Computers and Mathematics with Applications 61 (2011) 1241–1253

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

A stochastic dynamic programming approach-based yield management with substitution and uncertainty in semiconductor manufacturing

Guanghua Han [a], Ming Dong [b,*], Xiaofeng Shao [b]

[a] Sino-US Global logistics Institute, Shanghai Jiao Tong University, 1954 Huashan Road, Shanghai, 200030, PR China

[b] Department of Operations Management, Antai College of Economics & Management, Shanghai Jiao Tong University, Shanghai, 200052, PR China

ARTICLE INFO

Article history:
Received 19 June 2010
Received in revised form 26 December 2010
Accepted 4 January 2011

Keywords:
Yield management
Inventory
Multi-product
Upward substitution
Periodic depreciation

ABSTRACT

Yield management is important and challengeable in semiconductor industry for the quality uncertainty of the final products. The total yield rate of the semiconductor manufacturing process is uncertain, each product is graded into one of several quality levels according to performance before being shipped. A product originally targeted to satisfy the demand of one product may be used to satisfy the demand of other products when it conforms to their specifications. At the same time, the products depreciate in allocation periods, which mainly results from technical progresses. This paper studies the semiconductor yield management issue of a make-to-stock system with single input, multi-products, multi-demand periods, upward substitution and periodic depreciation. The whole time horizon of the system operation process can be divided into two stages: the production stage and the allocation stage. At the first stage, the firm invests in raw materials before any actual demand is known and produces multiple types of products with random yield rates. At the second stage, products are classified into different classes by quality and allocated in numbers of periods. The production and allocation problem are modeled as a stochastic dynamic program in which the objective is to maximize the profit of the firm. We show that the PRA (parallel allocation first, then upgrade) allocation policy is the optimal allocation policy and the objective function is concave in production input. An iterative algorithm is designed to find the optimal production input and numerical experiments are used to illustrate its effectiveness.

1. Introduction

The more challengeable environment of a semiconductor industry can be characterized by randomly periodic demand, high manufacturing lead time, the expensive set-up costs, and the rapid change of technology, all of which means a significant capital and big risk. The first step in the production of semiconductor chips is the drawing of ingots of either silicon or gallium arsenide. These ingots are sliced into wafers. After several layers of semiconductor materials are placed on the wafers, they are cut into individual chips. Depending on the complexity of the circuits involved, each wafer may yield between 10 and 100,000 chips. The individual chips can then be measured against one or more dimensions of electrical performance and classified as different products. A more detailed description of the production process can be found in [1,2]. In other words, the products have random yields and are used to satisfy the demands of many products. These products have specification requirements that overlap. A product originally targeted to satisfy the demand of one product may be used

* Corresponding address: Department of Operations Management, Antai College of Economics & Management, Shanghai Jiao Tong University. Antai Building, No. 535, FaHua Zhen Rd., Shanghai, 200052, PR China. Tel.: +86 21 52301193; fax: +86 21 52301193.

E-mail address: mdong@sjtu.edu.cn (M. Dong).

0898-1221/$ – see front matter
doi:10.1016/j.camwa.2011.01.003

to satisfy the demand of other products when it conforms to their specifications. The total yield rate of the semiconductor manufacturing process is probabilistic. Hence, the percentage of acceptable units and the relative proportions of products in each production lot could be different from run to run. Meanwhile, a large proportion of the product mix of a semiconductor firm has a relatively short life cycle (one or two years, typically), and a semiconductor chip may loses 60% of value within the first half year of its life cycle. All these means that yield management is important and challengeable in the semiconductor industry.

Some literatures have focused on the yield management of the make-to-stock production systems. Pasternack and Drezner [3] consider a stochastic model for two products which have a single-period inventory structure and which can be used as substitutes for each other. They prove that the expected profit function is concave, and it is possible to find optimal stocking levels for these two products. However, the deterministic yield rates assumption limits the applications of the proposed model. In practice, the yield rates are often random. Karmarkar and Lin [4], Moinzadeh and Lee [5], Lee and Yano [6], Henig and Gerchak [7] study the substitution and allocation problems with assumptions that the yield rate of one product is random and other yield rates are deterministic. Gerchak et al. [8] model a production system with random yield rates and two kinds of products, and the products can be substituted with each other. Their study focuses on identifying the properties of their model. Hsu and Bassok [9] first present a single-period, multi-product inventory model with upward substitution, and then determine the optimal production input and the allocation policy of n products for satisfying n demand classes. Ha [10–12] consider a make-to-stock production system with several product types and demand classes. The study mainly proves that the optimal stock rationing policy is closely related to the production limits, and storage level can be characterized by the corresponding customer class. Tomlin and Wang [13] assume that supplies, demands and yield rates are random, and they allow customers to choose the second-choice products if their first choice is not available. They investigate the pricing and allocation policies in a co-production system with two-class products. In all of these studies, only the single demand period is considered in the proposed models.

Replenishment is not allowed to occur within the allocation periods in several papers. Alstrup et al. [14] study a dynamic overbooking problem with substitution and two product types. The paper proposes a two-stage model: the booking stage and the allocation stage. All demands are realized at the beginning of the allocation periods and substitution is allowed in the allocation stage. Karaesmen and van Ryzin [15] give a more general model with multiple product classes. Our study is similar to their model but with a difference that the demands are realized at the beginning of each period instead of all demands being realized at the beginning of the whole allocation stage. Bitran and Dasu [16] consider a multi-product, multi-period model with demand substitution and discrete random yield. The paper gives the structure of the optimal inventory allocation policy for a two-product, two-period problem. Bitran and Leong [17] add a new service level constraint in the model and assume that the substitution decision is made before the demand is observed. Different from the discrete random yield assumption, our model assumes a continuous random yield and does not allow the backorder.

In yield management, the firm always looks for the optimal policies for allocating the inventory among customer demand classes. Thus, some literatures consider the inventory allocation problem with multi-demand periods. Van Mieghem and Rudi [18] present a newsvendor model with multiple demand periods, and their model allows the firm to replenish inventory in each demand period. Karmarkar [19], Robinson [20], Archibald et al. [21], Frank et al. [22] and Axsater [23] also use the same replenishment policy as Van Mieghem and Rudi [18]. Bassok et al. [24]prove that the substitution is beneficial in the multi-product inventory model. Shumsky and Zhang [25] examine a multi-period inventory allocation model with substitution and describe an optimal allocation policy, and then give an approximate solution for the optimal allocation quantity in each period.

A product-specific depreciation rate is usually based on the rate implicit in financial statements, and it is used in a production or allocation system as one of control variables sometimes [26–28]. In this paper, we consider a yield management problem with different application settings from the previous literatures. Our model can be seen as an extension of the single input, multi-products, single demand period model of Hsu and Bassok [9] to a new application with multi-demand periods. This paper studies the yield management issue of a make-to-stock system with single input, multi-products, multi-demand periods, upward substitution and periodic depreciation. We assume that a single input yields n different products and there are n corresponding demands. The continuous random yield rates of the products are denoted as $\eta_1, \eta_2, \ldots, \eta_n$, respectively. The allocation stage can be divided into several demand periods (T), and the firm allocates the inventory to customers after the demands within each period are observed. In each demand period, if a particular product is out of stock, the firm might upgrade the customer with a more expensive one. Each class of product has a periodic depreciation rate r_i. We consider the single input, n products, and demands, T periods make-to-stock system, as $1 \times n \times T$ system. The objective is to find the optimal production input quantity of the system. Because customers could arrive in any period, this means that the future demands for a particular product type are still unknown, the decision making on optimal allocation and the input quantity is very difficult.

The contributions of this paper are threefold. First, it extends the previous research to a multi-product, multi-demand period make-to-stock system with upward substitution and periodic depreciation rates. Second, an effective algorithm is designed to solve the proposed stochastic dynamic programming model. Third, we prove that the objective function is concave in input quantity and show that PRA allocation policy is more profitable than other propositional allocation policies.

The rest of the paper is organized as follows. Section 2 describes the basic model, which is a stochastic dynamic programming model. In Section 3, we prove that the objective function is concave in the production input quantities, give the available allocation framework and show that the PRA allocation policy is the optimal policy for allocating inventory

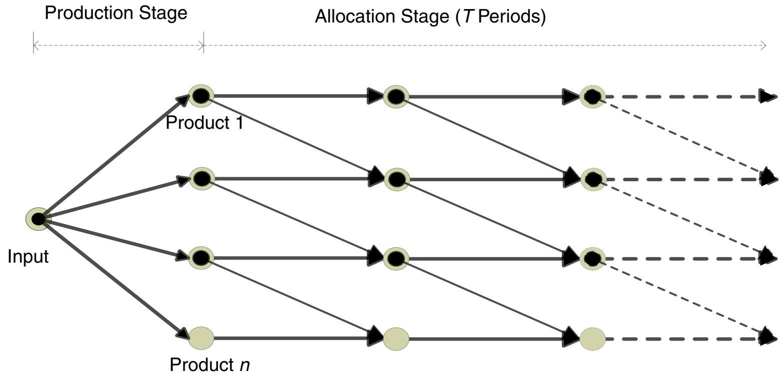

Fig. 1. Two stages of the whole time horizon.

among different customer classes. In Section 4, an effective iterative algorithm is designed to solve the proposed stochastic dynamic programming model and a numerical example is provided. Section 5 concludes the paper.

2. The model

For the convenience, the following assumptions are adopted in the study.

Assumption 1. The demands for all products in each period are random and independent, the PDF (probability density function) are known, the yield rate of each product is also random and its PDF is known. The periodic depreciation rate of each product class is constant and is known.

Assumption 2. The salvage value of each product is zero, and all unsatisfied demands cannot be backordered. One unsatisfied demand can be upgraded by one higher class product.

The whole time horizon can be divided into two stages: the production stage and the allocation stage (see Fig. 1). At the production stage, the firm determines the optimal production input, while at allocation stage, the firm allocates the products through T time periods.

2.1. Production stage

At the beginning of production stage, the production input Q_0 is determined, and the cost of each unit of the production input is c. The production outputs are proportional to the production input quantity with random yield rates, which are denoted by $\eta_1, \eta_2, \ldots, \eta_n$. If the production input quantity Q_0 and yield rates $\eta_1, \eta_2, \ldots, \eta_n$ are given, then the product quantities $(Q_1, Q_2, \ldots, Q_n) = (\eta_1, \eta_2, \ldots, \eta_n) \times Q_0$. There is a ranking among the products which is indexed based on quality. Here, the quality will decrease when the index increases. Therefore, if $i > j$, then demand i can be satisfied by product j.

2.2. Allocation stage

The initial inventory at the allocation stage is the output of the production stage. There are T periods in this stage, and the demand $d^t = (d_1^t, d_2^t, \ldots, d_n^t)$ is observed at the beginning of period t. Suppose that inventory $X^t = (x_1^t, x_2^t, \ldots, x_n^t)$ is available at the beginning of period t, so the initial inventory at the allocation stage is $X^1 = (\eta_1, \eta_2, \ldots, \eta_n) \times Q_0$. Let N^t be the difference between the actual demand at period t and available inventory at period t, then we have

$$N^t = (N_1^t, N_2^t, \ldots, N_n^t) = ((x_1^t - d_1^t), (x_2^t - d_2^t), \ldots, (x_n^t - d_n^t)).$$

Obviously, N_i^t $(i = 1, \ldots, n)$ can be positive, negative, or zero. The allocation decisions made in period t are based on both N^t and the unrealized demand in the following periods. For $i = 1, \ldots, n$, if $N_i^t > 0$ and $N_{i+1}^t < 0$, then $y_{i+1,i}^t$ units of product i can be offered for upgrading. The realized upgraded quantity is non-negative and does not exceed the inventory that product i can provide. That is

$$0 \le y_{i+1,i}^t \le \min(|N_{i+1}^t|, N_i^t).$$

The firm will make the allocation decisions at the beginning of each period, after the demand is realized. Each class of product depreciate with periods and excess inventory at the end of period T has no salvage value. So the whole cost in the time horizon contains four parts: the production input cost, penalty cost, usage cost and depreciation cost.

Let $\Pi(Q_0)$ be the profit function in the whole time horizon. The production input cost occurs at the production stage while product revenue, penalty cost and usage cost occur at the allocation stage. Our objective is to find the optimal production input quantity in order to maximize the profit function. This problem can be formulated as a dynamic program with $T+1$ steps (the corresponding model is considered as a dynamic model). In the production stage, the firm determines the optimal production input quantity, while in periods 1 through T the firm allocates its products inventory to maximize its revenue. Let p_i^t denote the price of product i at period t, v_j denote the penalty cost of product j if unsatisfied, and u_j denote the usage cost of product j per unit (if a unit of product i is sold to customer, then the firm must pay u_j). Let $\alpha_{i,j}^t$ be the contribution margin for satisfying a demand of class i with product i at period t. Similarly, $\alpha_{i,j}^t$ is the contribution margin for satisfying a demand of class i with product j at period t. So the dynamic model is as follows.

Production stage:

$$\Pi(Q_0) = \max_{Q_0}\{\theta^1(X^1) - cQ_0\} \tag{1}$$

$$\text{where, } X^1 = (x_1^1, x_2^1, \ldots, x_n^1) = (\eta_1, \eta_2, \ldots, \eta_n)Q_0. \tag{2}$$

Allocation stage ($1 \le t \le T$):

$$\theta^t(X^t) = E_{D^t}\{\max_{Y^t+X^{t+1}=X^t}[H^t(Y^t/D^t) + \theta^{t+1}(X^{t+1})]\} \tag{3}$$

$$\text{where, } H^t(Y^t/D^t) = \max_{Y^t}\left[\sum_{i,j}\alpha_{i,j}^t y_{i,j}^t - \sum_i v_i d_i^t\right] \tag{4}$$

s.t.

$$\alpha_{i,j}^t = \alpha_{i,j}(1-r_i)^{t-1} \tag{5}$$

$$\sum_j y_{i,j}^t \le d_i^t \tag{6}$$

$$\sum_i y_{i,j}^t \le y_i^t \tag{7}$$

$$y_{i,j}^t, Y^t, X^{t+1} \in N, \qquad Q_0 \in R^+. \tag{8}$$

In the dynamic model, $\theta^t(X^t)$ denotes the total profit of the $T-t+1$ periods (from period t to period T). Similarly, $\theta^1(X^1)$ in Eq. (1) is the whole profit of allocation stage. Thus, Eq. (1) is to maximize the total profit of the whole time horizon, which is equal to the value that the profit at the allocation stage minus the input cost at the production stage. Because the salvage value is zero, $\theta^{T+1}(X^{T+1})$ is equal to zero. Eq. (2) describes that the output at the production stage is the initial inventory at the allocation stage. Y^t denotes the available product inventory that will be allocated in period t, so $Y^t + X^{t+1} = X^t$ ensures that the sum of inventory that sold in period t and the inventory held over to the next period is equal to the inventory available at the beginning of period t. Eq. (3) states that $\theta^t(X^t)$ is the sum of the profit in period t and the profit obtained in the following periods. $y_{i,j}^t$ is the inventory of product j that can be used to substitute for the demand of product i. $H^t(Y^t/D^t)$ in Eq. (4) is the revenue in the single period t with substitution, given realized demand D^t. Eq. (5) is the contribution margin of allocating product j to satisfy the demand i at period t. Eqs. (6) and (7) are the demand constraint and supply constraint in period t, respectively. Obviously, Eq. (8) shows that variables $y_{i,j}^t$, Y^t and X^{t+1} are non-negative integers and Q_0 is a positive number (R^+).

Since the value $\theta^1(X^1)$ is accumulative value from period T to period 1, the calculating process for $\theta^1(X^1)$ is a dynamic stochastic programming model with T steps.

3. Model analysis and the optimal policy

An allocation decision depends on not only the inventory and demand information in the present period, but also the estimated demand information in the following periods. The demand variables are random and independent from each other, and the yield rate η_i is also random. Thus, the dynamic model is a $(T+1)$-step stochastic dynamic programming model:

Lemma 1. *$\Pi(Q_0)$ is concave in Q_0.*

Proof. PRA is the optimal allocation policy, and let $\Pi(Q_0)$ be the maximum profit of the system. Because the salvage value is zero, $\theta^{T+1}(X^{T+1}) = 0$. From the dynamic model, we have

$$\theta^T(X^T) = E_{D^T}\{\max[H^T(X^T/D^T)]\} \tag{9}$$

where $H^T(Y^T/D^T) = \max_{Y^T}\left[\sum_{i,j}\alpha_{i,j}y_{i,j}^T - \sum_i v_i d_i^T\right]$.

s.t.

$$\sum_j y_{i,j}^T \le d_i^T \tag{10}$$

$$\sum_i y_{i,j}^T \le X_i^T \tag{11}$$

$$X_i^T, y_{i,j}^T, Y^T, X^{T+1} \in R_n^+$$

$H^T(Y^T/D^T)$ is a linear program model with the constraints of Eqs. (10) and (11). Obviously, $H^T(Y^T/D^T)$ is concave in X^T because a linear program is concave in variables that determine the right-hand side of its constraints. Van Slyke and Wets [29] prove that concavity is preserved over the expectation operator, so $\theta^T(X^T)$ is concave in X^T.

Assume that $\theta^{t+1}(X^{t+1})$ is concave in X^{t+1}. Again, $Y^t = (y_1^t, \dots, y_n^t)$ determines the right-hand side of constraints in $H^t(Y^t/D^t)$, so function $H^t(Y^t/D^t)$ is concave in Y^t. Because of the constraint $Y^t + X^{t+1} = X^t$, $\theta^t(X^t)$, as the maximum value of sum of two concave functions (Eq. (9)), is concave in X^t [30].

From above derivations, it can be seen that $\theta^1(X^1)$ is concave in X^1. $X^1 = (\eta_1, \eta_2, \dots, \eta_n)Q_0$ is a positive linear function in Q_0, so $\theta^1(X^1)$, as the function of X^1, is also concave in Q_0. Because $-cQ_0$ is also a linear function in Q_0, so $\Pi(Q_0) = \theta^1(X^1) - cQ_0$ must be concave since the sum of concave functions is concave. □

3.1. The optimal allocation policy

We assume that the prices of product i at period t and period 1 are p_i^t and p_i^1, respectively, and let the depreciation rate of product i be r_i, then product i's corresponding price at period $t+1$ is

$$p_i^{t+1} = p_i^t - p_i^t r_i = p_i^1(1-r_i)^t. \tag{12}$$

Because of Eq. (9), $\alpha_{i,j}^{t+1}$ is equal to

$$\alpha_{i,j}^{t+1} = p_i^{t+1} + v_i - u_j = p_i^t(1-r_i) + v_i - u_j \le p_i^t + v_i - u_j = \alpha_{i,j}^t. \tag{13}$$

Eq. (13) shows that the contribution margin of one current satisfaction pattern with a certain direction is always more than that in any of the future periods.

Generally, higher classes of products have higher usage costs, so it is reasonable that usage cost u_j decreases with the index j. At period 1, higher classes of products have higher revenue because the depreciation happens after the demands are realized at this period. So $p_j^1 + v_j$ decreases with index j, and $\alpha_{i,j}^1$ increases with index j and decreases with index i. Then, we have

$$u_i < u_j, \qquad v_i < v_j, \qquad \alpha_{i,j}^1 < \alpha_{j,j}^1 \quad \text{and} \quad \alpha_{i,i}^1 < \alpha_{j,j}^1 \quad \text{for } j < i. \tag{14}$$

Single-step upgrade can deliver the most of benefit of more complex substitution schemes [31]. Then, some literatures consider the single-step upgrade as the optimal location policy, which means that contribution margin $\alpha_{i,j}$ becomes positive if class j products are used to satisfy class j demands or class $j+1$ demands, but it will become negative if class j products are used to satisfy other classes of demands (see Eq. (15)). At period 1, depreciations happen after the demands are realized. So,

$$\begin{cases} \alpha_{i,j}^1 > 0, & \text{for } j \le i \le j+1 \\ \alpha_{i,j}^1 < 0, & \text{otherwise.} \end{cases} \tag{15}$$

When $t > 1$ and $i > j+1$, we can obtain $\alpha_{i,j}^t < 0$ because of Eqs. (13) and (15). It indicates that the single-step upgrade still holds in this make-to-stock system with depreciation.

Because of Eqs. (12)–(15), $\alpha_{i,j}^t$ can be described as follows:

(1) If $i - j > 1$: $\alpha_{i,j}^t = p_i^1(1-r_i)^{t-1} + v_i - u_j \le p_i^1 + v_i - u_j = \alpha_{i,j}^1 < 0$.

(2) If $i - j \le 1$ and $v_i - u_j \ge 0$: $\alpha_{i,j}^t = p_i(1-r_i)^{t-1} + v_i - u_j > v_i - u_j \ge 0$.

(3) If $i - j \le 1$, $v_i - u_j < 0$, and $t \le \frac{\ln(u_j - v_i) - \ln pi}{\ln(1-r_i)} + 1$:

$$\alpha_{i,j}^t = p_i(1-r_i)^{t-1} + v_i - u_j \ge p_i(1-r_i)^{(\ln(u_j-v_i)-\ln p_i)/\ln(1-r_i)} + v_i - u_j = 0.$$

(4) If $i - j \le 1$, $v_i - u_j < 0$, and $t > \frac{\ln(u_j - v_i) - \ln p_i}{\ln(1-r_i)} + 1$:

$$\alpha_{i,j}^t = p_i(1-r_i)^{t-1} + v_i - u_j < p_i(1-r_i)^{(\ln(u_j-v_i)-\ln p_i)/\ln(1-r_i)} + v_i - u_j = 0.$$

Table 1
The signs of the contribution value $\alpha_{i,j}^t$.

Condition	$A > 1$	$A \le 1, B \ge 0$	$A \le 1, B < 0, t \le C$	$A \le 1, B < 0, t > C$
$\alpha_{i,j}^t$	$\alpha_{i,j}^t < 0$	$\alpha_{i,j}^t > 0$	$\alpha_{i,j}^t \ge 0$	$\alpha_{i,j}^t < 0$

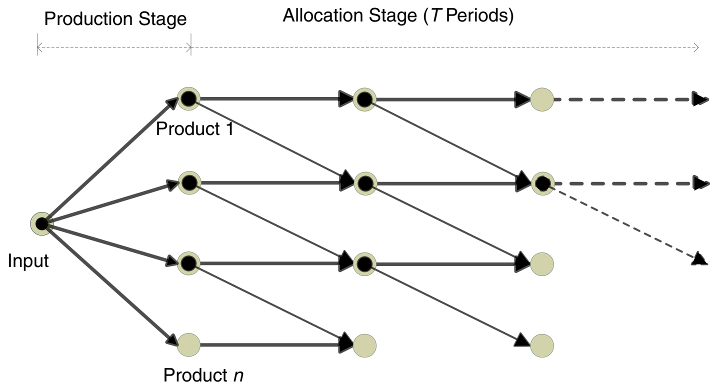

Fig. 2. Available production and allocation flows.

Let A, B and C represent $i - j$, $v_i - u_j$ and $\frac{\ln(u_j - v_i) - \ln p_i}{\ln(1-r_i)} + 1$, respectively. The sign of the contribution margin $\alpha_{i,j}^t$ can be summarized and shown in Table 1.

The price of the higher class products is higher than that of the lower class products in the same allocation period. This fact is easy to understand and it is a common regulation in semiconductor industry and other industries (for example, a high-profile computer is certainly expensive than a low-profile one at the same time). Thus, if $i > j$, we have

$$\alpha_{i,j}^t = p_i^t + v_i - u_j < p_j^t + v_i - u_j < p_j^t + v_j - u_j = \alpha_{j,j}^t. \tag{16}$$

Eq. (16) indicates that the parallel satisfaction is more profitable than the upgrade satisfaction of the products.

Because different products have different depreciation rates, sometimes high class products may not necessarily means high contributions to the producer. However, allocation may happen only at the condition that contribution value $\alpha_{i,j}^t$ is a non-negative value. Based on the above parameter analysis, two insights can be found: (1) When $N_i^t > 0$, $N_{i+1}^t < 0$, allocate as much product i as possible to satisfy demand i, then proper quantities of the remaining product i are allocated to satisfy demand $i+1$. (2) When contribution margin $\alpha_{i,j}^t \le 0$, no allocation that satisfies demand i with product j is allowed in the current and future periods.

Shumsky and Zhang [25] study an allocation system with upgrading and find that PRA is an optimal inventory allocation policy. According to the insights above, PRA allocation policy is still the optimal allocation policy to the make-to-stock production systems. So PRA allocation policy is adopted in solving the input quantity problem of the make-to-stock production systems.

To maximize the profit of the manufacturer, the contribution margins of all allocated products should be non-negative. An available production and allocation network of the make-to-stock system is shown in Fig. 2, where the arrows represent the possible production and allocation flows.

3.2. Optimal allocation quantity

PRA allocation policy is adopted in solving the dynamic allocation problem (see Section 3.1). Thus, if $N_i^t > 0$, $N_{i+1}^t < 0$ and the firm satisfies as much as class i demands with product i's inventory, then some excess inventory of product i can be used to satisfy class $i+1$ demands. The parallel satisfaction and upgrade satisfaction by PRA allocation policy are

$$y_{i,i}^{P-t} = \min(d_i^t, y_i^t), 0 \le y_{i+1,i}^{P-t} \le N_i^t + N_{i+1}^t. \tag{17}$$

The contribution margin of satisfying demand $i+1$ with product i in period t is

$$C(y_{i+1,i}^t) = \alpha_{i+1,i}^t. \tag{18}$$

If one of the upgraded products could not be realized and be retained to the next period, the contribution margin of the last unit of these retained products becomes

$$C'(y_{i+1,i}^t) = \theta^{t+1}(x_1^{t+1}, x_2^{t+1}, \ldots, N_i^{t+1} - (y_{i+1,i}^t - 1), \ldots, x_n^{t+1}) - \theta^{t+1}(x_1^{t+1}, x_2^{t+1}, \ldots, N_i^{t+1} - y_{i+1,i}^t, \ldots, x_n^{t+1})$$

where $y_{i+1,i}^t \ge 1$.

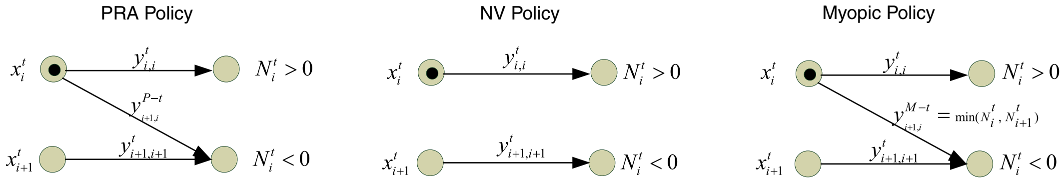

Fig. 3. Comparisons between PRA, NV and myopic allocation policies.

Because the retained products may not be allocated at period $t+1$, we have $C'(y^t_{i+1,i}) \leq \alpha^{t+1}_{i,j}$.

To enable the upgrading policy, the upgrading should bring more profits than retaining the products to the next period. Because $\theta^{t+1}(X^{t+1})$ is concave in X^{t+1} (see Lemma 1), $C'(y^t_{i+1,i})$ increases with $y^t_{i+1,i}$. Since $C(y^t_{i+1,i})$ is a constant parameter and $C'(y^t_{i+1,i})$ decreases with $y^t_{i+1,i}$, the values of $y^t_{i+1,i}$ that satisfy $C(y^t_{i+1,i}) \geq C'(y^t_{i+1,i})$ can be found. Thus, the maximal value of $y^t_{i+1,i}$ is the optimal upgrade quantity $y^{P-t*}_{i+1,i}$. However, if no $y^t_{i+1,i}$ satisfies inequality $C(y^t_{i+1,i}) \geq C'(y^t_{i+1,i})$, it means that the contribution margin of upgrading policy is always less than that of retaining policy to next period. So, the optimal upgrading quantity is zero.

In brief, if $N^t_i > 0$ and $N^t_{i+1} < 0$, the optimal upgrade quantity is

$$y^{P-t*}_{i+1,i} = \begin{cases} \max(y^t_{i+1,i}), & \text{if } C(y^t_{i+1,i}) \geq C'(y^t_{i+1,i}) \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

3.3. Lower bound of the input quantity

The PRA allocation policy is proved to be the optimal allocation policy at the allocation stage (see Section 3.2). Meanwhile, allocation decisions in each period depend on not only the current remaining inventories and demands but also the demands in the future periods. Thus, allocation decisions in each period can be made by using Eq. (19). However, since Eq. (19) takes the unrealized demands into considerations, the optimal upgrading quantities need a heavy burden computation.

There are two other similar and simple allocation policies: myopia allocation policy and newsboy allocation policy (myopia policy and NV policy, for simplicity). In a NV allocation policy, the products of one class can only be allowed to satisfy the demands of the same class at the allocation stage. Meanwhile, the myopic allocation policy at one period includes two steps. First, allocate the products of one class to satisfy the demands of the same class as much as possible. Second, if the products of one class are in stock and the products of the adjacent lower class are out of stock, then allocate the products of the higher class to satisfy the unmet demand of the adjacent lower class as much as possible. An illustrative example is shown in Fig. 3.

In Fig. 3, the upgrading quantity by myopia policy is $y^{M-t}_{i+1,i} = \min(N^t_i, -N^t_{i+1})$. Meanwhile, the upgrading quantity by the PRA allocation policy, $y^{P-t}_{i+1,i}$, is based on the present and future demands, so the corresponding numerical interval is $0 \leq y^{P-t}_{i+1,i} \leq \min(N^t_i, -N^t_{i+1})$. The upgrading quantity by myopic policy is no less than that by the PRA policy because of Eq. (17). Obviously, future demand information is not needed for NV and myopic policies. In other words, fewer computations are needed for these two policies than that for PRA allocation policy.

Lemma 2. *PRA policy is profitable than NV policy and myopic policy.*

Proof. Given a certain input quantity, some products are produced. After that, products are allocated to the periodic demands in T allocation periods. Let $\theta^{M-t}(X^t)$, $\theta^{P-t}(X^t)$ and $\theta^{\mathrm{NV}-t}(X^t)$ be the profits from period t to period T by myopic policy, PRA policy and NV policy, respectively.

Let Δ^t be the difference of upgrading quantity by myopia and by PRA policy at period t, and $\Delta^t = Y^{M-t}_{i+1,i} - Y^{P-t}_{i+1,i} \geq 0$. Because the contribution margin of upgrading is $\alpha^t_{i+1,i}$, the profit of Δ^t units of product i by myopia policy is $\sum \Delta^t \alpha^t_{i+1,i}$. The salvage costs of all products are zero, so we have $\theta^{M-T+1}(X^T) = \theta^{P-T+1}(X^T) = 0$. Because period T is the last period in the allocation stage, if product inventory is X^T at the beginning of period T, both the parallel allocation quantities and the upgrade allocation quantities by either myopic or PRA policies are the same: $\theta^{M-T}(X^T) = \theta^{P-T}(X^T)$. So the profit from period $T-1$ to period T by myopic policy is

$$\begin{aligned} \theta^{M-T-1}(X^{T-1}) &= \sum y^{T-1}_{i,i}\alpha^{T-1}_{i,i} + \sum (y^{P-T-1}_{i+1,i} + \Delta^{T-1})\alpha^{T-1}_{i+1,i} + \theta^{M-T}(X^T - \Delta^{T-1}) \\ &= \sum y^{T-1}_{i,i}\alpha^{T-1}_{i,i} + \sum y^{P-T-1}_{i+1,i}\alpha^{T-1}_{i+1,i} + \sum \Delta^{T-1}\alpha^{T-1}_{i+1,i} + \theta^{M-T}(X^T - \Delta^{T-1}) \\ &\leq \sum y^{T-1}_{i,i}\alpha^{T-1}_{i,i} + \sum y^{P-T-1}_{i+1,i}\alpha^{T-1}_{i+1,i} + \theta^{P-T}(X^T) - \theta^{P-T}(X^T - \Delta^{T-1}) + \theta^{M-T}(X^T - \Delta^{T-1}) \\ &= \sum y^{T-1}_{i,i}\alpha^{T-1}_{i,i} + \sum y^{P-T-1}_{i+1,i}\alpha^{T-1}_{i+1,i} + \theta^{P-T}(X^T) \\ &= \theta^{P-T-1}(X^{T-1}) \end{aligned} \tag{20}$$

Table 2
Parameters for policy comparisons.

Parameter	d_1^1	d_2^1	d_1^2	d_2^2	p_1	p_2	r_1	r_2
Distribution	$n(13, 20)$	$n(18, 23)$	$n(13, 20)$	$n(18, 23)$	$n(10, 30)$	$U[0, 1]p_1$	$U[0, 1]$	$U[0, 1]$
Parameter	u_1	u_2	η_1	η_2	v_1	v_2	c	
Distribution	$n(4, 18)$	$U[0, 1]u_1$	$B(24, 9)$	$U[0, 1]\eta_1$	$n(5, 12)$	$U[0, 1]v_1$	$n(2, 10)$	

where $\theta^{M-T-1}(X^{T-1})$ (see Eq. (20)) includes three parts: profit from the parallel satisfaction at period $T-1$, profit from the upgrading satisfaction at period $T-1$ and the profit obtained at period T.

Assuming that $\theta^{M-t+1}(X^{t+1}) \le \theta^{P-t+1}(X^{t+1})$, we then compare $\theta^{M-t}(X^t)$ with $\theta^{P-t}(X^t)$. That is

$$\begin{aligned}
\theta^{M-t}(X^t) &= \sum y_{i,i}^t \alpha_{i,i}^t + \sum (y_{i+1,i}^{P-t} + \Delta^t)\alpha_{i+1,i}^t + \theta^{M-t+1}(X^{t+1} - \Delta^t) \\
&= \sum y_{i,i}^t \alpha_{i,i}^t + \sum y_{i+1,i}^{P-t}\alpha_{i+1,i}^t + \sum \Delta^t \alpha_{i+1,i}^t + \theta^{M-t+1}(X^{t+1} - \Delta^t) \\
&\le \sum y_{i,i}^t \alpha_{i,i}^t + \sum y_{i+1,i}^{P-t}\alpha_{i+1,i}^t + [\theta^{P-t+1}(X^{t+1}) - \theta^{P-t+1}(X^{t+1} - \Delta^t)] + \theta^{M-t+1}(X^{t+1} - \Delta^t) \\
&\le \sum y_{i,i}^t \alpha_{i,i}^t + \sum y_{i+1,i}^{P-t}\alpha_{i+1,i}^t + \theta^{P-t+1}(X^T) \\
&= \theta^{P-t}(X^t).
\end{aligned} \tag{21}$$

Because of Eqs. (20) and (21), we have $\theta^{M-t}(X^t) \le \theta^{P-t}(X^t)$ for $\forall t \in \{1, 2, \dots, T\}$. Accordingly, we can obtain $\theta^{M-1}(X^1) \le \theta^{P-1}(X^1)$.

Meanwhile, no upgrading is allowed that allocated by NV policy. In Fig. 3, $y_{i+1,i}^{P-t}$ is the upgrading quantity from product i to satisfy demand $i+1$ with PRA policy. The upgrading quantity is based on the corresponding contribution margin (see Section 3.2). So let $\Delta Y^t = \{y_{2,1}^{P-t}, y_{3,2}^{P-t}, \dots, y_{n-1,n}^{P-t}\}$; the profit from period t to period T is

$$\begin{aligned}
\theta^{P-t}(X^t) &= \sum y_{i,i}^t \alpha_{i,i}^t + \sum y_{i+1,i}^{P-t}\alpha_{i+!,i}^t + \theta^{P-t+1}(X^{t+1}) \\
&\ge \sum y_{i,i}^t \alpha_{i,i}^t + [\theta^{P-t+1}(X^{t+1} + \Delta Y^t) - \theta^{P-t+1}(X^{t+1})] + \theta^{P-t+1}(X^{t+1}) \\
&= \sum y_{i,i}^t \alpha_{i,i}^t + \theta^{P-t+1}(X^{t+1} + \Delta Y^t) \\
&= \theta^{\mathrm{NV}-t}(X^t).
\end{aligned} \tag{22}$$

From Eq. (22), we can obtain $\theta^{P-1}(X^t) \ge \theta^{\mathrm{NV}-1}(X^t)$.

Because of $\theta^{P-1}(X^t) \ge \theta^{\mathrm{NV}-1}(X^t)$ and $\theta^{P-1}(X^1) \ge \theta^{M-1}(X^1)$, the profit by PRA allocation policy is higher than that by myopic and NV policies when a certain quantity of input is given. □

Observation 1. *The optimal input quantity by PRA policy is more than that by myopic.*

Model analysis shows that PRA policy is more profitable than myopic and NV policies due to its better flexibility. Let Q_0^{P*} be the optimal production input quantity generated by PRA policy, and Q_0^{M*} and $Q_0^{\mathrm{NV}*}$ be the optimal production input quantities generated by myopic and NV policies, respectively.

In the experiments with a single input, two outputs and two allocation periods, we suppose that each parameter (see Table 2) follows one type of random distribution. All demands follow the normal distributions truncated at 0 and rounded to the nearest integer. All the parameters are non-negative and are set based on the previous assumptions (see Section 3.1).

The values of parameters are the same in all comparable simulations when we perform the comparison, and then the corresponding optimal input quantities of the three allocation policies are calculated and compared. First, 50 scenarios are generated for prejudgment (Fig. 4). These simulations show that the optimal input quantity by NV policy is either more or less than that by myopic and PRA policies.

Prejudgment shows that the optimal input quantity by myopic policy is always less than that by PRA policy. To confirm this observation, more scenarios are generated to compare the optimal input quantities by myopic and PRA polices with the DOE method (design of experiment). A two-tailed test statistical experiment is designed, and its corresponding confidence level, test power and permissible error of the result are 0.95, 0.1 and 0.01, respectively. Totally 4000 scenarios are randomly generated, so that more extreme values and parameter combinations could be included in the experiment. The mean of the value $(Q_0^{P*} - Q_0^{M*})/Q_0^{P*}$ is 0.0317 and the standard variance is 0.0354, so the basic effective scenario size for the statistical evaluation is just 2554 by DOE theory [32]. In other words, adequate number of scenarios has been generated to effectively perform the experiment. Fig. 5 shows the values of difference rate $(Q_0^{P*} - Q_0^{M*})/Q_0^{P*}$ and difference value $Q_0^{P*} - Q_0^{M*}$ in each scenario.

All the difference values of $Q_0^{P*} - Q_0^{M*}$ are non-negative in the experiment (see Fig. 5). In other words, the optimal input quantity by PRA policy is always more than that by myopic policy. From the statistics collection of the scenarios, there are 3743 scenarios (about 93.57% among 4000 scenarios) that the difference rates $(Q_0^{P*} - Q_0^{M*})/Q_0^{P*}$ are included in the interval

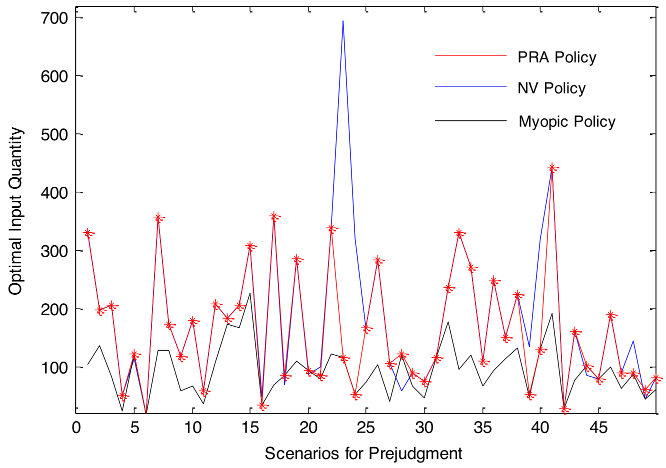

Fig. 4. Prejudgment on optimal input quantity by the three allocation policies.

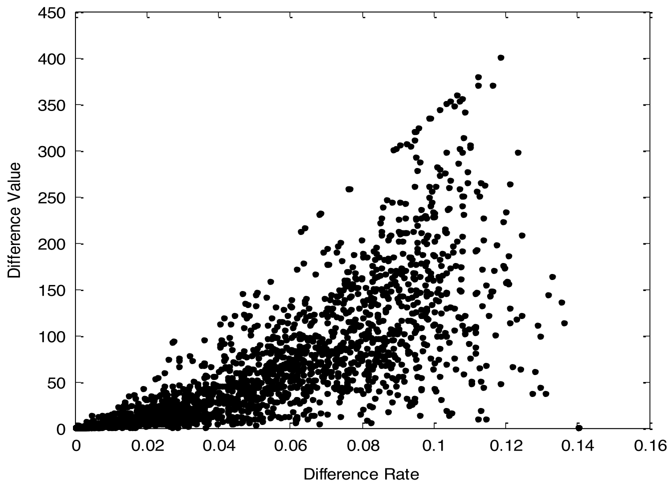

Fig. 5. Comparison on optimal input quantity by myopic and PRA policies.

[0, 0.1]. Then, the means and variances of all random parameters (see Table 2) are modified to perform more experiments. In all the experiments, the difference values of $Q_0^{P*} - Q_0^{M*}$ are non-negative, and among over 90% scenarios in each experiment, the values of $(Q_0^{P*} - Q_0^{M*})/Q_0^{P*}$ are less than 0.1. More experiments (in $1 \times 2 \times 3$, $1 \times 3 \times 3$ and $1 \times 4 \times 3$ systems) are performed to examine the above observation. We find that the optimal production input quantity by PRA allocation policy is always more than that by myopic allocation policy, and in most of scenarios the values of $(Q_0^{P*} - Q_0^{M*})/Q_0^{P*}$ are less than 0.1.

The objective function is a stochastic dynamic programming problem, and each allocation decision at each period is a dynamic programming model with all future demands as control variables. Although PRA is the optimal allocation policy of the make-to-stock system, finding the optimal production input quantity is a significant computational burden by this allocation policy. For example, given an initial production input quantity, at least n dynamic stochastic programs must be evaluated to obtain the optimal production input quantity for a $1 \times n \times T$ system. However, finding the optimal input quantity that allocated by myopic policy is much easier, because the future demands are not needed when an allocation decision is made. Since the production input quantity optimization by PRA policy has a heavy computational burden, Observation 1 becomes very useful to reduce the computation complexity for optimizing the production input quantity.

4. Model solution

4.1. Allocation decisions at the current period

Allocation decisions are made based on the current inventories, current demands and the future demands of each class of product by PRA policy. The allocation in each period becomes a transportation problem, and the framework of the corresponding transportation problem at period t is given in Table 3.

Table 3
Transportation framework at period t.

Current inventor y	Realized demands at period t				Expected demands in future periods								
	d_1^t	d_2^t	…	d_n^t	$E(d_1^{t+1})$	$E(d_2^{t+1})$	…	$E(d_n^{t+1})$	…	$E(d_1^T)$	$E(d_2^T)$	…	$E(d_n^T)$
x_1^t	$c_{1,1}^t$	$c_{2,1}^t$	…	$c_{n,1}^t$	$c_{1,1}^{t+1}$	$c_{2,1}^{t+1}$	…	$c_{n,1}^{t+1}$	…	$c_{1,1}^T$	$c_{2,1}^T$	…	$c_{n,1}^T$
x_2^t	$c_{1,2}^t$	$c_{2,2}^t$	…	$c_{n,2}^t$	$c_{1,2}^{t+1}$	$c_{2,2}^{t+1}$	…	$c_{n,2}^{t+1}$	…	$c_{1,2}^T$	$c_{2,2}^T$	…	$c_{n,2}^T$
…	…	…	…	…	…	…	…	…	…	…	…	…	…
x_n^t	$c_{1,n}^t$	$c_{2,n}^t$	…	$c_{n,n}^t$	$c_{1,n}^{t+1}$	$c_{2,n}^{t+1}$	…	$c_{n,n}^{t+1}$	…	$c_{1,n}^T$	$c_{2,n}^T$	…	$c_{n,n}^T$

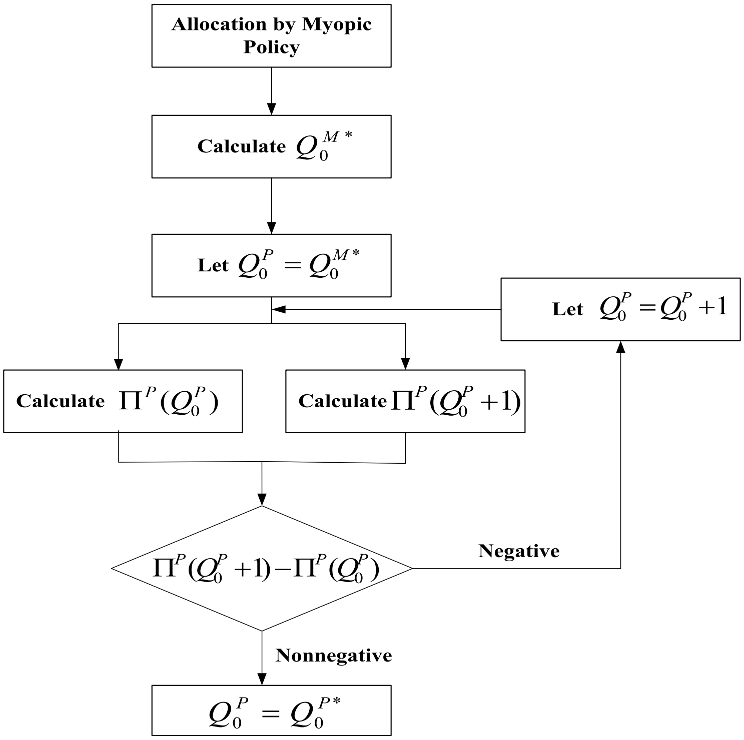

Fig. 6. Solution algorithm for the dynamic model.

Based on Table 3, the transport freight charge per unit product is

$$\begin{cases} c_{i,j}^t = +\infty & \text{for } \alpha_{i,j}^t < 0 \\ c_{i,j}^t = -\alpha_{i,j}^t & \text{for } \alpha_{i,j}^t \geq 0. \end{cases}$$

The optimal transport quantities $Y^t = \{y_{i,j}^t\}$ that allocated to demands $D^t = \{d_1^t, d_2^t, \ldots, d_2^t\}$ are the optimal allocation quantities in period t.

4.2. Searching algorithm for optimal input quantity

Since $\Pi(Q_0)$ is concave in Q_0 (based on Lemma 1), an optimal production input quantity Q_0^* exists. However, finding that the optimal production input quantity is a significant computational burden. So a searching algorithm is designed for the dynamic model. Since PRA is the optimal allocation policy, let $\Pi^P(Q_0)$ be the maximum profit of the system and Q_0^{p*} be the optimal input quantity. On the basis of Observation 1, we firstly allocate the products by myopic policy and obtain the optimal input Q_0^{M*}. Then, we take Q_0^{M*} as the initial value of Q_0^P and find the optimal input quantity by PRA policy with iterative operations. The flowchart of the proposed algorithm is as follows (see Fig. 6):

The basic procedure of the proposed algorithm is as follows:

Step 1. Solve the dynamic model by the myopic allocation policy and obtain the optimal input quantity Q_0^{M*}.

Step 2. Set the initial input quantity as Q_0^{M*}, that is, $Q_0^P = Q_0^{M*}$.

Step 3. Calculate $\Pi^P(Q_0^P)$ and $\Pi^P(Q_0^P + 1)$.

Step 4. If $\Pi^P(Q_0^P) \geq \Pi^P(Q_0^P + 1)$, then the optimal input quantity Q_0^{P*} is $Q_0^{P*} = Q_0^P$. Otherwise, let $Q_0^P = Q_0^P + 1$, then go to *step* 2.

Table 4
Calculations by the proposed algorithm.

Parameter	Value										
Q_0^P	93	94	95	96	97	98	99	100	101	102	103
$\Pi^P(Q_0^P)$	396.2128	397.1858	398.0670	398.8012	399.3899	399.8834	400.2531	400.5102	400.6517	400.7018	400.6528

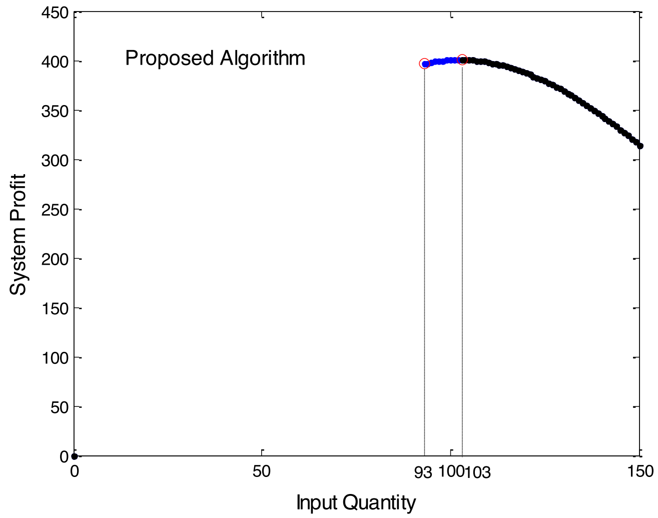

Fig. 7. Computational results by the proposed algorithm.

4.3. Numerical example

A numerical experiment with single input, two outputs and two allocation periods is implemented in order to illustrate the effectiveness of the proposed algorithm. The yield rate η_1 follows the beta distribution. All demands follow the normal distributions truncated at 0 and rounded to the nearest integer. The given parameters are as follows:

$$d_1^1 \sim n(18, 24), \quad d_1^2 \sim n(18, 24), \quad d_2^1 \sim n(12, 21),$$
$$d_2^2 \sim n(12, 21), \quad \eta_1 \sim B(5, 8), \quad \eta_2 = 1 - \eta_1,$$
$$p_1 = 8, \quad p_2 = 4, \quad v_1 = 5, \quad v_2 = 2, \quad c = 1,$$
$$r_1 = 0.24, \quad r_2 = 0.38, \quad u_1 = 1.5, \quad u_2 = 1.2.$$

We firstly compute the optimal production input quantity of the model that is allocated by myopic policy, then we can obtain the optimal value $Q_0^{M*} = 93$ and the corresponding profit is $\Pi^M(93) = 241.0178$. Taking $Q_0^P = Q_0^{M*} = 93$ as the initial value for the dynamic model, the corresponding initial profit by PRA policy is $\Pi^P(93) = 396.2128$. Then, the dynamic model is solved by the proposed algorithm.

The computations by the proposed algorithm are given in Table 4. Since $\Pi^P(Q^P = 102) > \Pi^P(Q^P = 101)$ and $\Pi^P(Q^P = 102) > \Pi^P(Q^P = 103)$, the optimal input quantity is $Q_0^{P*} = 102$. The maximum profit of the make-to-stock system is $\Pi^P(Q_0^{P*}) = \Pi(102) = 400.7018$. Fig. 7 shows that the new algorithm takes only 11 iterations to obtain the optimal production input quantity, while it will take 103 iterations (see Fig. 8) using the traditional search algorithm that takes zero as the initial value.

5. Conclusions

This paper is motivated by the high yield variability in semiconductor industry where the quality of the final products is uncertain and the products are graded into one of several quality levels according to their performances before being shipped. We study this dynamic multi-period yield management problem of a two-stage make-to-stock system with substitution faced by a semiconductor manufacturing firm. The objective is to determine the optimal production input quantity in order to maximize the firm's total profits. Demand can be classified into multiple classes corresponding to product levels and be upgraded when one type of product has been depleted. At the same time, products depreciate in allocation periods, which mainly results from technical progresses. Because the yield rate of each product level and the corresponding demand are random, the system can be modeled as a stochastic dynamic program. The PRA allocation policy is proved to be the optimal allocation policy, which states that satisfying as much as the parallel demands, then upgrade the demands by the one class higher product with the optimal quantities. We also show that the objective function of the

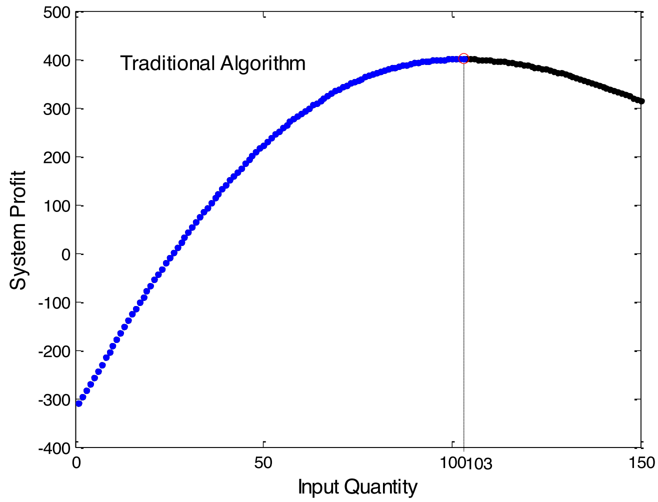

Fig. 8. Computation results by the traditional algorithm.

stochastic dynamic model is concave in production input quantity, and there exists an optimal production input quantity. The objective function of the stochastic dynamic model is proved to be concave in production input quantity. Two simple allocation policies (NV policy and myopic policy) are studied for comparative analysis with the PRA policy. Both model analysis and numerical experiments show that the optimal value of production input quantity by myopic policy is less than that by PRA policy and the difference rate is mostly less than 0.1. Based on these findings, a searching algorithm is designed for the dynamic model to reduce the computational burden. Both theoretical derivations and numerical experiments prove that the proposed algorithm requires much fewer computations and is effective to solve the proposed dynamic model.

Acknowledgements

The work presented in this paper has been supported by a grant from the National High-Tech Research and Development Program (863 Program) of China (2008AA04Z104) and a grant from the National Natural Science Foundation of China (70871077).

References

[1] V. Kothari, Silicon wafer manufacture, Unpublished Thesis, Sloan School of Management, MIT, Cambridge, Massachusetts, 1984.
[2] G.R. Bitran, D. Tirupati, Planning and scheduling for epitaxial wafer production facilities, Operations Research 36 (1) (1988) 34–49.
[3] B.A. Pasternack, Z. Drezner, Optimal inventory policies for substitutable commodities with stochastic demand, Naval Research Logistics 38 (2) (1991) 221–240.
[4] U. Karmarkar, S.C. Lin, Production planning with uncertain yields and demands, Working Paper, Simon Graduate School of Business Administration, University of Rochester, Rochester, NY, 1987.
[5] K. Moinzadeh, H.L. Lee, A continuous-review inventory model with constant re-supply time and defective items, Naval Research Logistics 34 (4) (1987) 457–467.
[6] H.L. Lee, C.A. Yano, Production control in multistage systems with variable yield losses, Operations Research 36 (2) (1988) 269–278.
[7] M. Henig, Y. Gerchak, The structure of periodic review policies in the presence of stochastic yield, Operations Research 38 (4) (1990) 634–643.
[8] Y. Gerchak, A. Tripathy, K. Wang, Co-production models with stochastic functionality yields, IIE Transactions 28 (5) (1996) 391–403.
[9] A. Hsu, Y. Bassok, Stochastic yield and stochastic demand in a production system with downward substitution, Operations Research 47 (2) (1999) 277–290.
[10] A.Y. Ha, Inventory rationing in a make-to-stock production system with several demand classes and lost sales, Management Science 43 (8) (1997) 1093–1103.
[11] A.Y. Ha, Stock rationing policy for a make-to-stock production system with two priority classes and backordering, Naval Research Logistics 44 (5) (1997) 457–472.
[12] A.Y. Ha, Stock rationing in an M/E/1 make-to-stock queue, Management Science 46 (1) (2000) 77–87.
[13] B. Tomlin, Y. Wang, Pricing and operational recourse in co-production systems, Management Science 54 (3) (2008) 522–537.
[14] J. Alstrup, S. Boas, O.B.G. Madsen, R.V.V. Vidal, Booking policy for flights with two types of passengers, European Journal of Operational Research 27 (3) (1986) 274–288.
[15] I. Karaesmen, G.J. van Ryzin, Overbooking with substitutable inventory classes, Operations Research 52 (1) (2004) 83–104.
[16] G.R. Bitran, S. Dasu, Ordering policies in an environment of stochastic yields and substitutable demands, Operations Research 40 (5) (1992) 999–1017.
[17] G.R. Bitran, T.Y. Leong, Deterministic approximations to co-production problems with service constraints and stochastic yields, Management Science 38 (5) (1992) 724–742.
[18] J.A. Van Mieghem, N. Rudi, Newsvendor networks: inventory management and inventory investment with discretionary activities, Manufacturing & Service Operations Management 4 (4) (2002) 313–335.
[19] U. Karmarkar, The multi-period multi-location inventory problem, Operations Research 29 (2) (1981) 215–228.
[20] L.W. Robinson, Optimal and approximate policies in multi-period, multi-product inventory models with transshipment, Operations Research 38 (2) (1990) 278–295.
[21] T.W. Archibald, S.A.E. Sassen, L.C. Thomas, An optimal policy for a two depot inventory problem with stock transfer, Management Science 43 (2) (1997) 173–183.

[22] K.C. Frank, R.Q. Zhang, I. Duenyas, Optimal inventory policies for inventory systems with priority demand classes, Operations Research 51 (6) (2003) 993–1002.
[23] S. Axsater, A new decision rule for lateral transshipments in inventory systems, Management Science 49 (9) (2003) 1168–1179.
[24] Y. Bassok, R. Anupindi, R. Akella, Single-period multi-product inventory models with substitution, Operations Research 47 (4) (1999) 632–642.
[25] R.A. Shumsky, F.Q. Zhang, Dynamic inventory management with substitution, Operations Research 57 (3) (2009) 671–684.
[26] B.H. Hall, The stock market's valuation of R&D investment during the 1980's, The American Economic Review 83 (2) (1993) 259–264.
[27] M.I. Nadiri, I. Prucha, Estimation of the depreciation rate of physical and R&D capital in the US total manufacturing sector, Economic Inquiry 34 (1) (1996) 43–56.
[28] H. Shane, M. Klock, The relation between patent citations and Tobin's Q in the semiconductor industry, Review of Quantitative Finance and Accounting 9 (2) (2010) 131–146.
[29] R. Van Slyke, R. Wets, Programming under uncertainty and stochastic optimal control, SIAM Journal on Control 4 (1) (1966) 179–193.
[30] R.T. Rockafellar, Convex Analysis, Princeton University Press, Princeton, New Jersey, 1970.
[31] W.C. Jordan, S.C. Graves, Principles on the benefit of manufacturing process flexibility, Management Science 41 (4) (1995) 577–594.
[32] R. Kuehl, Design of Experiments: Statistical Principles of Research Design and Analysis (Second Version), California, USA, Duxbury Press, 1999.