}U \subset \mathcal{L}X$ is open. Then $\mathcal{L}U$ admits a representation as the limit of a Cartesian ind-pro-system as in [23]:

$$\mathcal{L}U = \text{“}\varinjlim_{\epsilon \in \mathbf{E}}\text{”} \varprojlim_{n \geqslant 0} \mathcal{L}_n^\epsilon(\phi).$$

Here the scheme $\mathcal{L}_n^{\epsilon}(\phi)$ is defined as follows. First, consider the case when $U = \mathbb{A}^{d_1|d_2}$ with (even and odd) coordinates $a_1, \ldots, a_N$, $N = d_1 + d_2$, and $\phi = \mathrm{Id}$. In this case

$$\mathcal{L}_n^{\epsilon}(\mathrm{Id}) = \operatorname{Spec} \mathbb{C}\big[a_i[l];\ -N \leqslant n \leqslant l\big] / \big(\big(a_i[l]\big)^{1+\epsilon_l};\ l < 0\big),$$

where N is any number such that $\epsilon_l = 0$ for $l < -N$. This is a super-scheme of finite type mapping onto $\mathbb{A}^{d_1|d_2}$ via the homomorphism of rings $a_i \mapsto a_i[0]$. Next, for an arbitrary étale $\phi : U \to \mathbb{A}^{d_1|d_2}$ one has, as in [23, (1.7.3)], that

$$\mathcal{L}_{\phi}^{\epsilon} = \mathcal{L}_n^{\epsilon}\big(\mathbb{A}^{d_1|d_2}\big) \times_{\mathbb{A}^{d_1|d_2}} U.$$

5. Factorization structure on $\mathcal{L}X$

5.1. Reminder on $\mathcal{L}^0_{C^I} X$ and $\mathcal{L}_{C^I} X$

Let us extend the construction of the global formal loops space from [23] to the case of targets belonging to the super category. Let C be a (purely even) smooth algebraic curve and X be a smooth algebraic super-variety. Let **Fset**$^+$ be the category of nonempty finite sets and their surjections. Let I belong to **Fset**$^+$. Let S be a super-scheme and $c_I : S \to C^I$ be a morphism, so $c_I = (c_i : S \to C)_{i \in I}$. Let $\Gamma_i \subset S \times C$ be the graph of c_i. Let $\Gamma = \bigcup_{i \in I} \Gamma_i$ be the union. We denote by $\widehat{\mathcal{O}}_{\Gamma}$ the completion of $\mathcal{O}_{S \times C}$ along Γ, and by $\mathcal{K}_{\Gamma}$ the localization $\widehat{\mathcal{O}}_{\Gamma}[r^{-1}]$, where r is a local equation of Γ in $S \times C$. Finally, let $\Gamma_{\mathrm{red}} = \Gamma \cap (S_{\mathrm{red}} \times C)$, and $\mathcal{K}_{\Gamma}^{\sqrt{}} \subset \mathcal{K}_{\Gamma}$ be the subsheaf formed by sections whose restriction to $S_{\mathrm{red}} \times C$ lies in $\widehat{\mathcal{O}}_{\Gamma_{\mathrm{red}}}$.

Lemma 5.1.1. *Let $\underline{\Gamma}$ be the underlying topological space of the super-scheme Γ. Then $\widehat{\mathcal{O}}_{\Gamma}$ and $\mathcal{K}_{\Gamma}^{\sqrt{}}$ are sheaves of local super-commutative algebras on $\underline{\Gamma}$, so $(\underline{\Gamma}, \widehat{\mathcal{O}}_{\Gamma})$ and $(\underline{\Gamma}, \mathcal{K}_{\Gamma}^{\sqrt{}})$ are super-spaces.*

The proof is similar to that of Lemma 4.1.1.

Consider the functor

$$\lambda_{X,C^I} : S \longmapsto \big\{(c_I, \phi);\ c_I : S \longrightarrow C^I,\ \phi \in \mathrm{Hom}_{\mathbf{Ssp}}\big(\big(\underline{\Gamma}, \mathcal{K}_{\Gamma}^{\sqrt{}}\big), (\underline{X}, \mathcal{O}_X)\big)\big\}. \tag{5.1.2}$$

We define the functor λ^0_{X,C^I} is a similar way, with $\widehat{\mathcal{O}}_{\Gamma}$ instead of $\mathcal{K}_{\Gamma}^{\sqrt{}}$.

We denote by $\mathfrak{g}$ the Lie algebra $\operatorname{Der} \mathbb{C}[[t]]$ and by K the group scheme

$$\operatorname{Aut} \mathbb{C}[[t]] = \operatorname{Spec}\big(\mathbb{C}\big[a_1^{-1}, a_1, a_2, a_3, \ldots\big]\big).$$

So for a ring R an R-point of K is a formal change of coordinates

$$t \longmapsto a_1 t + a_2 t^2 + \cdots, \quad a_1 \in R^{\times},\ a_i \in R,\ i \geqslant 2.$$

The Lie algebra $\mathfrak{g}$ and the group scheme K form a *Harish-Chandra pair*, see [3, (2.9.7)]. By an action of $(\mathfrak{g}, K)$ on an ind-scheme Y we mean an action of K by automorphisms and an action of $\mathfrak{g}$ by derivations (infinitesimal automorphisms) which are compatible.

Let C be as before and $\widehat{C} \to C$ be the scheme whose points are pairs (c, t_c) where c is a point of C and t_c is a formal coordinate near c. The Harish-Chandra pair $(\mathfrak{g}, K)$ acts on $\widehat{C}$ with the action of K preserving the projection $\widehat{C} \to C$ and the action of the element d/dt of $\mathfrak{g}$ defining an integrable connection on $\widehat{C}$ along C.

Proposition 5.1.3.

(a) *The functor λ_{X,C^I} is represented by a super-ind-scheme $\mathcal{L}_{C^I}X$ over C^I, and λ^0_{X,C^I} by a super-subscheme $\mathcal{L}^0_{C^I}X$ over C^I.*
(b) *If $I = \{1\}$, the ind-scheme $\mathcal{L}_C X$ and the scheme $\mathcal{L}^0_C X$ are obtained by the principal bundle construction of Gelfand–Kazhdan, i.e.,*

$$\mathcal{L}_C X = \mathcal{L}X \times_K \widehat{C}, \qquad \mathcal{L}^0_C X = \mathcal{L}^0 X \times_K \widehat{C}.$$

Proof. This is quite similar to [23, (2.3-7)], so we indicate the main steps. First, we consider the case when $X = \mathbb{A}^1$ with coordinate t. As in [23, (2.7)] we see that for $S = \operatorname{Spec}(R)$ an affine super-scheme, a morphism $c_I : S \to C^I$ is given by an I-tuple of elements $(b_i \in R)_{i\in I}$. Then Γ_i is given by the equation $t = b_i$ and Γ is given by $\prod_{i\in I}(t - b_i) = 0$, so the completion of $\mathcal{O}_{S\times C}$ along Γ is described explicitly by

$$H^0(\Gamma, \widehat{\mathcal{O}}_\Gamma) = \varprojlim_{n\geqslant 0} R[t]/\prod_{i\in I}(t - b_i)^{n+1},$$

which is then identified with the set of formal series

$$\sum_{l\geqslant 0} a_l(t)\prod_{i\in I}(t - b_i)^l, \quad a_l(t) \in R[t],\ \deg(a_l) < |I|. \tag{5.1.4}$$

Similarly, $H^0(\Gamma, \mathcal{K}_\Gamma)$ is identified with the set of series

$$\sum_{l\gg -\infty} a_l(t)\prod_{i\in I}(t - b_i)^l, \quad a_l(t) \in R[t],\ \deg(a_l) < |I|. \tag{5.1.5}$$

The subring $H^0(\Gamma, \mathcal{K}^{\sqrt{}}_\Gamma)$ is specified by the condition that the coefficients of $a_l(t)$, $l < 0$, are nilpotent in R.

Therefore, if $X = \mathbb{A}^{d_1|d_2}$ then a morphism ϕ as in (5.1.2) is just given by specifying, for each $l \in \mathbb{Z}$, a vector-valued polynomial $a_l^{(\phi)}(t) \in R[t] \otimes \mathbb{C}^{d_1|d_2}$ with the condition each component of each coefficient has even parity, and the components of the coefficients of $a_l^{(\phi)}$ with $l < 0$, are nilpotent. This describes $\mathcal{L}_{C^I}\mathbb{A}^{d_1|d_2}$ explicitly, in terms of the polynomial and power series rings in these components considered as independent variables, as in [23, (2.7.2)]. Similarly for $\mathcal{L}^0_{C^I}\mathbb{A}^{d_1|d_2}$.

Next, if X is an affine super-scheme of finite type, then we realize X as a closed sub-super-scheme of some $\mathbb{A}^{d_1|d|2}$ and then realize $\mathcal{L}_{C^I}X$ inside $\mathcal{L}_{C^I}\mathbb{A}^{d_1|d_2}$ by imposing the equations of X identically on $d_1 + d_2$-tuple of indeterminate series (5.1.5). Similarly for $\mathcal{L}^0_{C^I}\mathbb{A}^{d_1|d_2}$.

To treat the case of an arbitrary super-scheme of finite type, we prove the analog of the gluing property of the functors λ_{X,C^I} and λ^0_{X,C^I} as in [23, Prop. 2.6.1]. This analog follows directly from the definition of the functors in terms of morphisms of super-spaces as in (5.1.2).

Finally, we pass from the case $C = \mathbb{A}^1$ to the case of an arbitrary smooth curve by using étale local coordinates on C. This proves part (a) of the proposition.

To prove part (b), notice that for $C = \mathbb{A}^1$, the choice of a coordinate t on C gives a section $C \to \widehat{C}$ and thus a splitting of the Gelfand–Kazhdan construction, identifying, say $\mathcal{L}X \times_K \widehat{C}$, with $\mathcal{L}X \times C$. In the presence of such identification, the identification of $\mathcal{L}_C X$ with $\mathcal{L}X \times C$ is immediate for $X = \mathbb{A}^{d_1|d_2}$ and thus for X closed in $\mathbb{A}^{d_1|d_2}$ from the explicit construction above (the polynomials a_l will have degree 0). As the statement is local, the canonical identification for any affine X that this produces, entails an identification for any X of finite type. The case of an arbitrary C can be treated by working locally on C. So we can assume that C has an étale coordinate t which again splits the Gelfand–Kazhdan construction and the argument is similar. $\square$

5.2. *Factorization structure*

The category $\mathbf{Fset}^+$ has a final object $\{1\}$ (a one-point set) and a monoidal structure $\sqcup$ (disjoint union) but no unit object for $\sqcup$. Let **Fset** be the category of all finite sets and all maps. This is a monoidal category with the unit object $\varnothing$.

If $p : J \to I$ and $p' : J' \to I'$ are two morphisms of **Fset**, we denote their disjoint union by

$$p \sqcup p' : I \sqcup I' \longrightarrow J \sqcup J'.$$

Let C be any super-scheme of finite type. For every morphism $p : J \to I$ in **Fset** we denote by C^p the open subset in C^J consisting of the J-tuples (c_j) such that $c_j \neq c_{j'}$ for $p(j) \neq p(j')$. We will write p_J, or simply J, for the unique map $J \to \{1\}$. Notice that $C^{p_J} = C^J$ so the two notations are compatible. We will also write $1_J : J \to J$ for the identity.

Let $K \xrightarrow{q} J \xrightarrow{p} I$ be a composable pair of morphisms of **Fset**. We have the *diagonal map*

$$\Delta_{p,q} : C^p \longrightarrow C^{pq}, \quad (c_j) \longmapsto (c_{q(k)}).$$

If q is surjective, then $\Delta_{p,q}$ is a closed embedding. We also have the *off-diagonal map*

$$j_{p,q} : C^q \longrightarrow C^{pq}, \quad (c_k) \longmapsto (c_k),$$

which is always an open embedding. For each p, p' we have also the map

$$i_{p,p'} : C^{p \sqcup p'} \longrightarrow C^p \times C^{p'}, \quad (c_k) \longmapsto (c_k),$$

which is also an open embedding. The above maps fit into the following commutative diagrams, existing for any composable triple $L \xrightarrow{r} K \xrightarrow{q} J \xrightarrow{p} I$ of morphisms of **Fset**:

$$\begin{tikzcd} C^p \arrow[r, "\Delta_{p,q}"] \arrow[dr, "\Delta_{p,qr}"'] & C^{pq} \arrow[d, "\Delta_{pq,r}"] \\ & C^{pqr}, \end{tikzcd}$$

$$\begin{tikzcd} C^r \arrow[r, "j_{q,r}"] \arrow[dr, "j_{pq,r}"'] & C^{qr} \arrow[d, "j_{p,qr}"] \\ & C^{pqr}, \end{tikzcd}$$

$$\begin{tikzcd} C^q \arrow[r, "j_{p,q}"] \arrow[d, "\Delta_{q,r}"'] & C^{pq} \arrow[d, "\Delta_{pq,r}"] \\ C^{qr} \arrow[r, "j_{p,qr}"] & C^{pqr}. \end{tikzcd} \tag{5.2.1}$$

Definition 5.2.2. Let $Y_C \to C$ be a super-ind-scheme formally smooth over C, equipped with an integrable connection along C. A *factorization semigroup* on Y_C is a system consisting of:

(a) for any morphism p of $\mathbf{Fset}^+$, a super-ind-scheme $\rho_p : Y_p \to C^p$ formally smooth over $C^{(p)}$, equipped with integrable connections along C^p, so that $Y_{\{1\}} = Y_C$,
(b) for any composable pair p, q in $\mathbf{Fset}^+$, morphisms of relative super-ind-schemes with connections

$$\varkappa_{p,q} : \Delta^*_{p,q}(Y_{pq}) \longrightarrow Y_p, \qquad \kappa_{p,q} : j^*_{p,q}(Y_{pq}) \longrightarrow Y_q$$

which are isomorphisms and satisfy the compatibility conditions lifting (5.2.1):

$$\varkappa_{p,qr} = \varkappa_{p,q} \circ \Delta^*_{p,q}(\varkappa_{pq,r}) : \Delta^*_{p,qr}(Y_{pqr}) \longrightarrow Y_p,$$

$$\kappa_{pq,r} = \kappa_{q,r} \circ j^*_{q,r}(\kappa_{p,qr}) : j^*_{pq,r}(Y_{pqr}) \longrightarrow Y_r,$$

$$\kappa_{p,q} \circ j^*_{p,q}(\varkappa_{pq,r}) = \varkappa_{q,r} \circ \Delta^*_{q,r}(\kappa_{p,qr}) : j^*_{p,q}\Delta^*_{pq,r}(Y_{pqr}) = \Delta^*_{q,r} j^*_{p,qr}(Y_{pqr}) \longrightarrow Y_q,$$

(c) for any pair p, p' in $\mathbf{Fset}^+$, isomorphisms

$$\sigma_{p,p'} : i^*_{p,p'}(Y_p \times Y_{p'}) \longrightarrow Y_{p \sqcup p'}.$$

Definition 5.2.3. A factorization semigroup $(\rho_p : Y_p \to C^p)$ is said to be *cocommutative* if, for any J, J' the maps $\varkappa$, κ factor through a morphism of $C^{J \sqcup J'}$-schemes $Y_{J \sqcup J'} \to Y_J \times Y_{J'}$. Here $Y_J = Y_{p_J}$.

Example 5.2.4. The collection (C^p) forms a cocommutative factorization semigroup which we call the *unit semigroup*.

In the remainder of this subsection we will assume that C is a purely even smooth algebraic curve.

Remarks 5.2.5 *(Semigroups versus monoids).* (a) Definition 5.2.2 is equivalent to [23, (2.2.1)]. Indeed, given a system (Y_p) as before, we define $Y_I = Y_{p_I}$. Then the Y_I satisfy the conditions of [23]. Conversely, given (Y_I) as in [23] and $p : J \to I$ a surjection, we define $Y_p = j^*_{p_I,p}(Y_J)$.

Then the Y_p satisfy the conditions of Definition 5.2.2. The reason for the definition chosen here is that it allows one to easily treat higher compatibility conditions, which become necessary when dealing with factorizing line bundles, factorizing gerbes, etc. This will be important in the subsequent paper.

(b) In this paper we changed the terminology of [23] by calling factorization semigroups what was there called *factorization monoids*. Indeed, it is more natural, following [3, (3.10.16)], to reserve the term "factorization monoid" to mean a similar structure, but with Y_p defined for any morphism p in **Fset**, the morphisms $\kappa_{p,q}$ and $\sigma_{p,p'}$ being always isomorphisms, and $\varkappa_{p,q}$ being an isomorphism for surjective q. A factorization monoid (Y_p) possesses a *unit section* which is a collection of sections $(e_p : C^p \to Y_p)$, $p : J \to I$, defined as follows. Take $q : \varnothing \to J$, then $C^{pq} = \{\bullet\}$, and the analog of the axiom (c) implies that $Y_{pq} = \{\bullet\}$ as well. Thus $\Delta^*_{p,q}(Y_{pq}) = C^p$, and $\varkappa_{p,q}$ is a morphism from C^p to Y_p. We define e_p to be this morphism. It then follows, in particular, that $(e_p : C^p \to Y_p)$ is a morphism of factorization semigroups. It also follows that for any local section s of $Y_C \to C$, the product $y_{\{1\}} \times s$ extends to a section of $Y_{\{1,2\}}$ (via κ, σ) whose restriction to the diagonal is identified with s (via $\varkappa$).

(c) One can compare our concept of a factorization monoid/semigroup with that of a chiral monoid/semigroup as introduced in [3, (3.10.16)]. The latter objects live on symmetric powers of C, not Cartesian powers. In addition, the authors impose a condition which (translated into the Cartesian power language) means that the closure in Y_I of the complement to the pre-image of the discriminant divisor in C^I equals Y_I.

(d) The map in Definition 5.2.3 goes in the direction opposite to the map in [3, (3.10.16)] in the axioms of commutative chiral monoids.

(e) The integrable connection of a factorization monoid can be recovered from the other axioms as follows. Assume that $p = p_{\{1\}}$, so $C^p = C$ and $Y_p = Y_C$. Let us show how to recover the connection on Y_p in this case. The general case is similar. We use the second description of integrable connections in Proposition 3.1.7. Set $J = \{1, 2\}$, so $C^{[J]}$ is the formal neighborhood of the diagonal in C^2, and let $q_1, q_2 : C^{[J]} \to C$ be the coordinate projections. We will construct an isomorphism of super-ind-$C^{[J]}$-schemes $q_1^*(Y_C) \to q_2^*(Y_C)$ which restricts to the identity of Y_C over the diagonal $C \subset C^J$. By definition of the unit, the maps $\mathrm{Id} \times y_{\{2\}}$, $y_{\{1\}} \times \mathrm{Id}$ yield isomorphisms $q_1^*(Y_C) \to (Y_J)|_{C^{[J]}}$, $q_2^*(Y_C) \to (Y_J)|_{C^{[J]}}$ which restrict to the identity over the diagonal. This gives a connection. Further, taking $I = \{1, 2, 3\}$ and using the unit property gives at once the integrability, see Section 3.4.7 of [3].

Now, let X be a smooth algebraic super-variety. Recall that C is a purely even smooth algebraic curve. Given a morphism $p : J \to I$ in $\mathbf{Fset}^+$, we denote by $\mathcal{L}_p X$, $\mathcal{L}_p^0 X$ the restrictions of $\mathcal{L}_{C^J} X$, $\mathcal{L}^0_{C^J} X$ to the subscheme C^p of C^J. We have the morphisms

$$X \xleftarrow{\pi_p} \mathcal{L}_p^0 X \xrightarrow{i_p} \mathcal{L}_p X \xrightarrow{\rho_p} C^p. \tag{5.2.6}$$

Proposition 5.2.7. *The systems $(\mathcal{L}_p^0 X)$, $(\mathcal{L}_p X)$ are structures of factorization semigroups on $\mathcal{L}_C^0 X$, $\mathcal{L}_C X$. Further, the factorization semigroup $(\mathcal{L}_C^0 X)$ is cocommutative.*

Proof. In the case of purely even X, the factorization structure was given in [23, (2.3.3)] and established at the level of the functors λ_{X,C^I} and λ^0_{X,C^I} represented by $\mathcal{L}_{C^I} X$ and $\mathcal{L}^0_{C^I} X$. This argument extends verbatim to the case when X is a smooth algebraic super-variety.

However, the integrable connections along C^p were not given in [23]. Since the factorization semigroups $(\mathcal{L}_p^0 X)$, $(\mathcal{L}_p X)$ have no units, in the sense of Remarks 5.2.5, these connections have to be defined separately. Here we supply the definition. We use the formulation of an integrable connection as a datum of type (1) in Proposition 3.1.7. We will construct the connection on $Y_I = Y_{p_I} \to C^I$, and the case of arbitrary Y_p, $p: J \to I$ will follow by restriction to an open subset $C^p \subset C^J$.

So let $c_I, c'_I : S \to C^I$ be two infinitely near maps, with components $c_i, c'_i : S \to C$, $i \in I$. Constructing the data in Proposition 3.1.7(1), we will explain how to canonically identify the pullback $c_I^* \mathcal{L}_{C^I} X$ with $c_I'^* \mathcal{L}_{C^I} X$, and similarly for the pullbacks $\mathcal{L}^0_{C^I} X$. Indeed, for each i we have that c_i and c'_i are infinitely near. Let $\Gamma, \Gamma' \subset S \times C$ be the graph unions for c_I and c'_I. Note that the underlying topological spaces of Γ and Γ' are the same. By definition (5.1.2), a morphism from S to $\mathcal{L}_{C^I} X$ covering c_I, is the same as a morphism of super-spaces $(\underline{\Gamma}, \mathcal{K}_\Gamma^{\sqrt{}}) \to (\underline{X}, \mathcal{O}_X)$. Similarly, a morphism $S \to \mathcal{L}^0_{C^I} X$ covering c_I, is the same a morphism $(\underline{\Gamma}, \widehat{\mathcal{O}}_\Gamma) \to (\underline{X}, \mathcal{O}_X)$. Therefore, in order to identify the pullbacks, it is enough to prove the following:

Lemma 5.2.8. *In the situation described, we have a canonical identification of sheaves on* $\underline{\Gamma} = \underline{\Gamma'}$*:*

$$\widehat{\mathcal{O}}_\Gamma \simeq \widehat{\mathcal{O}}_{\Gamma'}, \qquad \mathcal{K}_\Gamma^{\sqrt{}} \simeq \mathcal{K}_{\Gamma'}^{\sqrt{}}.$$

Proof. This statement is local on C. Choosing an étale coordinate on C we reduce to the case $C = \mathbb{A}^1$, so for each i we can see c_i, c'_i as elements of the coordinate ring $B := \mathbb{C}[S]$ such that $s_i = c_1 - c_2$ is a nilpotent element of B. Let n_0 be such that $s_i^{n_0} = 0$. Put $R = B[t] = \mathbb{C}[S \times \mathbb{A}^1]$ and let $r_i = t - c_i$, $r'_i = t - c'_i$, so $r_i - r'_i = c'i - c_i = -s_i$. The equation of Γ in $S \times C$ is then $r = \prod_i r_i$, while the equation of Γ' is $r' = \prod_i r'_i$. Then

$$\widehat{\mathcal{O}}_\Gamma = \varprojlim_n \mathcal{O}_{S\times C}/\big(r^n\big), \qquad \widehat{\mathcal{O}}_{\Gamma'} = \varprojlim_n \mathcal{O}_{S\times C}/\big(r'^n\big).$$

On the other hand, since $r_i - r'_i$ is nilpotent for each i, so is $r - r'$. This implies that the r_1-adic and the r_2-adic topologies on $\mathcal{O}_{S\times C}$ are equivalent to each other. This implies the first identification of the lemma.

To prove the second identification, we recall that $\mathcal{K}_\Gamma$ is obtained from $\widehat{\mathcal{O}}_\Gamma$ by inverting a local equation of Γ which we can take to be the element r above. Similarly for $\mathcal{K}_{\Gamma'}$ and r'. Let

$$\widehat{R} = \varprojlim R/\big(r^n\big) = \varprojlim R/\big(r'^n\big).$$

Then $r, r' \in \widehat{R}$ with $s = r - r'$ nilpotent, and it is enough to prove that

$$\widehat{R}\big[r^{-1}\big] = \widehat{R}\big[r'^{-1}\big].$$

To see this, let us write

$$\frac{1}{r} = \frac{1}{r'}\left(1 - \frac{s}{r'} + \frac{s^2}{r'^2} - \cdots\right)$$

(a terminating geometric series). So r is invertible in $\widehat{R}[r'^{-1}]$. Changing the order, we see that r' is invertible in $\widehat{R}[r^{-1}]$. This implies that $\mathcal{K}_\Gamma$ is identified with $\mathcal{K}_{\Gamma'}$. Further, the subsheaves $\mathcal{K}_\Gamma^{\sqrt{}}$ and $\mathcal{K}_{\Gamma'}^{\sqrt{}}$ are defined by the condition involving restriction to $S_{\mathrm{red}} \times C$, and are therefore identified as well. $\square$

We finally explain why $(\mathcal{L}_p^0 X)$ gives a cocommutative factorization semigroup structure. In other words, for each nonempty finite sets I, I' we construct morphisms of $C^{I \sqcup I'}$-ind-schemes

$$\mathcal{L}^0_{C^{I\sqcup I'}} X \longrightarrow \mathcal{L}^0_{C^I} X \times \mathcal{L}^0_{C^{I'}} X. \tag{5.2.9}$$

Indeed, let S be a super-scheme and $c_{I\sqcup I'} = (c_I, c_{I'})$ be a morphism from S to $C^{I\sqcup I'}$. Let Γ_I be the union of the graph of the components $c_i : S \to C$ of c_I. Similarly $\Gamma_{I'}$, $\Gamma_{I\sqcup I'}$. Now, the second datum of a morphism from S to $\mathcal{L}_{C^{I\sqcup I'}} X$ is a morphism ϕ from the formal neighborhood of $\Gamma_{I\sqcup I'}$ in $S \times C$, to X. Now, the formal neighborhoods of Γ_I and $\Gamma_{I'}$ are each contained in that of $\Gamma_{I\sqcup I'}$, so by restricting ϕ we get morphisms of these formal neighborhoods into X which, together with the c_I, $c_{I'}$ give morphisms $S \to \mathcal{L}^0_{C^I} X$ and $S \to \mathcal{L}^0_{C^{I'}} X$. This finishes the proof of Proposition 5.2.7. $\square$

Remark 5.2.10. The same proof as in Proposition 4.2.7 implies that, for any super-scheme X of finite type, we have an isomorphism of super-ind-schemes

$$\mathcal{L}_{C^p} \mathcal{S} X = \mathcal{S} \mathcal{L}_{C^p} X.$$

Indeed, both represent the functor

$$S \longmapsto \big\{(c_p, \phi);\ c_p : S \longrightarrow C^p,\ \phi \in \mathrm{Hom}_{\mathbf{Ssp}}\big(\big(\underline{\Gamma}, \mathcal{K}_\Gamma^{\sqrt{}}[\eta]\big), (\underline{X}, \mathcal{O}_X)\big)\big\}$$

where η is an odd generator.

5.3. Factorization of $\mathcal{L}\mathcal{S}^N X$ on super-curves

Fix an integer $N \geqslant 0$. Let now C be a smooth super-curve of pure dimension $(1|N)$. For every $\mathbb{C}$-point $c \in C$ the completed local ring $\widehat{\mathcal{O}}_{C,c}$ is isomorphic to $\mathbb{C}[[t]][\eta_1, \ldots, \eta_N]$. More generally, let $c : S \to C$ be a point of C with values in a super-scheme S. Denoting $\Gamma_c \subset S \times C$ the graph of c, we have the completion $\widehat{\mathcal{O}}_c$ of $\mathcal{O}_{S\times C}$ along Γ, and we call a *formal coordinate system* at c an isomorphism of sheaves of topological local rings

$$\mathcal{O}_\Gamma[[t]][\eta_1, \ldots, \eta_N] \longrightarrow \widehat{\mathcal{O}}_c.$$

As in Section 5.1, we have a super-scheme $\widehat{C} \to C$ whose S-points are data $(c, t, \eta_1, \ldots, \eta_N)$ consisting of an S-point $c : S \to C$ and a formal coordinate system $(t, \eta_1, \ldots, \eta_N)$ at c. We also have the Harish-Chandra pair $(\mathfrak{g}_{1|N}, K_{1|N})$. Here $\mathfrak{g}_{1|N}$ is the Lie super-algebra $\mathrm{Der}\,\mathbb{C}[[t]][\eta_1, \ldots, \eta_N]$ while $K_{1|N} = \mathrm{Aut}\,\mathbb{C}[[t]][\eta_1, \ldots, \eta_N]$ is the group super-scheme whose points in a super-commutative algebra R are invertible formal changes of coordinates

$$t \longmapsto \sum_{i \geqslant 0; J} a_{i,J} t^i \eta^J, \qquad \eta_\nu \longmapsto \sum_{i \geqslant 0; J} b^\nu_{i,J} t^i \eta^J.$$

Here J runs over subsets $J = \{1 \leqslant j_1 < \cdots < j_p \leqslant N\}$, the element $a_{i,J} \in R$ is of parity $|J|$, the element $b^{\nu}_{i,J}$ is of parity $|I| + 1$, and we have

$$\eta^J = \eta_{j_1}\eta_{j_2}\cdots\eta_{j_p}.$$

It is further required that $a_{0,\varnothing} = 0$, $a_{1,\varnothing} \in R^*_{\bar{0}}$, and the matrix $\|b^{\nu}_{0,\{\mu\}}\|^N_{\mu,\nu=1}$ is invertible. This Harish-Chandra pair acts on $\widehat{C} \to C$ as in the even case. As in Proposition 4.2.7 the ind-scheme $\mathcal{L}\mathcal{S}^N X = \mathcal{S}^N \mathcal{L}X$ represents the functor

$$S \longmapsto \mathrm{Hom}_{\mathbf{Ssp}}\big(\big(\underline{S}, \mathcal{O}_S((t))^{\sqrt{}}[\eta_1, \ldots, \eta_N]\big), (\underline{X}, \mathcal{O}_X)\big),$$

where $\eta_1, \ldots, \eta_N$ are odd generators. Similarly for $\mathcal{L}^0\mathcal{S}^N X$ and $\mathcal{O}_S[[t]][\eta_1, \ldots, \eta_N]$. The Harish-Chandra pair $(\mathfrak{g}_{1|N}, K_{1|N})$ also acts on the super-scheme $\mathcal{L}^0\mathcal{S}^N X$ and the super-ind-scheme $\mathcal{L}\mathcal{S}^N X$, thus giving a super-scheme and a super-ind-scheme

$$\mathcal{L}^0_C X = \mathcal{L}^0\mathcal{S}^N X \times_{K_{1|N}} \widehat{C} \longrightarrow C, \qquad \mathcal{L}_C X = \mathcal{L}\mathcal{S}^N X \times_{K_{1|N}} \widehat{C} \longrightarrow C \tag{5.3.1}$$

with integrable connections along C. These integrable connections are given by the action of $\partial/\partial t, \partial/\partial\eta_\nu \in \mathfrak{g}_{1|N}$.

Proposition 5.3.2. *For any $N \geqslant 0$ and any smooth super-curve C of dimension $(1|N)$ there exist factorization semigroups $(\mathcal{L}^0_p X)$, respectively $(\mathcal{L}_p X)$ on the super-ind-schemes $\mathcal{L}^0_C X$, respectively $\mathcal{L}_C X$ given by* (5.3.1).

Proof. The construction is similar to that of Section 5.1. That is, for any $I \in \mathbf{Fset}^+$ and any $c_I : S \to C^I$ with components $c_i : S \to C$, we denote by $\Gamma \subset S \times C$ the union of the graphs of the c_i and construct three sheaves $\widehat{\mathcal{O}}_\Gamma$, $\mathcal{K}_\Gamma$, and $\mathcal{K}^{\sqrt{}}_\Gamma$ on the underlying topological space $\underline{\Gamma}$. Of these, $\widehat{\mathcal{O}}_\Gamma$ is just the completion of $\mathcal{O}_{S\times C}$ along Γ (so its construction does not use the specifics of C being a super-survey). Next, the definition of $\mathcal{K}_\Gamma$ is based on the following lemma.

Lemma 5.3.3. *Let $(r, \xi_1, \ldots, \xi_N)$ and $(r', \xi'_1, \ldots, \xi'_N)$ be two systems of local equations for Γ in $S \times C$, with r, r' being even and ξ_ν, ξ'_ν being odd. Then r' is invertible in $\widehat{\mathcal{O}}_\Gamma[r^{-1}]$, and r is invertible in $\widehat{\mathcal{O}}_\Gamma[r'^{-1}]$.*

Proof. Proof follows from the nilpotency of $r - r'$, as in Lemma 5.2.8. □

The lemma implies that we have a well-defined sheaf $\mathcal{K}_\Gamma = \widehat{\mathcal{O}}_\Gamma[r^{-1}]$, and we define $\mathcal{K}^{\sqrt{}}_\Gamma \subset \mathcal{K}_\Gamma$ as in Section 5.1. After this we define the functor λ_{X,C^I} as in (5.1.2), using $\mathcal{K}^{\sqrt{}}_\Gamma$ and similarly for λ^0_{X,C^I} using $\widehat{\mathcal{O}}_\Gamma$. The proof of representability of these functors is completely analogous to the proof of Proposition 5.1.3(a). Finally the proof that these functors yield factorization semigroups is completely analogous to the proof of Proposition 5.2.7. We leave the remaining details to the reader. □

Remark 5.3.4. Factorization semigroups on $(1|N)$-dimensional super-curves are nonlinear analogs of $N_W = N$ SUSY vertex algebras as defined in [21]. More precisely, recall that the categories of factorization algebras and chiral algebras on a curve are equivalent [3]. One can define a category of factorization algebras on a given $(1|N)$-dimensional super-curve C which is equivalent to the category of chiral algebras on C considered in [21]. Further, factorization semigroups on C yield natural examples of factorization algebras on C, and $N_W = N$ SUSY vertex algebras yield chiral algebras on C according to [21]. In particular, Proposition 5.3.2 provides a geometric reason for the observation of [5] that Ω_X^{ch}, the chiral de Rham complex of any manifold, is a sheaf of $N_K = 1$ SUSY vertex algebras. Indeed, Ω_X^{ch} can be seen as a sheaf of chiral differential operators on $\mathcal{S}X$ and can be recovered from $\mathcal{LS}X$ and its $(1|1)$-dimensional factorization structure.

6. The transgression

6.1. Definition of the transgression

Recall from Section 3.2 that for every super-ind-scheme Y we have a sheaf Ω_Y^m on the topological space $\underline{Y}$. In particular, if $Y = \mathcal{L}X$, then $\underline{Y} = \underline{\mathcal{L}^0 X}$. We define

$$\Omega^m_{\mathcal{L}X|\mathcal{L}^0X} = \operatorname{Ker}\bigl\{\Omega^m_{\mathcal{L}X} \longrightarrow \Omega^m_{\mathcal{L}^0X}\bigr\}.$$

In particular, for $m = 0$ we write

$$\mathcal{O}_{\mathcal{L}X|\mathcal{L}^0X} = \Omega^0_{\mathcal{L}X|\mathcal{L}^0X}.$$

Let $R = \varprojlim_{\alpha\in A} R_\alpha$ be a super-commutative pro-algebra, or, what is the same, a topological super-algebra represented as a filtering projective limit of discrete super-commutative algebras R_α. The ring of Laurent series with coefficients in R is defined by

$$R((t)) = \varprojlim_{\alpha\in A} R_\alpha((t)) = \left\{\sum_{n=-\infty}^{\infty} a_n t^n \;\Big|\; a_n \in R,\ \lim_{n\to-\infty} a_n = 0\right\}. \tag{6.1.1}$$

As in [24, (6.2)], we have the evaluation map which is a morphism of ringed spaces

$$\mathrm{ev} : \bigl(\underline{\mathcal{L}^0 X}, \mathcal{O}_{\mathcal{L}X}((t))\bigr) \longrightarrow (\underline{X}, \mathcal{O}_X). \tag{6.1.2}$$

Its underlying morphism of topological spaces is

$$\mathrm{ev}_\flat = \pi : \underline{\mathcal{L}^0 X} \longrightarrow \underline{X}.$$

In terms of the identification of $\pi_*\mathcal{O}_{\mathcal{L}X}$ given in Proposition 4.3.8(b), the morphism of sheaves of rings corresponding to ev is

$$\mathrm{ev}^\sharp : \pi^{-1}\mathcal{O}_X \longrightarrow \mathcal{O}_{\mathcal{L}X}((t)), \quad \mathrm{ev}^\sharp(a) = \sum_{n=-\infty}^{\infty} a[n]t^n. \tag{6.1.3}$$

Remark 6.1.4. Let S^1 be the unit circle $|t| = 1$ in the complex plane, M be a complex analytic manifold, and $LM = C^\infty(S^1, M)$ be the space of C^∞-maps from S^1 to M. The map ev is the algebraic analog of the canonical map $S^1 \times LM \to M$.

Let us now construct, similarly to [25, (1.3)], the *transgression map*

$$\tau : \Omega^m_X \longrightarrow \pi_* \Omega^{m-1}_{\mathcal{L}X|\mathcal{L}^0 X}, \tag{6.1.5}$$

compatible with the differential. For a topological super-commutative algebra R as above we have the *residue homomorphism*

$$\mathrm{Res} : \Omega^m\bigl(R((t))\bigr) \longrightarrow \Omega^{m-1}(R),$$

see e.g., [25, (1.3.4)] for the commutative case, the super-commutative case is given by the same formulas. Now, the map τ is the composition of

$$\mathrm{Res} : \pi_* \Omega^m\bigl(\mathcal{O}_{\mathcal{L}X}((t))\bigr) \longrightarrow \pi_* \Omega^{m-1}(\mathcal{O}_{\mathcal{L}X}) = \pi_* \Omega^{m-1}_{\mathcal{L}X},$$

and the pullback with respect to the evaluation map

$$\mathrm{ev}^* : \Omega^m_X = \Omega^m(\mathcal{O}_X) \longrightarrow \pi_* \Omega^m\bigl(\mathcal{O}_{\mathcal{L}X}((t))\bigr).$$

We now assume that C is a purely even smooth algebraic curve and use the factorization semigroups $(\mathcal{L}_p X)$ and $(\mathcal{L}^0_p X)$ from Section 5.1.

Definition 6.1.6. Let $\xi \in \Omega^m_{\mathcal{L}X|\mathcal{L}^0 X}$ be a globally defined m-form vanishing on $\mathcal{L}^0 X$. We say that ξ is *additive*, if, first of all, it is $(\mathfrak{g}, K)$-invariant and so gives rise to a relative m-form $\xi_C \in \Omega^m_{(\mathcal{L}_C X|\mathcal{L}^0_C X)/C}$. Second, we require that there exists a family ξ_p of relative forms on $\mathcal{L}_p X$ over C^p, vanishing along $\mathcal{L}^0_p X$ and satisfying the conditions:

(a) For $p = \{1\}$ (the identity map of a 1-element set), we have $\xi_p = \xi_C$.
(b) For any two composable morphisms p, q of $\mathbf{Fset}^+$ we have

$$\Delta^*_{p,q}(\xi_{pq}) = \varkappa^*_{p,q}(\xi_p), \qquad j^*_{p,q}(\xi_{pq}) = \kappa^*_{p,q}(\xi_q).$$

(c) For any two morphisms p, p' of $\mathbf{Fset}^+$ we have

$$i^*_{p,p'}(\xi_p \boxplus \xi_{p'}) = \sigma^*_{p,p'}(\xi_{p \sqcup p'}),$$

where $\boxplus$ means the differential form on the Cartesian product obtained by adding the pull-backs of two forms from the factors.

Note that the forms ξ_p, if they exist, are uniquely defined by the conditions above. We denote by $\mathcal{A}dd^m(X)$ the space of additive m-forms on $\mathcal{L}X$, and by $\mathcal{A}dd^m_X$ the sheaf

$$U \longmapsto \mathcal{A}dd^m(U) \tag{6.1.7}$$

on the Zariski topology of X. Note that the differential of an additive form is again additive, so we have the de Rham complex $\mathcal{A}dd^\bullet_X$ of additive forms. For $m = 0$ we will speak of *additive functions* and denote by $\mathcal{A}dd_X = \mathcal{A}dd^0_X$ the sheaf of such functions.

Proposition 6.1.8. *For any m-form $\eta \in \Omega^m(U)$, the $(m-1)$-form $\tau(\eta)$ is additive. We have therefore a morphism of complexes of sheaves*

$$\tau : \Omega_X^\bullet \longrightarrow \mathcal{A}dd_X^{\bullet-1}.$$

Proof. First of all, the fact that $\tau(\eta)$ is $(\mathfrak{g}, K)$-invariant, is clear, as integration of differential forms is an invariant procedure. Further, generalizing [25, (1.6)], we construct the "global" version of the transgression map

$$\tau_I : \Omega_X^m \longrightarrow \pi_{I*}\big(\Omega^{m-1}_{(\mathcal{L}_{C^I}X/C^I)|\mathcal{L}^0_{C^I}X}\big). \tag{6.1.9}$$

Here the sheaf in the right-hand side consists of relative $(m-1)$-forms on $\mathcal{L}_{C^I}X$ over C^I, vanishing along $\mathcal{L}^0_{C^I}X$, and π_I is the canonical projection

$$\pi_I : \mathcal{L}^0_{C^I}X \longrightarrow X.$$

Let η be a local section of Ω_X^m. Restricting X if necessary, we can assume that η is a global section. To define $\tau_I(\eta)$, we need to define, for each super-scheme S and each morphism $h : S \to \mathcal{L}_{C^I}X$, an $(m-1)$-form $h^*\tau_I(\eta)$ on S in a compatible way. Let h correspond to a datum (c_I, ϕ) with respect to the graph subscheme Γ, as in (5.1.2). We then get a section

$$\phi^*\eta \in H^0\big(\Gamma, \mathcal{K}_\Gamma^{\sqrt{}} \otimes \Omega^m_{S\times C}\big).$$

Let $q : S \times C \to S$ be the projection. Then

$$\Omega^1_{S\times C} = \Omega^1_{S\times C/S} \oplus q^*\Omega^1_S,$$

which implies that

$$\Omega^m_{S\times C} = \bigoplus_{i+j=m} \Omega^i_{S\times C/S} \otimes q^*\Omega^j_S.$$

Let

$$v : \Omega^m_{S\times C} \longrightarrow \Omega^1_{S\times C/S} \otimes q^*\Omega^{m-1}_S \tag{6.1.10}$$

be the projection to the summand with $i = 1$, $j = m-1$.

Let us denote the projection $\Gamma \to S$ by the same letter q. Our statement now follows from the next lemma.

Lemma 6.1.11. *For each super-scheme S and each morphism of super-schemes $c_I : S \to C^I$, $c_I = (c_i : S \to C)_{i\in I}$, there is a morphism*

$$\mathrm{Res}_{\Gamma/S} : q_*\big(\mathcal{K}_\Gamma \otimes \Omega^1_{S\times C/S}\big) \longrightarrow \mathcal{O}_S$$

of sheaves on S, and these morphisms satisfy the following properties:

(a) *Compatibility with base change for any morphism of super-schemes $S' \to S$.*
(b) *Additivity: Let I', I'' be two nonempty finite sets, and $c_{I'}: S \to C^{I'}$ and $c_{I''}: S \to C^{I''}$ be two morphisms whose graph unions Γ', Γ'' are disjoint. Denote $I = I' \sqcup I''$ and let $c_I = (c_{I'}, c_{I''}): S \to C^I$ be the combined morphism whose graph union is $\Gamma = \Gamma' \sqcup \Gamma''$. Then, with respect to the identification*

$$q_*\big(\mathcal{K}_\Gamma \otimes \Omega^1_{S\times C/S}\big) = q'_*\big(\mathcal{K}_{\Gamma'} \otimes \Omega^1_{S\times C/S}\big) \oplus q''_*\big(\mathcal{K}_{\Gamma''} \otimes \Omega^1_{S\times C/S}\big),$$

we have

$$\mathrm{Res}_{\Gamma/S}\big(\omega' \oplus \omega''\big) = \mathrm{Res}_{\Gamma'/S}\big(\omega'\big) + \mathrm{Res}_{\Gamma''/S}\big(\omega''\big).$$

(c) *Normalization: If $|I| = 1$, so that $q: \Gamma \to S$ is an isomorphism, and t is a local equation of Γ in $S \times C$, then*

$$\mathrm{Res}_{\Gamma/S}\left(\sum_{n\gg-\infty}^{\infty} u_n t^n dt\right) = u_{-1}.$$

Indeed, suppose we know the lemma. We then define the form

$$h^*\tau_I(\eta) = (\mathrm{Res}_{\Gamma/S} \otimes \Sigma)\big(v\big(\phi^*\eta\big)\big)$$

on S for each S and each $h: S \to \mathcal{L}_{C^I}X$. Here $\Sigma: q_*q^*\Omega^{m-1}_S \to \Omega^{m-1}_S$ is the "trace" morphism (summation over the fibers). By part (a) of the lemma, this means that we have the form $\tau_I(\eta)$ on $\mathcal{L}_{C^I}X$, as in (6.1.9). Let $p: J \to I$ be a morphism of $\mathbf{Fset}^+$. We define the $(m-1)$-form $\tau_p(\eta)$ on $\mathcal{L}_pX$ to be the restriction of $\tau_J(\eta)$ to the open part $\mathcal{L}_pX \subset \mathcal{L}_{C^J}X$. After that, the condition (a) of Definition 6.1.6 follows from part (c) of the lemma, condition (b) follows from the definition of $\tau_p\eta$ as the restriction, while condition (c) follows from part (b) of the lemma. So $\tau(\eta)$ is indeed an additive form. The fact that τ is a morphism of complexes, i.e.,

$$\tau_p\big(\eta + \eta'\big) = \tau_p(\eta) + \tau_p\big(\eta'\big), \qquad \tau_p(d\eta) = d\tau_p(\eta),$$

follows from the corresponding properties of τ and from the fact that the form $\tau_p(\eta)$ is defined by $\tau(\eta)$ and by the conditions (a)–(c) of Definition 6.1.6 uniquely (the question being only its existence).

Proof of Lemma 6.1.11. *Step 1: S is a scheme.* In this case the construction of $\mathrm{Res}_{\Gamma/S}$ is deduced from the Grothendieck duality theory [9], as described in [25, (1.6)]. That is, we have the principal part morphism

$$P: \mathcal{K}_\Gamma \otimes \Omega^1_{S\times C/S} \longrightarrow \mathcal{K}_\Gamma \otimes \Omega^1_{S\times C/S}/\widehat{\mathcal{O}}_\Gamma \otimes \Omega^1_{S\times C/S} = \underline{H}^1_\Gamma\big(\Omega^1_{S\times C/S}\big),$$

which we compose with the trace map of the Grothendieck duality

$$\mathrm{tr}_{\Gamma/S}: q_*\underline{H}^1_\Gamma\big(\Omega^1_{S\times C/S}\big) \longrightarrow R^1q_*\big(\Omega^1_{S\times C/S}\big) \longrightarrow \mathcal{O}_S.$$

Now, compatibility of the trace map with arbitrary base change for schemes was established in [9, (1.1.3)], by reduction to the case of Noetherian base (S in our case). This is possible because locally, over an affine $S = \mathrm{Spec}(R)$ any section of $\underline{H}^1$ is given by finitely many data, so the situation is pulled back from the spectrum of a finitely generated subring. For the same reason, it suffices to establish the additivity and normalization properties (b) and (c) in the case of Noetherian S, in which case they are basic properties of the residue symbol, formulated in [9, (A.1.5)] and proved there afterwards.

Step 2: the even part. Let S be an arbitrary super-scheme. Then both the source and target of the desired morphism $\mathrm{Res}_{\Gamma/S}$ are $\mathbb{Z}/2$-graded, so we need to construct the even component

$$\mathrm{Res}_{\Gamma/S,\bar{0}} : q_*\big(\mathcal{K}_\Gamma \otimes \Omega^1_{S\times C/S}\big)_{\bar{0}} \longrightarrow \mathcal{O}_{S,\bar{0}},$$

as well as the odd component $\mathrm{Res}_{\Gamma/S,\bar{1}}$. Notice that we have the ordinary scheme $\widetilde{S} = (\underline{S}, \mathcal{O}_{S,\bar{0}})$. Further, since C is a purely even curve, the $\mathbb{Z}/2$-grading in $\Omega^1_{S\times C/S}$ is induced by that on $\mathcal{O}_S$, which means that

$$\big(\Omega^1_{S\times C/S}\big)_{\bar{0}} = \Omega^1_{\widetilde{S}\times C/\widetilde{S}}.$$

So we define

$$\mathrm{Res}_{\Gamma/S,\bar{0}} = \mathrm{Res}_{\widetilde{\Gamma}/\widetilde{S}},$$

where $\widetilde{\Gamma}$ is the union of the graphs of the morphisms $\widetilde{c}_i : \widetilde{S} \to C$. Parts (b) and (c) of the lemma for $\mathrm{Res}_{\Gamma/S,\bar{0}}$ follow.

Step 3: the odd part. We now reduce to the previous case by using a version of the “even rules” method of [10, §1.7]. Let $\Lambda[\xi]$ be the exterior algebra in one variable, so $\operatorname{Spec}\Lambda[\xi] = \mathbb{A}^{0|1}$. For any super-commutative algebra R, its odd part $R_{\bar{1}}$ can be identified with a subspace of the even part $(R\otimes\Lambda[\xi])_{\bar{0}}$, to be precise, with $R_{\bar{1}}\cdot\xi$, which is the same as the kernel of the multiplication by ξ in $(R\otimes\Lambda[\xi])_{\bar{0}}$. Therefore, in order to define $\mathrm{Res}_{\Gamma/S,\bar{1}}$, we consider $S^\dagger = S\times\mathbb{A}^{0|1}$ and morphisms $c_i^\dagger : S^\dagger \to S \to C$, with the union of their graphs being the super-scheme $\Gamma^\dagger = \Gamma\times\mathbb{A}^{0|1}$. We then define $\mathrm{Res}_{\Gamma/S,\bar{1}}$ to be the restriction of $\mathrm{Res}_{\Gamma^\dagger/S^\dagger,\bar{0}}$ on the kernel of the multiplication with ξ in its source and target. Parts (b) and (c) of the lemma for $\mathrm{Res}_{\Gamma/S,\bar{1}}$ follow from their validity for $\mathrm{Res}_{\Gamma^\dagger/S^\dagger,\bar{0}}$.

It remains to show the compatibility of $\mathrm{Res}_{\Gamma/S}$ defined in terms of its even and odd components, with arbitrary base change of super-schemes $S'\to S$. It is enough to assume that $S = \operatorname{Spec}(R)$, $S' = \operatorname{Spec}(R')$, so we have a morphism of super-commutative algebras $R\to R'$. For $\mathrm{Res}_{\Gamma/S,\bar{0}}$ this follows from the compatibility of the Grothendieck duality with the base change for $R_{\bar{0}}\to R'_{\bar{0}}$, while for $\mathrm{Res}_{\Gamma/S,\bar{1}}$ it follows from compatibility with the base change for $R[\xi]\to R'[\xi]$. This finishes the proof of Lemma 6.1.11 and of Proposition 6.1.8. □

6.2. *Additive functions on $\mathcal{L}X$ and the Radon transform*

We start with several versions of the Poincaré lemma.

Lemma 6.2.1. *Let Y be a smooth algebraic super-variety, $Z\subset Y$ be a smooth sub-super-variety with sheaf of ideals $I_Z\subset\mathcal{O}_Y$, and*

$$\widehat{\Omega}^\bullet_Y = \varprojlim_{m\geqslant 0} \Omega^\bullet_Y / I_Z^{m+1}\Omega^\bullet_Y$$

be the completion of the de Rham complex of Y along Z. Then the complex $\widehat{\Omega}^\bullet_{Y|Z} = \mathrm{Ker}\{\widehat{\Omega}^\bullet_Y \to \Omega^\bullet_Z\}$ is exact everywhere on each affine open set of Z.

Proof. Denote by $N^* = I_Z/I_Z^2$ the conormal bundle of Z in Y. Filtering by powers of I_Z, we equip $\widehat{\Omega}^\bullet_{Y|Z}$ with a decreasing complete filtration whose quotients are nothing but the homogeneous pieces of the Koszul complex:

$$S^p N^* \longrightarrow S^{p-1}N^* \otimes_{\mathcal{O}_Z} \Lambda^1 N^* \longrightarrow S^{p-2}N^* \otimes_{\mathcal{O}_Z} \Lambda^2 N^* \longrightarrow \cdots, \quad p \geqslant 1.$$

Each such quotient is exact on each affine open set. □

Lemma 6.2.2. *Let X be a smooth super-manifold. The relative de Rham complex*

$$\pi_* \Omega^\bullet_{\mathcal{L}X|\mathcal{L}^0 X} = \left\{ \pi_* \mathcal{O}_{\mathcal{L}X|\mathcal{L}^0 X} \xrightarrow{d} \pi_* \Omega^1_{\mathcal{L}X|\mathcal{L}^0 X} \xrightarrow{d} \cdots \right\}$$

is exact everywhere on the Zariski topology of X.

Proof. The statement being local, we can assume that X admits an étale coordinate system $\phi : X \to \mathbb{A}^{d_1|d_2}$. We then have a realization of $\mathcal{L}X$ as a double ind-pro-limit of the schemes $\mathcal{L}^\epsilon_n(\phi)$, as in Remark 4.3.11. Fixing $m > 0$, let

$$\mathcal{L}^m_n(X) = \underset{\epsilon_i = 0,\, i < -m}{\text{“}\varinjlim\text{”}} \mathcal{L}^\epsilon_n X,$$

where the limit is taken over those $\epsilon \in \mathbf{E}$ which have $\epsilon_i = 0$ for $i < -m$. Then $\mathcal{L}^m_n(X)$ is isomorphic to the formal neighborhood of the smooth super-algebraic variety $\mathcal{L}^0_n X$ inside the product of $\mathcal{L}^0_n X$ with an affine super-space of dimension $d_1 m | d_2 m$. So $\Omega^\bullet_{\mathcal{L}X|\mathcal{L}^0 X}$ is a complex of the kind considered in Lemma 6.2.1 and therefore it is exact on each affine open set. Now,

$$\pi_* \Omega^\bullet_{\mathcal{L}X|\mathcal{L}^0 X} = \varprojlim_m \varinjlim_n \pi_{n*} \Omega^\bullet_{\mathcal{L}^m_n X|\mathcal{L}^0_n X},$$

where $\pi_n : \mathcal{L}^0_n X \to X$ is the projection. Further, the ind-pro-system has the maps in the ind-direction injective and the maps in the pro-direction surjective. So the double limit is exact as well. □

For a closed 2-form ω on X we have a closed 1-form $\tau(\omega)$ in $\Omega^{1,\mathrm{cl}}_{\mathcal{L}X|\mathcal{L}^0 X}$. Let $d^{-1}(\tau(\omega))$ be its unique pre-image under the de Rham differential which lies in $\mathcal{O}_{\mathcal{L}X|\mathcal{L}^0 X}$.

Theorem 6.2.3. *The correspondence $\omega \mapsto d^{-1}\tau(\omega)$ defines a morphism of sheaves $d^{-1}\tau : \Omega^{2,\mathrm{cl}}_X \to \mathcal{A}dd_X$, which is an isomorphism.*

This theorem was proved in [25] when X is an even manifold using the results of [13]. Here we give an independent proof in the more general context of super-manifolds. The morphism $d^{-1}\tau$ can be called the *Radon transform* on the space of formal loops. If ω is a symplectic form on X, the function $d^{-1}\tau(\omega)$ is the formal loop space version of the *symplectic action functional*.

To prove Theorem 6.2.3, we associate to any additive function f on $\mathcal{L}X$ a 2-form as follows. Consider the embedding of constant loops

$$\epsilon : X \hookrightarrow \mathcal{L}^0 X \hookrightarrow \mathcal{L}X.$$

We will study the behavior of f on the first and second infinitesimal neighborhoods of X in $\mathcal{L}X$. First, let us introduce the following notation

$$\Omega^1_{\mathcal{L}X}|_X = \epsilon^{-1}\big(\Omega^1_{\mathcal{L}X}\big) \otimes_{\epsilon^{-1}(\mathcal{O}_{\mathcal{L}X})} \mathcal{O}_X.$$

For a section ω of $\Omega^1_{\mathcal{L}X}$ we denote by $\omega|_X$ its image in $\Omega^1_{\mathcal{L}X}|_X$ and call it the *restriction* of ω to X.

Lemma 6.2.4.

(a) *We have* $\Omega^1_{\mathcal{L}X}|_X = \Omega^1_X((t^{-1}))$.
(b) *Dually, defining* $\Theta_{\mathcal{L}X}|_X = \mathrm{Der}(\mathcal{O}_{\mathcal{L}X}, \mathcal{O}_X)$, *the sheaf of continuous derivations, we have* $\Theta_{\mathcal{L}X}|_X = \Theta_X((t))$.

Proof. Part (a). Let f be a local section of $\mathcal{O}_X$. Then, for any $m \in \mathbb{Z}$, we have that $f[m]$ is a local section of $\mathcal{O}_{\mathcal{L}X}$, and so $d(f[m])$ is a local section of $\Omega^1_{\mathcal{L}X}$. Our identification maps $f[m]d(g[n])$ to $(f dg)t^{m+n}$.

Part (b). Let ξ be a local section of Θ_X. We denote by ∂_ξ the corresponding derivation of $\mathcal{O}_X$. Let's now define $\partial_{\xi[n]}$ to be the derivation $\mathcal{O}_{\mathcal{L}X} \to \mathcal{O}_X$ given by

$$\partial_{\xi[n]}\big(f[m]\big) = \delta_{m,n}(\partial_\xi f).$$

This define a subsheaf $\Theta_X[n]$ of $\Theta_{\mathcal{L}X}|_X$. Our identification maps $\Theta_X[n]$ to $\Theta_X t^n$. $\square$

The group $\mathbb{G}_m \subset K$ acts on $\mathcal{L}X$ by *the rotation of the loop* $t \mapsto \lambda t$. So it acts also on the pro-sheaves $\Omega^1_{\mathcal{L}X}|_X$ and $\Theta_{\mathcal{L}X}|_X$. The homogeneous components of degree n are respectively $\Omega^1_X[n] = \Omega^1_X t^n$ and $\Theta_X[n] = \Theta_X t^n$.

Lemma 6.2.5. *If* $\omega \in \mathcal{A}dd^1_X$ *is an additive* 1*-form on* $\mathcal{L}X$ *then the restriction* $\omega|_X$ *is equal to* 0*. In particular, if* f *is an additive function on* $\mathcal{L}X$ *then the differential* $d_x f$ *vanishes along* x.

Proof. It is enough to prove the first claim. Since ω is additive, it is, in particular, $(\mathfrak{g}, K)$-invariant. Thus ω is invariant under the subgroup $\mathbb{G}_m$, and so is $\omega|_X \in \Omega_X((t^{-1}))$. Since $\mathbb{G}_m$ acts on $\Omega_x \cdot t^n$ via the character $\lambda \mapsto \lambda^n$, we conclude that $\omega|_X$ should lie in the subspace $\Omega_X \cdot t^0$. But the t^0-component should also vanish, since the condition that $\omega = 0$ on

$$\Theta_{\mathcal{L}^0 X}|_X = \prod_{n \geqslant 0} \Theta_X[n],$$

is also included in the property of being additive. $\square$

We now continue our argument. As the value and the differential of f vanishes identically along X, we have the invariantly defined Hessian, which is a quadratic form on the restriction of the tangent bundle to X:

$$H(f): S^2\Theta_{\mathcal{L}X|_X} \longrightarrow \mathcal{O}_X.$$

Let $B(f)$ be the corresponding symmetric bilinear form. From the $\mathbb{G}_m$-invariance of f and thus of $B(f)$ we conclude that the only possibly non-trivial homogeneous components of $B(f)$ are the pairings

$$B^n(f): \Theta_X[-n] \otimes \Theta_X[n] \longrightarrow \mathcal{O}_X, \quad n \neq 0.$$

By identifying each $\Theta_X[n]$ with Θ_X, we can associate to $B^n(f)$ a contravariant 2-tensor $\omega^n \in H^0(X, \Omega_X^1 \otimes \Omega_X^1)$:

$$\omega^n(v, w) = B^1(f)\big(v[-n], w[n]\big). \tag{6.2.6}$$

Here v, w are vector fields on X. For example, let $X = \mathbb{A}^{d_1|d_2}$, so tangent vectors v, w to X at any point x with values in a super-commutative algebra R can be seen as elements of $R \otimes \mathbb{C}^{d_1|d_2}$. Then

$$\omega_x^n(v, w) = \frac{1}{2}\frac{d^2}{d\varepsilon^2}\bigg|_{\varepsilon=0} f\big[\varepsilon v t^{-n} + x + \varepsilon w t^n\big] \in R. \tag{6.2.7}$$

Proposition 6.2.8. *For each $n > 0$ we have $\omega^{\pm n} = n\omega^{\pm 1}$.*

Proof. Consider the morphism

$$\Psi^n: \mathcal{L}X \longrightarrow \mathcal{L}X$$

induced by the change of variable $t = u^n$ in the formal series. More precisely, this change of variable induces, for any super-scheme S, a morphism of super-spaces

$$\big(\underline{S}, \mathcal{O}_S((t))^{\surd}\big) \longrightarrow \big(\underline{S}, \mathcal{O}_S((u))^{\surd}\big),$$

and thus we have an endomorphism of the functor representing $\mathcal{L}X$. It is clear that Ψ^n is identical on X and its differential has the following form on $\Theta_X[1]$:

$$d\Psi^n: \Theta_X[1] \longrightarrow \Theta_X[n], \quad v[1] \longmapsto v[n].$$

So our statement would follow from the next lemma.

Lemma 6.2.9. *Any additive m-form $\omega \in \mathcal{A}dd_X^m$ satisfies $(\Psi^n)^*(\omega) = n\omega$.*

Proof. Our change of variable gives a morphism

$$D = \operatorname{Spec}\big(\mathbb{C}[u]\big) \longrightarrow C = \operatorname{Spec}\big(\mathbb{C}[t]\big).$$

Let $\mathbb{Z}_n$ be the cyclic group of order n acting on D^n and thus on $\mathcal{L}_{D^n}X$ by cyclic permutations. The morphism $\Psi^n : \mathcal{L}X \to \mathcal{L}X$ extends to a morphism of global loop spaces

$$\widetilde{\Psi}^n : \mathcal{L}_C X \longrightarrow (\mathcal{L}_{D^n} X)/\mathbb{Z}_n,$$

as the pre-image of a nonzero point $t_0 \in C$ consists on n points defined up to a cyclic permutation.

Now, the m-form $\omega_{D^n} \in \Omega^m_{\mathcal{L}_{D^n}X/D^n}$ is invariant under all permutations, in particular, under $\mathbb{Z}_n$ and so descends to an m-form $\widetilde{\omega}_{D^n}$ on $(\mathcal{L}_{D^n}X)/\mathbb{Z}_n$. Consider the m-form $(\widetilde{\Psi}^n)^*(\widetilde{\omega}_{D^n})$ on the ind-scheme $\mathcal{L}_C X$. The fiber of $\mathcal{L}_C X$ over each $t \in C$ is identified with $\mathcal{L}X$ canonically up to the action of the group scheme K. Now, for $t \neq 0$ the restriction of the m-form $(\widetilde{\Psi}^n)^*(\widetilde{\omega}_{D^n})$ to this fiber is equal to $n\omega$ because each of the n pre-images of t in D will contribute a summand equal to ω, in virtue of the additivity of ω. On the other hand, for $t = 0$ the restriction is equal to $(\Psi^n)^*(\omega)$ by definition. This proves the lemma. □

Proposition 6.2.10. *Let $f \in \mathcal{A}dd_X$ and let ω^1 be defined as above. Then:*

(a) *The tensor ω^1 is skew symmetric in the super-sense, yielding a differential 2-form on X.*
(b) *We have $\omega^n = n\omega^1$ for all $n \neq 0$.*
(c) *The 2-form ω^1 is closed: $d\omega^1 = 0$.*

Proof. It is enough to assume that X is affine and is equipped with an étale morphism $\phi : X \to \mathbb{A}^{d_1|d_2}$. We denote by $x_1, \dots, x_N \in \mathcal{O}(X)$, $N = d_1 + d_2$, the pullbacks under ϕ of the (odd and even) coordinate functions on $\mathbb{A}^{d_1|d_2}$. Then $dx_1, \dots, dx_N$ form an $\mathcal{O}_X$-basis of Ω^1_X, and we denote by $\partial/\partial x_1, \dots, \partial/\partial x_N$ the dual basis of Θ_X. We can then use Taylor expansions of functions on X and $\mathcal{L}(X)$ in the same way as if X was a Zariski open subset in $\mathbb{A}^{d_1|d_2}$. The identification

$$\mathcal{L}(X) \simeq \mathcal{L}\big(\mathbb{A}^{d_1|d_2}\big) \times_{\mathbb{A}^{d_1|d_2}} X,$$

see Remark 4.3.11 and [23, Prop. 1.6.1], means that we have the functions $x_{i,n}$ on $\mathcal{L}(X)$, with $i = 1, \dots, N$ and $n \in \mathbb{Z}$ which we can think as the coefficients of N indeterminate Laurent series

$$x_i(t) = \sum_{n=-m}^{\infty} x_{i,n} t^n, \quad i = 1, \dots, N. \tag{6.2.11}$$

Thus we can expand the function f in the pro-algebra $\mathcal{O}(\mathcal{L}X)$ near each $\mathbb{C}$-point of $X \subset \mathcal{L}(X)$ as a series in these coordinates.

Let us consider only Laurent series starting with terms with t^{-1} and write $x_n = (x_{1,n}, \dots, x_{N,n})$ for the vector of the nth coefficients. Then we can write

$$\begin{aligned} f&\big[x_{-1}t^{-1} + x_0 + x_1 t + x_2 t^2 + \cdots\big] \\ &= \sum_{i,j} \omega_{ij}(x_0) x_{i,-1} x_{j,1} + \sum_{(i \leqslant j),k} \psi_{ijk}(x_0) x_{i,-1} x_{j,-1} x_{k,2} \\ &\quad + \sum_{(i \leqslant j),(k \leqslant l)} \phi_{ijkl}(x_0) x_{i,-1} x_{j,-1} x_{k,1} x_{l,1} + \cdots. \end{aligned} \tag{6.2.12}$$

Note that the above expansion is quasi-homogeneous of degree 0 in the $x_{i,n}$, because of the $\mathbb{G}_m$-invariance of f. We identify the coordinate $x_{i,0}$ on $\mathcal{L}(X)$ with the coordinate x_i on X. Now, the $\omega_{ij}(x_0)$ are nothing but the coefficients of the tensor ω^1. More precisely, we set

$$\omega^1 = \sum_{i,j} \omega_{ij}\, dx_i \otimes dx_j, \quad \omega^1_{ij} = (-1)^{d_i}\omega_{ij}(x_0).$$

So our first task is to prove the antisymmetry of the $\|\omega_{ij}\|$ in the super-sense, i.e., that

$$\omega_{ij} = (-1)^{(1+d_i)(1+d_j)}\omega_{ji}, \quad d_i = \deg(x_i) \in \mathbb{Z}/2.$$

We now explain a method allowing us to exploit the additivity of f in order to obtain information about the coefficients such as $\omega_{ij}(x)$. Fix a $\mathbb{C}$-point $o \in X$ and assume that the functions x_i vanish at o, so we think of o as the origin of coordinates and study the behavior of f near $o \in X \subset \mathcal{L}(X)$. Let $a = (a_1, \ldots, a_N)$ and $b = (b_1, \ldots, b_N)$ be two vectors of independent variables of the same parities as $(x_1, \ldots, x_N)$ which we eventually suppose to be nilpotent of some degree d, so we define

$$R = \mathbb{C}[a_i, b_i \mid i = 1, \ldots, N] / \big(a_i^d, b_i^d \bigm| i = 1, \ldots, N\big).$$

Consider the rational loop

$$\gamma(t) = \frac{a}{t} + \frac{b}{\lambda - t}, \tag{6.2.13}$$

where $\lambda \in \mathbb{C}$ is a parameter. To be precise, $\gamma(t)$ is the unique $R[\lambda]$-point of $\mathcal{L}_{\mathbb{A}^2}X$ whose image under $\phi : X \to \mathbb{A}^{d_1|d_2}$ is the rational loop in the right-hand side of (6.2.13). Note that the canonical map $\mathcal{L}_{\mathbb{A}^2}X \to \mathbb{A}^2$ takes $\gamma(t)$ to the $R[\lambda]$-point $(0, \lambda)$ of $\mathbb{A}^2$. Since f is additive, we have the function $f_{\mathbb{A}^2}$, whose value at $\gamma(t)$ is an element of $R[\lambda]$. On the other hand, we can expand $\gamma(t)$ at each of the two poles, which gives

$$\gamma(t) = at^{-1} + \frac{b}{\lambda}t^0 + \frac{b}{\lambda^2}t + \frac{b}{\lambda^3}t^2 + \cdots$$

near $t = 0$. Now, near $t = \lambda$ we have the coordinate $s = \lambda - t$, and

$$\gamma(t) = bs^{-1} + \frac{a}{\lambda}s^0 + \frac{a}{\lambda^2}s^1 + \frac{a}{\lambda^3}s^2 + \cdots.$$

We see that the coefficients of each individual expansion become singular as $\lambda \to 0$, but the value

$$f_{\mathbb{A}^2}\big[\gamma(t)\big] = f\left[at^{-1} + \frac{b}{\lambda}t^0 + \frac{b}{\lambda^2}t + \frac{b}{\lambda^3}t^2 + \cdots\right] + f\left[bs^{-1} + \frac{a}{\lambda}s^0 + \frac{a}{\lambda^2}s^1 + \frac{a}{\lambda^3}s^2 + \cdots\right]$$

must be regular at $\lambda = 0$. So expanding each summand into a Taylor series using (6.2.12), we have that the coefficients at each negative power of λ must cancel, which provides a system of constraints on the coefficients ω_{ij}, ψ_{ijk}, etc. Thus, we have

$$f\left[at^{-1} + \frac{b}{\lambda}t^0 + \frac{b}{\lambda^2}t + \frac{b}{\lambda^3}t^2 + \cdots\right] = \sum_{i,j}\omega_{ij}\left(\frac{a}{\lambda}\right)\frac{a_i b_j}{\lambda^2} + \sum_{(i \leqslant j),k}\psi_{ijk}\left(\frac{b}{\lambda}\right)\frac{a_i a_j b_k}{\lambda^3} + \cdots$$

where dots stand for terms with $1/\lambda^4$ and higher. To arrive at the precise coefficients at the powers of λ, we need to further expand $\omega_{ij}(b/\lambda)$, $\psi_{ijk}(b/\lambda)$, etc. near the point o, using the Taylor formula, which gives

$$f\left[at^{-1}+\frac{b}{\lambda}t^0+\frac{b}{\lambda^2}t+\frac{b}{\lambda^3}t^2+\cdots\right]=\sum_{i,j}\omega_{ij}(o)\frac{a_ib_j}{\lambda^2}+\sum_{i,j,k}\frac{\partial\omega_{ij}}{\partial x_k}(o)\frac{b_ka_ib_j}{\lambda^3}+\sum_{(i\leqslant j),k}\psi_{ijk}(o)\frac{a_ia_jb_k}{\lambda^3}+\cdots,$$

and similarly for the other summand. So the cancellation of the terms with $1/\lambda^2$ in $f_{\mathbb{A}^2}[\gamma(t)]$ implies that

$$\omega_{ij}(o)+(-1)^{\deg(a_i)\deg(b_j)}\omega_{ji}(o)=0.$$

Since $d_i=\deg(a_i)$, $d_j=\deg(b_j)$ and o can be any point of X, this proves the antisymmetry of ω^1 and thus parts (a) and (b) of Proposition 6.2.10.

Continuing further, for $j\leqslant k$, cancellation of the coefficients at $a_ib_jb_k/\lambda^3$ gives

$$(-1)^{d_id_k+d_jd_k}\frac{\partial\omega_{ij}}{\partial x_k}(o)+(-1)^{d_id_j}\frac{\partial\omega_{ik}}{\partial x_j}(o)+(-1)^{d_id_j+d_id_k}\psi_{jki}(o)=0. \tag{6.2.14}$$

So the terms with the derivatives of ω_{ij} become mixed with the terms with ψ_{ijk}. To avoid this mixing, we modify our approach by considering the rational loop with three poles

$$\delta(t)=\frac{a}{t}+\frac{b}{\lambda-t}+\frac{c}{\lambda+t}, \tag{6.2.15}$$

where $c=(c_1,\dots,c_N)$ is a third group of nilpotent independent variables of the same parities as $(x_1,\dots,x_N)$. As before, $f_{\mathbb{A}^3}[\delta(t)]$ is the sum of values of f at the three expansions of $\delta(t)$: near $t=0$ where it is

$$\delta(t)=at^{-1}+\frac{b+c}{\lambda}t^0+\frac{b-c}{\lambda^2}t+\frac{b+c}{\lambda^3}t^2+\cdots,$$

near $t=\lambda$, where the expansion in $s=\lambda-t$ is

$$\delta(t)=bs^{-1}+\frac{a+c/2}{\lambda}s^0+\frac{a+c/4}{\lambda^2}s+\frac{a+c/8}{\lambda^3}s^2+\cdots,$$

and near $t=-\lambda$, where the expansion in $u=t+\lambda$ is

$$\delta(t)=cu^{-1}+\frac{-a+b/2}{\lambda}s^0+\frac{-a+b/4}{\lambda^2}s+\frac{-a+b/8}{\lambda^3}u^2+\cdots.$$

The sum of the values of f at these three expansions should not have terms with negative powers of λ. As before, the cancellation of the terms with $1/\lambda^2$ gives the antisymmetry of the ω_{ij}, while the coefficient at $1/\lambda^3$ is found by using the Taylor formula to be

$$\sum_{i,j} \frac{\partial \omega_{ij}}{\partial x_k}(o)(b_k + c_k)a_i(b_j - c_j) + \sum_{(i \leqslant j),k} \psi_{ijk}(o)a_i a_j(b_k - c_k)$$
$$+ \sum_{i,j,k} \frac{\partial \omega_{ij}}{\partial x_k}(o)(a_k + c_k/2)b_i(a_j + c_j/4) + \sum_{(i \leqslant j),k} \psi_{ijk}(o)b_i b_j(a_k - c_k/4)$$
$$+ \sum_{i,j,k} \frac{\partial \omega_{ij}}{\partial x_k}(o)(-a_k + b_k/2)c_i(-a_j + b_j/4) + \sum_{(i \leqslant j),k} \psi_{ijk}(o)c_i c_j(-a_k + b_k/4).$$

In this sum we concentrate on the mixed monomials of the form $a_i b_j c_k$. The coefficient at such a monomial is found to be

$$\frac{1}{2}\left((-1)^{d_i+d_id_k+d_jd_k} \frac{\partial \omega^1_{ij}}{\partial x_k}(o) + (-1)^{1+d_i+d_id_j} \frac{\partial \omega^1_{ik}}{\partial x_j}(o) + (-1)^{d_j} \frac{\partial \omega^1_{jk}}{\partial x_i}(o)\right).$$

So vanishing of such coefficients implies that ω^1 is closed, because

$$d\omega^1 = \sum_{i,j,k}\left((-1)^{(1+d_k)(d_i+d_j)} \frac{\partial \omega^1_{ij}}{\partial x_k}(o) + (-1)^{(1+d_i)(1+d_j)} \frac{\partial \omega^1_{ik}}{\partial x_j}(o) + \frac{\partial \omega^1_{jk}}{\partial x_i}(o)\right)dx_i dx_j dx_k.$$

Proposition 6.2.10 is proved. □

We will denote the 2-form ω^1 simply by ω and call it the *tangential* 2*-form* of f. To emphasize its dependence on f, we will write $\omega = Df$.

Lemma 6.2.16. *Let $\omega' \in \Omega^{2,\mathrm{cl}}(X)$ be a given closed* 2*-form and let $f = d^{-1}\tau(\omega')$. Then $Df = \omega'$.*

Proof. As earlier, it is enough to prove the statement in the formal neighborhood of any point $x \in X$, and so the statement reduces to that for the formal completion of $\mathbb{A}^{d_1|d_2}$ at 0. Because of the formal Poincaré lemma, we can assume that $\omega' = d\eta$ is exact, so $f = \tau(\eta)$. Let $x_1, \ldots, x_N$ be the (even and odd) coordinates in $\mathbb{A}^{d_1|d_2}$, so we write $\eta = \sum_{i=1}^N \eta_i(x)dx_i$. Here $\eta_i(x) \in \mathbb{C}[[x_1, \ldots, x_N]]$ is a formal power series in the even variables with coefficients being elements of the exterior algebra in the odd variables. Then

$$\omega' = \sum_{i<j} \omega'_{ij}(x)dx_i dx_j, \quad \omega'_{ij}(x) = \frac{\partial \eta_i}{\partial x_j} - \frac{\partial \eta_j}{\partial x_i}.$$

Let $\omega = Df$, so we need to prove that $\omega = \omega'$. Let $e_1, \ldots, e_N$ be the basis of $\mathbb{C}^{d_1|d_2}$ corresponding to the coordinate system $x_1, \ldots, x_N$. We view $\mathbb{C}^{d_1|d_2}$ as the tangent space to $\mathbb{A}^N$ at any $\mathbb{C}$-point. Then we need to prove that

$$\omega'_{ij}(x) = \omega_x(e_i, e_j) := \frac{1}{2}\frac{d^2}{d\varepsilon^2}\bigg|_{\varepsilon=0} f\left[\varepsilon e_i t^{-1} + x + \varepsilon e_j t\right].$$

For the purposes of such a proof all the coordinates x_k, $k \neq i, j$ appear as parameters (constants with respect to the differentiation), so we can assume that $N = 2$, $i = 1$, $j = 2$. By splitting η into two summands and switching the roles of x_1 and x_2, it is enough to assume that $\eta = \eta_2(x_1, x_2)dx_2$. Further, by decomposing η_2 into monomials, we reduce to the case $\eta = x_1^a x_2^b dx_2$. The formal loop

$$\gamma(t) = \gamma_\varepsilon(t) = \varepsilon e_1 t^{-1} + x + \varepsilon e_2 t$$

has the coordinates

$$x_1\big[\gamma(t)\big] = \varepsilon t^{-1} + x_1, \qquad x_2\big[\gamma(t)\big] = x_2 + \varepsilon t.$$

So we have

$$f\big[\gamma(t)\big] = \operatorname{Res}_{t=0}\big[\big(\varepsilon t^{-1} + x_1\big)^a (x_2 + \varepsilon t)^b d(x_2 + \varepsilon t)\big] = \varepsilon^2 a x_1^{a-1} x_2^b,$$

see [25, Ex. (1.3.8)]. This is exactly ε^2 times the coefficient at $dx_1 dx_2$ of

$$d\big(x_1^a x_2^b dx_2\big) = d\eta = \omega'. \qquad \square$$

We have proved that the morphism of sheaves

$$D : \mathcal{A}dd_X \longrightarrow \Omega_X^{2,\mathrm{cl}}$$

is left inverse to $d^{-1}\tau$, so D is surjective and $d^{-1}\tau$ is injective. To prove that they are mutually inverse isomorphisms, it suffices to prove the following.

Proposition 6.2.17. *If an additive function f is such that $Df = 0$ identically, then $f = 0$ identically.*

Proof. *First step.* We prove that $f[x(t)] = 0$ if $x(t)$ is any formal loop whose expansion begins with terms with t^{-1}, and so is given by an expansion as in (6.2.12). Suppose that $\omega = Df$ vanishes identically. Then the first group of terms in the right-hand side of (6.2.12) vanishes. We prove inductively that all the coefficients in this expansion vanish, using the vanishing of the coefficient at each negative power of λ in $f[\gamma(t)]$. Indeed, identical vanishing of each ω_{ij} implies, by (6.2.14), that each $\psi_{ijk}(o) = 0$. Here o can be any point, so $\psi_{ijk} \equiv 0$. Next, comparing coefficients at $1/\lambda^4$ in $f[\gamma(t)]$, we get a relation between the values of ϕ_{ijkl}, the first derivatives of the ψ_{ijk} and the second derivatives of the ω_{ij} at any given point o. This implies that each $\phi_{ijkl} \equiv 0$, and so on.

Second step. Any formal loop, i.e., each R-point of $\mathcal{L}(X)$

$$x(t) = \sum_{n=-M}^{\infty} x_n t^n, \quad x_n = (x_{1,n}, \dots, x_{N,n}), \; x_{i,n} \in R,$$

with order of pole $M \geqslant 2$, can be deformed into a 1-parameter family of rational loops each having M poles of first order, by considering the $R[\lambda]$-point of $\mathcal{L}_{\mathbb{A}^M}(X)$ given by

$$x_\lambda(t) = \sum_{p=2}^{M} \frac{x_{-p}}{t(t+\lambda)\cdots(t+(p-1)\lambda)} + \sum_{n=-1}^{\infty} x_n t^n.$$

Then the value $f_{\mathbb{A}^M}[x_\lambda(t)] \in R[\lambda]$ vanishes for $\lambda \neq 0$, since then $x_\lambda(t)$ has only first order poles. Therefore the specialization of $f_{\mathbb{A}^M}[x_\lambda(t)]$ to $\lambda = 0$, i.e., $f[x(t)]$, vanishes as well. $\square$

This finishes the proof of Theorem 6.2.3.

6.3. *Additive forms on $\mathcal{L}X$*

The goal of this subsection is to prove the following theorem.

Theorem 6.3.1. *The morphism of complexes $\tau_X^{\geqslant 2} : \Omega_X^{\geqslant 2} \to \mathcal{A}dd_X^{\geqslant 1}[-1]$ is a quasi-isomorphism.*

Proof. Since $\Omega^\bullet_{\mathcal{L}X} = \mathcal{O}_{\mathcal{S}\mathcal{L}X} = \mathcal{O}_{\mathcal{L}\mathcal{S}X}$, and similarly for $\mathcal{L}_{C^p}X$ for any p, see Remark 5.2.10, we conclude that $\mathcal{A}dd^\bullet_X = \mathcal{A}dd^0_{\mathcal{S}X}$. Further, the de Rham differential in $\mathcal{A}dd^\bullet_X$ is just the action on $\mathcal{A}dd^0_{\mathcal{S}X}$ of the vector field D discussed in Section 2.2. Next, by Theorem 6.2.3 applied to $\mathcal{S}X$ we have a sheaf isomorphism

$$d^{-1}\tau : \Omega^{2,\mathrm{cl}}_{\mathcal{S}X} \longrightarrow \mathcal{A}dd^0_{\mathcal{S}X},$$

given by the transgression on $\mathcal{S}X$. Now, by Corollary 2.4.2 (case $p = 2$), we have a derived category isomorphism

$$q : \mathfrak{t}_{\geqslant 3}(\Omega^\bullet_X) \longrightarrow \Omega^{2,\mathrm{cl},\bullet}_{\mathcal{S}X},$$

represented by the diagram (2.3.12) of quasi-isomorphisms of complexes, in our case by

$$\mathfrak{t}_{\geqslant 3}(\Omega^\bullet_X) \longleftarrow W^\bullet \longrightarrow \Omega^{2,\mathrm{cl},\bullet}_{\mathcal{S}X}, \tag{6.3.2}$$

read from left to right. Here $W^\bullet$ is the total complex of the double complex $W^{\bullet\bullet}$ defined in (2.3.11), and the arrows are the projections to the two edges.

We now regard the transgression as a morphism of truncated complexes

$$\tau^{\mathfrak{t}}_X : \mathfrak{t}_{\geqslant 3}(\Omega^\bullet_X) \longrightarrow (\mathfrak{t}_{\geqslant 2}\mathcal{A}dd^\bullet_X)[-1] = \{\mathrm{Ker}(d) \longrightarrow \mathcal{A}dd^1_X \xrightarrow{d} \mathcal{A}dd^2_X \longrightarrow \cdots\}.$$

Note that $\mathrm{Ker}(d)$ above is identified with the sheaf of additive functions, as we discussed already just before the statement of Theorem 6.2.3. Therefore we will view $\tau^{\mathfrak{t}}_X$ as a morphism of complexes

$$\tau^{\mathfrak{t}}_X : \mathfrak{t}_{\geqslant 3}(\Omega^\bullet_X) \longrightarrow \mathcal{A}dd^\bullet_X[-1].$$

Lemma 6.3.3. *The following diagram commutes in the derived category:*

$$\begin{tikzcd}
\mathfrak{t}_{\geqslant 3}(\Omega^\bullet_X) \arrow[r, "q"] \arrow[d, "\tau^{\mathfrak{t}}_X"'] & \Omega^{2,\mathrm{cl},\bullet}_{\mathcal{S}X} \arrow[d, "d^{-1}\tau_{\mathcal{S}X}"] \\
\mathcal{A}dd^\bullet_X[-1] \arrow[r, "s"] & \mathcal{A}dd^{0,\bullet}_{\mathcal{S}X}[-1].
\end{tikzcd}$$

The lemma implies that τ_X^{t} is a quasi-isomorphism, since q is an isomorphism in the derived category and the other two arrows in the diagram are isomorphisms of complexes. Further, we deduce that $\tau_X^{\geqslant 2}$ is a quasi-isomorphism. Indeed, the only difference between $\mathrm{t}_{\geqslant 3}(\Omega_X^\bullet)$ and $\Omega_X^{\geqslant 2}$ is the lowest degree term $\Omega_X^{2,\mathrm{cl}}$ attached on the left. However, the two projections in (6.3.2) are in fact isomorphisms on this lowest degree term, which allows us to conclude that τ_X^{t} will still induce a quasi-isomorphism after discarding the lowest degree terms. This induced quasi-isomorphism is $\tau_X^{\geqslant 2}$.

Proof of Lemma 6.3.3. Consider the transgression for differential forms on $\mathcal{S}X$

$$\tau_{\mathcal{S}X} : \Omega_{\mathcal{S}X}^{\bullet\bullet} \longrightarrow \mathcal{A}dd_{\mathcal{S}X}^{\bullet-1,\bullet}.$$

Here the first grading is by the degree of differential forms on $\mathcal{S}X$ or $\mathcal{LS}X$, while the second degree is induced by the $\mathbb{G}_m$-action on $\mathcal{S}X$. It is clear that $\tau_{\mathcal{S}X}$ is in fact a morphism of double complexes of degree $(-1,0)$. Indeed, we saw already that it commutes with D_1, the de Rham differentials on forms on $\mathcal{S}X$ and $\mathcal{S}X$. The commutativity with D_2 which is the action of the homological vector field D on $\mathcal{S}X$, follows by naturality of transgression. Note that $\mathcal{A}dd_{\mathcal{S}X}^{\bullet\bullet}$ is an $N = 2$ supersymmetric complex, since it is the additive part of the double de Rham complex of $\mathcal{L}X$.

Now, the quasi-isomorphism q is induced by the two edge projections (6.3.2) of the double complex $W^{\bullet\bullet}$ obtained from $\Omega_{\mathcal{S}X}^{\bullet\bullet}$ by truncating in degrees $\geqslant 2$ for the first grading and adding $\mathrm{Ker}(D_1)$, as described in (2.3.11). Applying $\tau_{\mathcal{S}X}$ to $W^{\bullet\bullet}$ term by term, we map it into a similar double complex formed out of $\mathcal{A}dd_{\mathcal{S}X}^{\bullet-1,\bullet}$ by truncating in degrees $\geqslant 1$ for the first grading and adding $\mathrm{Ker}(D_1)$. Denote this complex by $W_{\mathcal{S}}^{\bullet-1,\bullet}$ and its total complex by $W_{\mathcal{S}}^\bullet$. We have the diagram of edge projections

$$\mathrm{t}_{\geqslant 2}\mathcal{A}dd_X \longleftarrow W_{\mathcal{S}}^\bullet \longrightarrow \mathcal{A}dd^{0,\bullet}. \tag{6.3.4}$$

Since $\mathcal{A}dd_{\mathcal{S}X}^{\bullet\bullet}$ is an $N = 2$ supersymmetric complex, these projections are quasi-isomorphisms. Moreover, the morphism in the derived category obtained by reading this diagram from left to right is the same as the isomorphism of complexes s. Now, to prove the commutativity of the diagram in the lemma, involving q and s, it suffices to note that $\tau_{\mathcal{S}X}$ gives a morphism of the diagram (6.3.2) defining q, to the diagram (6.3.4) defining s. This finishes the proof of Lemma 6.3.3 and Theorem 6.3.1. □

Corollary 6.3.5. *The transgression defines a quasi-isomorphism of the 2-term complexes*

$$\{\Omega_X^2 \longrightarrow \Omega_X^{3,\mathrm{cl}}\} \xrightarrow{\tau} \{\mathcal{A}dd_X^1 \longrightarrow \mathcal{A}dd_X^{2,\mathrm{cl}}\}.$$

Note that τ is not an isomorphism of complexes. For example, any (not necessarily antisymmetric) contravariant 2-tensor on X can be transgressed to an additive 1-form.

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