Journal of Computational and Applied Mathematics 44 (1992) 219–233
North-Holland

CAM 1202

Evaluation of a C-table

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Received 19 June 1991
Revised 28 October 1991

Abstract

Paszkowski, S., Evaluation of a C-table, Journal of Computational and Applied Mathematics 44 (1992) 219–233.

A structure of the Padé table of power series is fully determined by the so-called C-table. Each element of it is a determinant composed of the series coefficients. Froissart and Gilewicz have proved an identity permitting to evaluate the whole C-table of arbitrary power series. The paper contains a new, simpler form of this identity and its new, purely algebraic proof. Details of algorithm based, in part, on the identity are given. An identity analogous to the Sylvester one is proved for determinants in question.

Keywords: C-table; Froissart–Gilewicz identity; generalized Sylvester identity; Padé table.

1. Introduction

For a (formal) power series

$$F(x)=\sum_{k=0}^{\infty} a_k x^k,$$

the *C-table* can be defined. It is composed of the numbers $C_F(l/m)$, l, $m=0, 1, \ldots$, where l and m indicate rows and columns of the table, respectively, such that

$$C_F(l/0) := 1, \tag{1}$$

$$C_F(l/m) := \begin{vmatrix} a_{l-m+1} & a_{l-m+2} & \cdots & a_l \\ a_{l-m+2} & a_{l-m+3} & \cdots & a_{l+1} \\ \vdots & \vdots & & \vdots \\ a_l & a_{l+1} & \cdots & a_{l+m-1} \end{vmatrix}, \quad m>0,$$

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where $a_k := 0$ for $k < 0$. In particular,

$$C_F(l/1) = a_l, \qquad C_F(0/m) = (-1)^{m(m-1)/2} a_0^m. \tag{2}$$

We will abbreviate $C_F(l/m)$ to $C(l/m)$ when no confusion can arise.

If $a_0 = \cdots = a_{j-1} = 0$, $a_j \neq 0$, $G(x) := x^{-j} F(x)$, then

$$C_F(l/m) = 0, \quad 0 \leqslant l < j, \qquad C_F(l/m) = C_G(l-j/m), \quad l \geqslant j.$$

It will therefore be sufficient to consider only the case $a_0 \neq 0$.

The C-table is closely related to the Padé table of the series F and has at least two applications.

(i) Positions of zero entries in the C-table determine a block structure of the corresponding Padé table. Indeed, the Padé table contains a square block such that for some integers λ, $\mu \geqslant 0$ and for an n natural or infinite,

$$[l/m] = [\lambda/\mu], \qquad l \geqslant \lambda,\ m \geqslant \mu,\ l+m \leqslant \lambda+\mu+n-1,$$
$$[l/m] \text{ does not exist}, \quad \lambda \leqslant l \leqslant \lambda+n-1,\ \mu \leqslant m \leqslant \mu+n-1,\ l+m > \lambda+\mu+n-1,$$

iff the C-table contains the square *block* $\mathscr{C}(\lambda, \mu, n)$ composed of the numbers

$$C(l/m), \quad l = \lambda, \lambda+1, \ldots, \lambda+n-1, \quad m = \mu, \mu+1, \ldots, \mu+n-1, \tag{3}$$

such that the numbers $C(\lambda/\mu+j)$, $C(\lambda+j/\mu)$, $j = 0, 1, \ldots, n-1$, and (if $n < \infty$) the numbers $C(\lambda+n/\mu+j)$, $C(\lambda+j/\mu+n)$, $j = 0, 1, \ldots, n$, adjacent to it are different from 0 and the remaining numbers (3) vanish [1, Theorems 1.4.2 and 1.4.3].

(ii) The determinants $C(l/m)$ allow to estimate a local accuracy of Padé approximants: for any block defined above,

$$F(x) - [\lambda/\mu](x) = rx^{\lambda+\mu+n} + \cdots, \quad \text{where } r = \frac{C(\lambda+n/\mu+1)}{C(\lambda+n-1/\mu)} = \frac{C(\lambda+1/\mu+n)}{C(\lambda/\mu+n-1)}$$

(cf. [1, proof of Theorem 1.1.1]).

The determinants $C(l/m)$ satisfy the Sylvester identity

$$C(l-1/m)C(l+1/m) - C(l/m-1)C(l/m+1) = C(l/m)^2. \tag{4}$$

If $C(l/m) \neq 0$ for any l, m, then formulae (1), (2), (4) permit to evaluate the whole C-table, namely, in succession,

$$\begin{array}{lll} C(0/0), & & \\ C(1/0), & C(0/1), & \\ \vdots & & \\ C(k/0), & C(k-1/1), \ldots, C(0/k), & \\ \vdots & & \end{array}$$

(*ascending* algorithm). Another possibility consists in the use of (1), the second formula of (2), (4) and

$$C(1/m) = -\sum_{j=1}^{m} (-1)^{jm+j(j-1)/2} a_0^{j-1} a_j C(1/m-j).$$

Then the antidiagonals of the C-table are evaluated in the opposite direction:

$$\begin{array}{llll} C(0/0), & & & \\ C(0/1), & C(1/0), & & \\ \vdots & & & \\ C(0/k), & C(1/k-1),\ldots,C(k/0), & & \\ \vdots & & & \end{array}$$

(*descending* algorithm; cf. [3]). However, in some cases none of these algorithms suffices to evaluate the whole C-table and extra formulae are needed (e.g., if $C(3/4) = C(5/2) = 0$, then $C(5/4)$ must be calculated by such a formula). They were given by Gilewicz [2] (as results obtained by him together with Froissart). The main goal of this paper is to give a new, simpler form (12) of the Froissart–Gilewicz formula [2, p.374, (85)], which permits to calculate $C(l/m)$ when $C(l/m-2) = 0$ and its new, purely algebraic proof. In addition, an identity (26) similar to (4) is proved. A particular case of (12) results also from (26).

2. Froissart–Gilewicz identity

From now on, we consider a block $\mathscr{C}(\lambda, \mu, n)$ such that $1 < n < \infty$. Using the ascending algorithm we need extra formulae allowing to evaluate the entries $C(\lambda + j/\mu + n)$, if $n > 2$, and $C(\lambda + j/\mu + n + 1)$ for $j = 1, 2, \ldots, n-1$. They are given in [2] where considerations were phrased in terms of the Toeplitz determinants rather than of the Hankel ones:

$$C_m^l := \begin{vmatrix} a_l & a_{l-1} & \cdots & a_{l-m+1} \\ a_{l+1} & a_l & \cdots & a_{l-m+2} \\ \vdots & \vdots & & \vdots \\ a_{l+m-1} & a_{l+m-2} & \cdots & a_l \end{vmatrix}.$$

Obviously, the unique difference between C_m^l and $C(l/m)$ concerns their signs:

$$C_m^l = (-1)^{m(m-1)/2} C(l/m). \tag{5}$$

Let us introduce, for a fixed $k = 1, 2, \ldots, n-1$, the following notations:

$$\begin{array}{ll} N := C(\lambda/\mu + k), & N^{\pm} := C(\lambda/\mu + k \pm 1), \\ N' := C(\lambda - 1/\mu + k), & \\ W := C(\lambda + k/\mu), & W^{\pm} := C(\lambda + k \pm 1/\mu), \\ W' := C(\lambda + k/\mu - 1), & \\ S := C(\lambda + n/\mu + n - k), & S^{\pm} := C(\lambda + n/\mu + n - k \mp 1), \\ S' := C(\lambda + n + 1/\mu + n - k), & \\ E := C(\lambda + n - k/\mu + n), & E^{\pm} := C(\lambda + n - k \mp 1/\mu + n), \\ E' := C(\lambda + n - k/\mu + n + 1). & \end{array} \tag{6}$$

The symbols $N, N^{\pm}, \ldots, E'$ denote the corresponding determinants C_m^l. The entries $N, N^{\pm}, \ldots, E'$ surround the square block of zeros in the manner shown in Fig. 1. In particular,

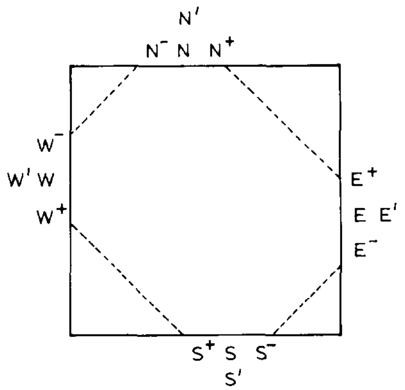

Fig. 1.

W', W^-, N^-, N' lie on the same antidiagonal (S', S^-, E^-, E' too), whereas N', N^+, E^+, E' lie on the same diagonal (W', W^+, S^+, S' too).

The Sylvester formula for C_m^l implies easily the existence of constants (ratios of geometric sequences) $\hat{N}$, $\hat{W}$, $\hat{S}$, $\hat{E}$ such that

$$N = \hat{N}^k C_\mu^\lambda, \qquad W = \hat{W}^k C_\mu^\lambda, \qquad S = \hat{S}^k C_{\mu+n}^{\lambda+n}, \qquad E = \hat{E}^k C_{\mu+n}^{\lambda+n} \tag{7}$$

[2, p.371]. Gilewicz proved that

$$\hat{N}\hat{S} = (-1)^{n-1}\hat{W}\hat{E} \tag{8}$$

[2, p.372, (83)]. Since

$$\frac{NS}{WE} = \frac{(\hat{N}\hat{S})^k C_\mu^\lambda C_{\mu+n}^{\lambda+n}}{(\hat{W}\hat{E})^k C_\mu^\lambda C_{\mu+n}^{\lambda+n}} = (-1)^{k(n-1)}, \tag{9}$$

it follows from (5) that

$$NS = WE. \tag{10}$$

A generalized ascending algorithm uses this identity (when $n > 2$) for evaluating E.

Besides (4) and (10) another identity is needed, namely one for evaluating the element E' for any $k = 1, 2, \ldots, n-1$. The identity in question was found by Froissart and Gilewicz [2, p.374, (85), (85')]. In our notations,

$$\frac{1}{\hat{N}\hat{E}}\left((-1)^n \frac{\hat{N}N'}{N} + (-1)^k \frac{\hat{E}E'}{E}\right) = \frac{1}{\hat{S}\hat{W}}\left((-1)^n \frac{\hat{S}S'}{S} + (-1)^k \frac{\hat{W}W'}{W}\right).$$

It may be considerably simplified as in the following theorem.

Theorem 1. *If the C-table contains a block $\mathscr{C}(\lambda, \mu, n)$ with $1 < n < \infty$, then for $k = 1, 2, \ldots, n-1$, in the notation of* (6),

$$\begin{vmatrix} N' & N^+ & E' \\ W^- & 0 & S^- \\ W' & W^+ & S' \end{vmatrix} = 0, \tag{11}$$

that is,

$$E' = S^-\left[\frac{N'}{W^-} + \frac{N^+}{W^+}\left(\frac{S'}{S^-} - \frac{W'}{W^-}\right)\right]. \tag{12}$$

If $\mu = 0$, then (11) *and* (12) *remain true for $W' := 0$ and, in particular,*

$$E' = N'S^- + N^+S'. \tag{13}$$

Identities (11) and (12) contain eight entries of the C-table, lying on its two diagonals and two antidiagonals. It is remarkable that these entries are arranged in (11) as in the C-table, up to a rotation of 45°; the central zero corresponds here in a sense to the square of zeros in this table. One should remark that all the denominators W^-, W^+, S^- in (12) are, by definition, different from 0.

Proof. Formulas (7) imply that $\hat{N} = N/N^-$, $\hat{E} = E/E^-$ and so on; then

$$\frac{\hat{S}\hat{W}}{\hat{N}\hat{E}}\left((-1)^n\frac{N'}{N^-} + (-1)^k\frac{E'}{E^-}\right) = \left((-1)^n\frac{S'}{S^-} + (-1)^k\frac{W'}{W^-}\right). \tag{14}$$

In view of (8) and (7),

$$\frac{\hat{S}\hat{W}}{\hat{N}\hat{E}} = \frac{\hat{N}\hat{S}\hat{W}}{\hat{N}^2\hat{E}} = (-1)^{n-1}\frac{\hat{W}^2}{\hat{N}^2} = (-1)^{n-1}\frac{W^+N^-}{N^+W^-},$$

and the left-hand side of (14) equals

$$\frac{W^+}{N^+}\left(-\frac{N'}{W^-} + (-1)^{k+n-1}\frac{N^-E'}{W^-E^-}\right).$$

Applying (9) to $N^-, \ldots, E^-$ rather than to $N, \ldots, E$ we obtain

$$W^-E^- = (-1)^{(k-1)(n-1)}N^-S^-,$$

and (14) takes the form

$$\frac{W^+}{N^+}\left(-\frac{N'}{W^-} + (-1)^{kn}\frac{E'}{S^-}\right) = (-1)^n\frac{S'}{S^-} + (-1)^k\frac{W'}{W^-}. \tag{15}$$

This identity simplifies still more after replacing the determinants C_m^l by $C(l/m)$. Of course, it suffices to modify only the signs of all the terms in (15). In particular, $\boldsymbol{W}^+/\boldsymbol{N}^+ = (-1)^\sigma W^+/N^+$, where

$$\sigma \equiv -\tfrac{1}{2}\mu(\mu-1) + \tfrac{1}{2}(\mu+k+1)(\mu+k) \equiv \tfrac{1}{2}(2\mu+k)(k+1).$$

The symbol $\equiv$ denotes here "has the same parity as". Analogously, we have to precede each other expression by $(-1)^\sigma$ where σ corresponding to it is shown on the right:

$$\frac{N'}{W^-}: \qquad \tfrac{1}{2}(\mu+k)(\mu+k-1)-\tfrac{1}{2}\mu(\mu-1)\equiv\tfrac{1}{2}(2\mu+k-1)k,$$

$$-\frac{W^+}{N^+}\frac{N'}{W^-}: \qquad 1+\tfrac{1}{2}(2\mu+k)(k+1)-\tfrac{1}{2}(2\mu+k-1)k\equiv\mu+k+1,$$

$$(-1)^{kn}\frac{E'}{S^-}: \qquad kn+\tfrac{1}{2}(\mu+n+1)(\mu+n)-\tfrac{1}{2}(\mu+n-k+1)(\mu+n-k)$$
$$\equiv\tfrac{1}{2}(2\mu-k+1)k,$$

$$\frac{W^+}{N^+}(-1)^{kn}\frac{E'}{S^-}: \qquad \tfrac{1}{2}(2\mu+k)(k+1)-\tfrac{1}{2}(2\mu+k-1)k\equiv\mu+k,$$

$$(-1)^n\frac{S'}{S^-}: \qquad n+\tfrac{1}{2}(\mu+n-k)(\mu+n-k-1)-\tfrac{1}{2}(\mu+n-k+1)(\mu+n-k)$$
$$\equiv k-\mu,$$

$$(-1)^k\frac{W'}{W^-}: \qquad k+\tfrac{1}{2}(\mu-1)(\mu-2)-\tfrac{1}{2}\mu(\mu-1)\equiv k-\mu+1.$$

Consequently,

$$\frac{W^+}{N^+}\left(\frac{N'}{W^-}-\frac{E'}{S^-}\right)=-\frac{S'}{S^-}+\frac{W'}{W^-}.$$

Thus, it is possible to evaluate E' by (12) which implies easily (11).

If $a_\lambda\neq0$, $a_{\lambda+1}=\cdots=a_{\lambda+n-1}=0$, $a_{\lambda+n}\neq0$ for some λ, n, then the C-table contains the block $\mathscr{C}(\lambda, 0, n)$. Hence the entry W' does not exist, but, as we can verify, identities (11), (12) remain true. It suffices to put there $W'=0$ and, according to (1), $W^-=W^+=1$. Then (12) simplifies to (13). □

3. General ascending algorithm

Creating a C-table we evaluate its successive antidiagonals. On the jth one we find successively the entries

$$C(j/0), C(j-1/1),\ldots,C(1/j-1), C(0/j).$$

The first, second and last ones equal 1, a_{j-1} and $(-1)^{j(j-1)/2}a_0^j$, respectively. Moreover, for $j\geqslant3$ and $m=2,\ldots,j-1$, the entries $C(l/m)$ with $l:=j-m$ must be calculated.

If $C(l/m-2)\neq0$, then $C(l/m)$ is evaluated by the Sylvester formula:

$$C(l/m)=\frac{C(l-1/m-1)C(l+1/m-1)-C(l/m-1)^2}{C(l/m-2)}.$$

Otherwise one should distinguish several cases.

(i) If $C(l/m-1) \neq 0$, then a square $\mathscr{S}$ of zeros, containing $C(l/m-2)$, is placed to the left of $C(l/m)$. Thus, the above Sylvester formula is not applicable. Using the known entries of the C-table, the square $\mathscr{S}$ must be localized. To this end, starting from $C(l-1/m-3)$ and moving along the diagonal to the left we find the first nonzero entry $C(p/q)$. It belongs to the row immediately preceding $\mathscr{S}$. In (12) we have

$$N^{+}=C(p/q+2), \qquad N'=C(p-1/q+1).$$

Next, starting from $C(p+2/q)$ and moving along the antidiagonal to the left we find the first nonzero entry $C(t/u)$. It belongs to the column immediately preceding $\mathscr{S}$. We have

$$W^{+}=C(t/u), \qquad W'=C(t-1/u-1), \qquad W^{-}=C(t-2/u).$$

Since $C(l/m-2)$ belongs to the last column of $\mathscr{S}$, the values of m, p, t, u determine the location of remaining entries in (12):

$$S^{-}=C(m+p-u-1/m+p-t+1), \qquad S'=C(m+p-u/m+p-t).$$

Of course, all the above entries are needed only if $u>0$ when the general formula (12) is applied. Otherwise the simplified formula (13) is used.

(ii) If $C(l/m-1)=0$ and $C(l-1/m)=0$, then $C(l/m)$ belongs to $\mathscr{S}$.

(iii) If $C(l/m-1)=0$ and $C(l-1/m) \neq 0$, then either

(iii.1) $C(l/m)$ belongs to $\mathscr{S}$, or

(iii.2) $C(l/m)$ is placed immediately on the right of $\mathscr{S}$.

The first of the next examples corresponds to case (iii.1) and the other ones to (iii.2):

≠ ≠ ≠ ≠ ≠	≠ ≠ ≠ ≠ ≠ ≠	≠ ≠ ≠ ≠ ≠ ≠
≠0 0 0 ?	≠0 0 0 0 ≠	≠0 0 0 0 ?
≠0 0 0	≠0 0 0 0 ?	≠0 0 0 0
≠0 0	≠0 0 0 0	≠0 0 0
≠0	≠0 0 0	≠0 0
≠	≠ ≠ ≠	≠ ≠

(the signs ?, ≠ denote any $C(l/m)$ and a nonzero entry, respectively).

The two subcases are distinguished as follows:

(1) Starting from $C(l+1/m-1)$ and moving along the antidiagonal to the left we find the first nonzero entry $C(p/q)$. Of course, $p+q=l+m$.

(2) If $C(p-1/q) \neq 0$ (as in the first example above), then (iii.1) holds and $C(l/m)=0$.

(3) If $C(p-1/q)=0$ (as in the second and third examples), then (iii.2) holds. Starting from $C(p-1/q-1)$ and moving along the diagonal to the left, we find the first nonzero element $C(t/u)$. We evaluate $C(l/m)$ by identity (9), where $E=C(l/m)$, $S=C(p/q)$, $W=C(t/u)$ and $N=C(l-p+t/m-p+t)$.

Example 2. Let

$$F(x) := 1+2x+x^3-x^4+x^5-x^6-x^7+x^8+x^9-x^{10}-x^{11} + 2x^{14}-8x^{15}+13x^{16}+\cdots.$$

The determinants $C(l/m)$ of Table 1 depend on these coefficients only.

Table 1
Determinants $C(l/m)$

l	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$	...
0	1	1	−1	−1	1	1	−1	−1	...
1	1	2	−4	−9	21	49	−114	−263	...
2	1	0	2	3	0	7	−109	−85	...
3	1	1	−1	−1	1	1	−99	−427	...
4	1	−1	0	0	2	−14	−86	−94	...
5	1	1	0	0	4	24	−88	−208	...
6	1	−1	−2	4	8	−16	−32	152	...
7	1	−1	−2	0	0	0	16	8	...
8	1	1	−2	0	0	0	−8	36	...
9	1	1	−2	0	0	0	4	96	...
10	1	−1	−2	2	2	−2	−2	160	
11	1	−1	−1	1	5	−25	−79		
12	1	0	0	−2	0	−115			
13	1	0	0	4	46				
14	1	2	−4	−8					
15	1	−8	−38						
16	1	13							
17	1								

If we apply the ascending algorithm, then $C(2/3)$ is the first entry noncalculable by the Sylvester formula. As $C(2/1)=0$, $C(2/2)\neq 0$, case (i) holds. One can verify that $p=1$, $q=0$, $t=3$, $u=0$. Since $u=0$, it suffices to use the entries

$$N^{+}=C(1/2)=-4, \qquad N'=C(0/1)=1,$$
$$S^{-}=C(3/2)=-1, \qquad S'=C(4/1)=-1,$$

which occur in the simplified formula (13):

$$C(2/3)=1\cdot(-1)+(-4)(-1)=3.$$

The general formula (12) is needed, e.g., to evaluate $C(5/5)$ because $C(5/3)=0$, $C(4/3)\neq 0$. In this case,

$$p=3, \qquad q=1, \qquad N^{+}=C(3/3)=-1, \qquad N'=C(2/2)=2,$$
$$t=5, \qquad u=1, \qquad W^{+}=C(5/1)=1, \qquad W'=C(4/0)=1, \qquad W^{-}=C(3/1)=1,$$
$$S^{-}=C(6/4)=8, \qquad S'=C(7/3)=0,$$
$$C(5/5)=8\left[\frac{2}{1}+\frac{-1}{1}\left(\frac{0}{8}-\frac{1}{1}\right)\right]=24.$$

For $l=7$, $m=5$, case (iii.1) holds because $C(7/3)=0$, $C(7/4)=0$, $C(6/5)\neq 0$, $p=10$, $q=2$, $C(9/2)\neq 0$. Hence $C(7/5)=0$.

Finally, if $l=m=4$, then the case (iii.2) holds as $C(4/2)=0$, $C(4/3)=0$, $C(3/4)\neq 0$, $p=6$, $q=2$, $C(5/2)=0$. Since $t=5$, $u=1$, we use identity (9) with $S=C(6/2)=-2$, $W=C(5/1)=1$, $N=C(3/3)=-1$. Hence $C(4/4)=(-1)(-2)/1=2$.

4. Algebraic proof of the Froissart–Gilewicz identity

To prove their main identity [2, p.374, (85)] Froissart and Gilewicz change the series coefficients a_k continuously in such a manner that a block $\mathscr{C}(\lambda+1, \mu+1, n-2)$ be transformed into $\mathscr{C}(\lambda, \mu, n)$. Our proof of the equivalent Theorem 1 is purely algebraic. We suppose here a block $\mathscr{C}(\lambda, \mu, n)$ being such that $1 < n < \infty$. Then in particular $C(\lambda/\mu) \neq 0$ and we can define the quantities

$$\bar{a}_j := \frac{1}{C(\lambda/\mu)} \begin{vmatrix} a_{\lambda-\mu+1} & a_{\lambda-\mu+2} & \cdots & a_{\lambda+1} \\ \vdots & \vdots & & \vdots \\ a_\lambda & a_{\lambda+1} & \cdots & a_{\lambda+\mu} \\ a_{j-\mu} & a_{j-\mu+1} & \cdots & a_j \end{vmatrix}, \quad j = 0, 1, \ldots . \tag{16}$$

In the simplest case, i.e., when $\mu = 0$, they are identical with the a_j's.

Lemma 3.

$$\bar{a}_j = 0, \quad j = \lambda+1, \lambda+2, \ldots, \lambda+\mu+n-1.$$

Proof. According to the Jacobi formula for a Padé approximant $[\lambda/\mu] = P_{\lambda\mu}/Q_{\lambda\mu}$ [1, Section 1.1] with the denominator normed so that $Q_{\lambda\mu}(0) = C(\lambda/\mu)$, we have

$$Q_{\lambda\mu}(x)F(x) = C(\lambda/\mu) \sum_{j=0}^{\infty} \bar{a}_j x^j.$$

Due to the existence of $\mathscr{C}(\lambda, \mu, n)$ the approximant is such that

$$Q_{\lambda\mu}(x)F(x) - P_{\lambda\mu}(x) = O(x^{\lambda+\mu+n}),$$

and Lemma 3 is true. □

Definition (16) implies the existence of numbers $\beta_0, \beta_1, \ldots, \beta_n$ such that for every j,

$$\bar{a}_j = \sum_{i=0}^{\mu} \beta_i a_{j-i}. \tag{17}$$

In particular,

$$\beta_0 = 1, \qquad \beta_\mu = (-1)^\mu \frac{C(\lambda+1/\mu)}{C(\lambda/\mu)} \tag{18}$$

(obviously, the denominator $Q_{\lambda\mu}$ normed so that $Q_{\lambda\mu}(0) = 1$ equals $\beta_0 + \beta_1 x + \cdots + \beta_\mu x^\mu$; cf. [1, Section 1.1, (18)]). Consequently, in each determinant $C(l/m)$ for $m > \mu$ we can, without affecting its value, replace the elements a_j of its rows, from the $(\mu+1)$st to the mth one, by the numbers $\bar{a}_j$. Then, due to Lemma 3, some elements vanish and evaluation of certain determinants becomes easier.

Lemma 4. *For $k = 1, 2, \ldots, n-1$ and in the notation of* (6),

$$N^+ = (-1)^{(\mu+k+2)+\cdots+(\mu+2)} \bar{a}_\lambda^{k+1} C(\lambda/\mu), \tag{19}$$

$$S^- = (-1)^{(\mu+1+\mu+n-k+1)+\cdots+(\mu+1+\mu+1)} \bar{a}_{\lambda+\mu+n}^{n-k-1} W^-. \tag{20}$$

Proof. After the mentioned transformation the rows from the $(\mu+1)$st to the $(\mu+k+1)$st one in the determinant $N^+ := C(\lambda/\mu+k+1)$ are equal to

$$\begin{matrix} \bar{a}_{\lambda-k} & \cdots & \bar{a}_{\lambda} \\ \vdots & \iddots & \mathbf{O} \\ \bar{a}_{\lambda} & & \end{matrix}$$

Indeed, in these rows the numbers $\bar{a}_j$ for $\lambda-k \leqslant j \leqslant \lambda+\mu+k$ occur. As, however, $\lambda+\mu+k \leqslant \lambda+\mu+n-1$, we have $\bar{a}_j = 0$ for $j > 1$. Expanding the transformed determinant successively by the $(\mu+k+1)$st, $(\mu+k)$th,...,$(\mu+1)$st rows we obtain (19).

Changing the $n-k+1$ last rows of the determinant $S^- := C(\lambda+n/\mu+n-k+1)$ with the aid of (17), we transform them into

$$\begin{matrix} & & \bar{a}_{\lambda+\mu+n} \\ \mathbf{O} & \iddots & \vdots \\ \bar{a}_{\lambda+\mu+n} & \cdots & \bar{a}_{\lambda+\mu+2n-k} \end{matrix}$$

Indeed, in these rows the numbers $\bar{a}_j$ for $\lambda+k \leqslant \lambda+\mu+2n-k$ occur. As, however, $\lambda+k > \lambda$, we have $\bar{a}_j = 0$ for $j \leqslant \lambda+\mu+n-1$. Expanding the transformed determinant by the last rows we obtain (20). □

Let us introduce now other auxiliary quantities:

$$D_{kp} := \begin{vmatrix} a_{p-\mu+1} & a_{p-\mu+2} & \cdots & a_{p+n-k} \\ \vdots & \vdots & & \vdots \\ a_p & a_{p+1} & \cdots & a_{p+\mu+n-k-1} \\ \bar{a}_{\lambda+k+2} & \bar{a}_{\lambda+k+3} & \cdots & \bar{a}_{\lambda+\mu+n+1} \\ \vdots & \vdots & & \vdots \\ \bar{a}_{\lambda+n+1} & \bar{a}_{\lambda+n+2} & \cdots & \bar{a}_{\lambda+\mu+2n-k} \end{vmatrix}, \quad \begin{matrix} k = 1, 2, \ldots, n-1, \\ p = \lambda, \lambda+1, \ldots, \lambda+k+1. \end{matrix} \tag{21}$$

This determinant of degree $\mu+n-k$ is composed of two parts. The upper part counts μ rows and contains (if $\mu > 0$) the coefficients a_j of the power series F, whereas the lower one contains the auxiliary coefficients $\bar{a}_j$ defined by (16). By virtue of Lemma 3 some items in the lower part vanish and it is of the form

$$\begin{matrix} & & \bar{a}_{\lambda+\mu+n} & \bar{a}_{\lambda+\mu+n+1} \\ \mathbf{O} & \iddots & \vdots & \\ \bar{a}_{\lambda+\mu+n} & & \cdots & \bar{a}_{\lambda+\mu+2n-k} \end{matrix} \tag{22}$$

Lemma 5. *For $k = 1, 2, \ldots, n-1$ and in the notation of* (6),

$$D_{k\lambda} = \left(S' - \frac{W'S^-}{W^-} \right) \frac{C(\lambda/\mu)}{W^+},$$

where for $\mu = 0$ one should assume that $W' = 0$.

Proof. For $\mu = 0$ it suffices to prove that $D_{k\lambda} = S'$. This is true because in this case

$$D_{k\lambda} = \begin{vmatrix} \bar{a}_{\lambda+k+2} & \cdots & \bar{a}_{\lambda+n+1} \\ \vdots & & \vdots \\ \bar{a}_{\lambda+n+1} & \cdots & \bar{a}_{\lambda+2n-k} \end{vmatrix},$$

and definition (16) simplifies into the form $\bar{a}_j := a_j$.

Let $\mu \neq 0$ now. We apply the formula

$$a_j = \bar{a}_j - \sum_{i=1}^{\mu} \beta_i a_{j-i}, \tag{23}$$

resulting from (16), (17), to the μth row of D_{kp}. If $\lambda < p \leqslant \lambda + k$, then this row contains the numbers a_j such that $p \leqslant j \leqslant p + \mu + n - k - 1$, thus $\lambda < j \leqslant \lambda + \mu + n - 1$. By virtue of Lemma 3 we then have $\bar{a}_j = 0$. Therefore this row is the sum of the row $-\beta_\mu a_{p-\mu}$, $-\beta_\mu a_{p-\mu+1}, \ldots, -\beta_\mu a_{p+n-k-1}$ and of a linear combination of the former rows:

$$D_{kp} = -\beta_\mu \begin{vmatrix} a_{p-\mu+1} & a_{p-\mu+2} & \cdots & a_{p+n-k} \\ \vdots & \vdots & & \vdots \\ a_{p-1} & a_p & \cdots & a_{p+\mu+n-k-2} \\ a_{p-\mu} & a_{p-\mu+1} & \cdots & a_{p+n-k-1} \\ \bar{a}_{\lambda+k+2} & \bar{a}_{\lambda+k+3} & \cdots & \bar{a}_{\lambda+\mu+n+1} \\ \vdots & \vdots & & \vdots \\ \bar{a}_{\lambda+n+1} & \bar{a}_{\lambda+n+2} & \cdots & \bar{a}_{\lambda+\mu+2n-k} \end{vmatrix}$$

$$= \frac{C(\lambda+1/\mu)}{C(\lambda/\mu)} D_{k,p-1}, \quad p = \lambda+1,\ \lambda+2, \ldots, \lambda+k. \tag{24}$$

In a similar manner $D_{k,\lambda+k+1}$ can be transformed. However, it should be taken into account that after applying (23) the last element of the μth row of the determinant contains the term $\bar{a}_{\lambda+\mu+n}$ probably different from 0. We bear in mind also a form of the lower part of the determinant given before this lemma:

$$D_{k,\lambda+k+1} = \frac{C(\lambda+1/\mu)}{C(\lambda/\mu)} D_{k,\lambda+k} + \begin{vmatrix} a_{\lambda-\mu+k+2} & a_{\lambda-\mu+k+3} & \cdots & a_{\lambda+n+1} \\ \vdots & \vdots & & \vdots \\ a_{\lambda+k} & a_{\lambda+k+1} & \cdots & a_{\lambda+\mu+n-1} \\ & & & \bar{a}_{\lambda+\mu+n} \\ \mathbf{O} & & \cdot^{\cdot^{\cdot}} & \vdots \\ & \bar{a}_{\lambda+\mu+n} & \cdots & \bar{a}_{\lambda+\mu+2n-k} \end{vmatrix}$$

$$= \frac{C(\lambda+1/\mu)}{C(\lambda/\mu)} D_{k,\lambda+k} + (-1)^{(\mu+\mu+n-k)+\cdots+(\mu+\mu)} \bar{a}_{\lambda+\mu+n}^{\,n-k-1} W'.$$

Using this result, (24) and (20) we obtain

$$D_{k,\lambda+k+1} = \left(\frac{C(\lambda+1/\mu)}{C(\lambda/\mu)} \right)^{k+1} D_{k\lambda} + \frac{S^-}{W^-} W'.$$

On the other hand, replacing in the definition of $D_{k,\lambda+k+1}$ all the $\bar{a}_j$ by the sums (17), we obtain $S' := C(\lambda+n+1/\mu+n-k)$. Finally, the identity of (7) implies the existence of a constant $\alpha \neq 0$ such that

$$C(\lambda+1/\mu) = \alpha C(\lambda/\mu), \qquad W^+ := C(\lambda+k+1/\mu) = \alpha^{k+1} C(\lambda/\mu).$$

Consequently,

$$\left(\frac{C(\lambda+1/\mu)}{C(\lambda/\mu)} \right)^{k+1} = \alpha^{k+1} = \frac{W^+}{C(\lambda/\mu)}, \qquad S' = \frac{W^+}{C(\lambda/\mu)} D_{k\lambda} + \frac{S^-}{W^-} W'. \qquad \square$$

We are now able to give a new, purely algebraic (without taking limits) proof of Theorem 1.

Proof of Theorem 1. Consider the determinant $E' := C(\lambda+n-k/\mu+n+1)$. Transforming its rows from the $(\mu+1)$st to the $(\mu+n+1)$st one by formula (17) (as in Lemma 4) and bearing in mind Lemma 3, we obtain

$$E' = \begin{vmatrix} a_{\lambda-\mu-k} & a_{\lambda-\mu-k+1} & \cdots & a_{\lambda+n-k} \\ \vdots & \vdots & & \vdots \\ a_{\lambda-k-1} & a_{\lambda-k} & \cdots & a_{\lambda+\mu+n-k-1} \\ \bar{a}_{\lambda-k} & \cdots & \bar{a}_{\lambda} & \\ \vdots & ⋰ & & \\ \bar{a}_{\lambda} & \mathbf{O} & & \bar{a}_{\lambda+\mu+n} \\ & & ⋰ & \vdots \\ & \bar{a}_{\lambda+\mu+n} & \cdots & \bar{a}_{\lambda+\mu+2n-k} \end{vmatrix}.$$

All the elements of the $(\mu+k+1)$st row vanish except the first ($\bar{a}_\lambda$) and the last ($\bar{a}_{\lambda+\mu+n}$) ones. We expand E' by this row and take into account a special form of resulting determinants:

$$E' = (-1)^{(\mu+k+2)+\cdots+(\mu+2)} \bar{a}_\lambda^{k+1} \begin{vmatrix} a_{\lambda-\mu+1} & a_{\lambda-\mu+2} & \cdots & a_{\lambda+n-k} \\ \vdots & \vdots & & \vdots \\ a_{\lambda} & a_{\lambda+1} & \cdots & a_{\lambda+\mu+n-k-1} \\ & & \bar{a}_{\lambda+\mu+n} & \bar{a}_{\lambda+\mu+n+1} \\ \mathbf{O} & & ⋰ & \vdots \\ & \bar{a}_{\lambda+\mu+n} & \cdots & \bar{a}_{\lambda+\mu+2n-k} \end{vmatrix}$$

$$+ (-1)^{(\mu+k+1+\mu+n+1)+\cdots+(\mu+k+1+\mu+k+1)} \bar{a}_{\lambda+\mu+n}^{n-k+1}$$

$$\times\begin{vmatrix} a_{\lambda-\mu-k} & a_{\lambda-\mu-k+1} & \cdots & a_{\lambda-1} \\ \vdots & \vdots & & \vdots \\ a_{\lambda-k-1} & a_{\lambda-k} & \cdots & a_{\lambda+\mu-2} \\ \bar{a}_{\lambda-k} & \cdots & \bar{a}_{\lambda} & \\ \vdots & & \cdot^{\cdot^{\cdot}} & \mathbf{O} \\ \bar{a}_{\lambda-1} & \bar{a}_{\lambda} & & \end{vmatrix}.$$

The first determinant here is equal to $D_{k\lambda}$ (cf. (21), (22)) and its cofactor simplifies due to (19). The second one, after returning from $\bar{a}_j$ to a_j in its lower part, gives $C(\lambda - 1/\mu + k)$, i.e., N', and its cofactor simplifies due to (20):

$$E' = \frac{N^+}{C(\lambda/\mu)} D_{k\lambda} + \frac{S^-}{W^-} N'.$$

Hence, by virtue of Lemma 5,

$$E' = \frac{N^+}{W^+}\left(S' - \frac{W'S^-}{W^-}\right) + \frac{S^-}{W^-} N'.$$

Of course, this is equivalent to (12) and therefore to (11). □

5. A generalization of the Sylvester identity

The Sylvester identity (4) can be expressed in a determinant form:

$$\begin{vmatrix} C(l-1/m) & C(l/m+1) \\ C(l/m-1) & C(l+1/m) \end{vmatrix} = C(l/m)^2. \tag{25}$$

It is natural to try to find another form for a more general determinant of degree $n+1>1$:

$$D(l,\ m,\ n) := \begin{vmatrix} C(l-n/m) & C(l-n+1/m+1) & \cdots & C(l/m+n) \\ C(l-n+1/m-1) & C(l-n+2/m) & \cdots & C(l+1/m+n-1) \\ \vdots & \vdots & & \vdots \\ C(l/m-n) & C(l+1/m-n+1) & \cdots & C(l+n/m) \end{vmatrix}.$$

In view of the definition of $C(l/m)$ it seems that $D(l,\ m,\ n)$ depends on all the coefficients a_j such that $l-m-n-1 \leqslant j \leqslant l+m+n-1$. Let us remark, however, that $a_{l-m-n+1}$ occurs only in the first row of $D(l,\ m,\ n)$ and has as cofactors the numbers $C(l-n+1/m-1)$, $C(l-n+2/m), \ldots, C(l+1/m+n-1)$, i.e., the elements of the second row. Consequently, $D(l,\ m,\ n)$ does not depend on $a_{l-m-n+1}$. Nor depends it on $a_{l+m+n-1}$ (occurring only in the last column of $D(l,\ m,\ n)$). We thus may hope that $D(l,\ m,\ n)$ expresses only in terms of $C(\lambda/\mu)$'s such that $\lambda-\mu \geqslant l-m-n+1$, $\lambda+\mu \leqslant l+m+n-1$. This is the case for $n=1$; cf. (25). We will prove an analogous identity for $n=2$.

Theorem 6. *For l, $m = 1, 2, \ldots,$*

$$D(l, m, 2) = C(l/m)[C(l-1/m)C(l+1/m) + C(l/m-1)C(l/m+1)]. \tag{26}$$

Proof. Let us assume for the moment that $C(l/m) \neq 0$. We multiply the first row of $D(l, m, 2)$ by $C(l/m)$ and subtract from it the second one multiplied by $C(l-1/m+1)$. We multiply the third row by $C(l/m)$ and subtract from it the second one multiplied by $C(l+1/m-1)$:

$$C(l/m)^2 D(l, m, 2) = \begin{vmatrix} d_{11} & 0 & d_{13} \\ C(l-1/m-1) & C(l/m) & C(l+1/m+1) \\ d_{31} & 0 & d_{33} \end{vmatrix},$$

where

$$d_{11} := C(l-2/m)C(l/m) - C(l-1/m-1)C(l-1/m+1),$$
$$d_{13} := C(l/m)C(l/m+2) - C(l-1/m+1)C(l+1/m+1),$$
$$d_{31} := C(l/m-2)C(l/m) - C(l-1/m-1)C(l+1/m-1),$$
$$d_{33} := C(l/m)C(l+2/m) - C(l+1/m-1)C(l+1/m+1).$$

Hence

$$\begin{aligned} C(l/m)^2 & D(l, m, 2) \\ &= C(l/m)\left[C(l-1/m)^2 C(l+1/m)^2 - C(l/m-1)^2 C(l/m+1)^2\right] \\ &= C(l/m)^3 [C(l-1/m)C(l+1/m) + C(l/m-1)C(l/m+1)], \end{aligned}$$

and Theorem 6 is true provided that $C(l/m) \neq 0$. But every $C(\lambda/\mu)$ is a polynomial in the a_k's, i.e., a continuous function of them; then the same identity remains true for $C(l/m) = 0$. □

In particular, if $C(l/m) = 0$, then $D(l, m, 2) = 0$. It is a particular case of (11) for $\lambda = l$, $\mu = m$, $n = 2$, $k = 1$).

The determinant $D(l, m, 3)$ was recently re-expressed in [4]:

$$\begin{aligned} D(l, m, 3) &\\ = C(l/m) &\\ \times\{&C(l-1/m-1)[C(l-1/m+1)C(l+2/m) + C(l+1/m-1)C(l/m+2)] \\ &+ C(l+1/m+1)[C(l-1/m+1)C(l/m-2) \\ &\qquad + C(l+1/m-1)C(l-2/m)]\} \\ + [&C(l-1/m)C(l+1/m) + C(l/m-1)C(l/m+1)] \\ &\times [C(l-2/m)C(l+2/m) - C(l/m-2)C(l/m+2)]. \end{aligned}$$

It is rather difficult to predict a general expression of $D(l, m, n)$ for any n.

Acknowledgements

The author wishes to express his thanks to the referees for their valuable comments.

References

[1] G.A. Baker Jr. and P. Graves-Morris, *Padé Approximants, Part I: Basic Theory* (Addison-Wesley, Reading, MA, 1981).
[2] J. Gilewicz, *Approximants de Padé,* Lecture Notes in Math. **667** (Springer, Berlin, 1978).
[3] J. Gilewicz and E. Leopold, Subroutine CTABLE for best Padé approximant detection, Rep. CPT-81/P.1324, Centre de Phys. Théor., CNRS, Luminy, 1981.
[4] W. Karczewski, Private communication, 1989.