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Subclasses of harmonic mappings defined by convolution

Rosihan M. Ali [a], B. Adolf Stephen [b], K.G. Subramanian [c,*]

[a] *School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM Penang, Malaysia*

[b] *Department of Mathematics, Madras Christian College, Chennai 600 059, India*

[c] *School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM Penang, Malaysia*

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ABSTRACT

Two new subclasses of harmonic univalent functions defined by convolution are introduced. The subclasses generate a number of known subclasses of harmonic mappings, and thus provide a unified treatment in the study of these subclasses. Sufficient coefficient conditions are obtained that are shown to be also necessary when the analytic parts of the harmonic functions have negative coefficients. Growth estimates and extreme points are also determined.

1. Introduction

Harmonic mappings in a domain D of the complex plane are univalent complex-valued harmonic functions $f = u + iv$ where both u and v are real harmonic. These mappings are important in the study of minimal surfaces. Harmonic mappings have been investigated as generalizations of conformal mappings. The seminal works of Clunie and Sheil-Small [1] and Sheil-Small [2] showed that while certain classical results for conformal mappings have analogues for harmonic mappings, many other basic questions remain unsolved.

Every harmonic function f in a simply connected domain can be expressed in the form $f = h + \overline{g}$, where both h and g are analytic. The function h is called the analytic part while g is the co-analytic part of f. A necessary and sufficient condition [1] for f to be locally univalent and sense preserving in D is for $|g'(z)| < |h'(z)|$ in D.

Let S_H denote the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense preserving in the unit disk $U = \{z : |z| < 1\}$ and normalized by the conditions $h(0) = 0 = h'(0) - 1$, and $g(0) = 0$. Denote by S_H^0 the subclass of S_H for which $g'(0) = 0$. A function $f \in S_H$ belongs to the classical normalized class of univalent analytic functions S if the co-analytic part of f is zero.

Of late, various subclasses of S_H have been introduced and studied by several authors [3–11]. We shall show in this note that these subclasses are special cases of the general class $S_H^0(\phi, \sigma, \alpha)$ given in the following definition.

Definition 1.1. Let σ be a real constant and $\phi(z) = z + \sum_{n=2}^{\infty} \phi_n z^n$ be a given analytic function in U. A harmonic function $f = h + \overline{g} \in S_H^0$ where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=2}^{\infty} b_n z^n, \tag{1.1}$$

* Corresponding author.
E-mail addresses: rosihan@cs.usm.my (R.M. Ali), adolfmcc2003@yahoo.co.in (B.A. Stephen), kgs@usm.my, kgsmani1948@yahoo.com (K.G. Subramanian).

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belongs to the class $S_H^0(\phi, \sigma, \alpha)$ if

$$\Re\left\{\frac{z(h*\phi)'(z) - \sigma\overline{z(g*\phi)'(z)}}{(h*\phi)(z) + \sigma\overline{(g*\phi)(z)}}\right\} > \alpha, \quad (0 \le \alpha < 1; z \in U). \tag{1.2}$$

Here $*$ is the convolution operator.

Equivalently, with $F(z) = (\phi+\sigma\overline{\phi})*(h(z)+\overline{g(z)})$, the function $f \in S_H^0(\phi, \sigma, \alpha)$ provided $\frac{\partial}{\partial\theta}\arg(F(re^{i\theta})) \ge \alpha$ on $|z| = r$. Several subclasses of harmonic functions are special cases of the new class $S_H^0(\phi, \sigma, \alpha)$ for suitable choices of ϕ and σ. It is obvious that $S_H^0\left(\frac{z}{1-z}, 1, 0\right)$ and $K_H^0\left(\frac{z}{(1-z)^2}, -1, 0\right)$ are respectively the well-known classes S_H^{*0} of harmonic starlike and K_H^0 harmonic convex functions investigated by Silverman [10]. The general classes $S_H^0\left(\frac{z}{1-z}, 1, \alpha\right)$ and $K_H^0\left(\frac{z}{(1-z)^2}, -1, \alpha\right)$ coincide with $S_H^0(\alpha)$ and $K_H^0(\alpha)$ studied by Jahangiri in [7,6]. If $\sigma = (-1)^l$ and $\phi(z) = z + \sum_{n=2}^{\infty} n^l z^n$, then the class $S_H^0(\phi, \sigma, \alpha)$ reduces to the class $H(l, \alpha)$ involving the modified Salagean operator investigated in [8]. Another example is when $\sigma = 1$ and $\phi = \frac{z}{(1-z)^{\lambda+1}}$, $\lambda > -1$. In this case, $S_H^0(\phi, \sigma, \alpha)$ becomes the class $R_H(\lambda, \alpha)$ involving the Ruscheweyh derivative operator [9]. If $\sigma = (-1)^l$ and $\phi = z + \sum_{n=2}^{\infty} n^l C(\lambda, n) z^n$, where $C(\lambda, n) = \frac{(\lambda+1)_{n-1}}{(n-1)!}$, $(\lambda+1)_{n-1} = (\lambda+1)(\lambda+2)\cdots(\lambda+n-1)$, then $S_H^0(\phi, \sigma, \alpha)$ reduces to the class $M_H(l, \lambda, \alpha)$ recently investigated by Al-Shaqsi and Darus [4]. It is clear that the class $S_H^0(\phi, \sigma, \alpha)$ generates a number of known subclasses and thus provides a unified treatment of these subclasses of harmonic mappings.

In Section 2 of this note, a necessary and sufficient convolution condition is obtained for $S_H^0(\phi, \sigma, \alpha)$ and the class $SP_H^0(\phi, \sigma, \alpha)$. Sufficient coefficient conditions are obtained for these two classes, which in Section 3 will also be shown to be necessary when f has negative coefficients. Section 3 is also devoted to determining growth estimates and extreme points for the class $TS_H^0(\phi, \sigma, \alpha)$.

2. Main results

We now derive a convolution characterization for functions in the class $S_H^0(\phi, \sigma, \alpha)$.

Theorem 2.1. *Let $f = h + \overline{g} \in S_H^0$. Then $f \in S_H^0(\phi, \sigma, \alpha)$ if and only if*

$$(h*\phi) * \left[\frac{z + \frac{x+2\alpha-1}{2-2\alpha}z^2}{(1-z)^2}\right] - \sigma\overline{(g*\phi)} * \left[\frac{\frac{x+\alpha}{1-\alpha}\overline{z} - \frac{x+2\alpha-1}{2-2\alpha}\overline{z}^2}{(1-\overline{z})^2}\right] \ne 0, \quad |x| = 1, \ |z| \ne 0. \tag{2.1}$$

Proof. A necessary and sufficient condition for $f = h + \overline{g}$ to be in the class $S_H^0(\phi, \sigma, \alpha)$, with h and g of the form (1.1), is given by (1.2). Since

$$\frac{z(h*\phi)'(z) - \sigma\overline{z(g*\phi)'(z)}}{(h*\phi)(z) + \sigma\overline{(g*\phi)(z)}} = 1$$

at $z = 0$, the condition (1.2) is equivalent to

$$\frac{1}{(1-\alpha)}\left\{\frac{z(h*\phi)'(z) - \sigma\overline{z(g*\phi)'(z)}}{(h*\phi)(z) + \sigma\overline{(g*\phi)(z)}} - \alpha\right\} \ne \frac{x-1}{x+1}; \quad |x| = 1, \ x \ne -1, \ 0 < |z| < 1. \tag{2.2}$$

By a simple algebraic manipulation, (2.2) yields

$$\begin{aligned} 0 \ne\ & (x+1)\left[z(h*\phi)'(z) - \sigma\overline{z(g*\phi)'(z)}\right] - \alpha(x+1)\left[(h*\phi)(z) + \sigma\overline{(g*\phi)(z)}\right] \\ & - (x-1)(1-\alpha)\left[(h*\phi)(z) + \sigma\overline{(g*\phi)(z)}\right] \\ =\ & (h*\phi) * \left[\frac{2(1-\alpha)z + (x-1+2\alpha)z^2}{(1-z)^2}\right] - \sigma\overline{(g*\phi)} * \left[\overline{\frac{2(\overline{x}+\alpha)z - (\overline{x}+2\alpha-1)z^2}{(1-z)^2}}\right]. \end{aligned}$$

The latter condition together with (1.2) establishes the result (2.1) for all $|x| = 1$. □

Necessary coefficient conditions for the harmonic starlike functions and harmonic convex functions were obtained in [1] and [2]. Using the convolution characterization, we now derive a sufficient coefficient condition for harmonic functions to belong to the class $S_H^0(\phi, \sigma, \alpha)$.

Theorem 2.2. *Let $f = h + \overline{g} \in S_H^0$. Then $f \in S_H^0(\phi, \sigma, \alpha)$ if*

$$\sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha}|a_n|\,|\phi_n| + |\sigma| \sum_{n=2}^{\infty} \frac{n+\alpha}{1-\alpha}|b_n|\,|\phi_n| \le 1.$$

Proof. For h and g given by (1.1), (2.1) gives

$$\left|(h*\phi)*\left[\frac{z+\frac{x+2\alpha-1}{2-2\alpha}z^2}{(1-z)^2}\right] - \sigma\overline{(g*\phi)}*\left[\frac{\frac{x+\alpha}{1-\alpha}\overline{z}-\frac{x+2\alpha-1}{2-2\alpha}\overline{z}^2}{(1-\overline{z})^2}\right]\right|$$

$$= \left|z + \sum_{n=2}^{\infty}\left[n+(n-1)\frac{x+2\alpha-1}{2-2\alpha}\right]a_n\phi_n z^n - \sigma\sum_{n=2}^{\infty}\left[n\frac{x+\alpha}{1-\alpha}-(n-1)\frac{x+2\alpha-1}{2-2\alpha}\right]\overline{b_n\phi_n z^n}\right|$$

$$> |z|\left[1 - \sum_{n=2}^{\infty}\frac{n-\alpha}{1-\alpha}|a_n|\,|\phi_n| - |\sigma|\sum_{n=2}^{\infty}\frac{n+\alpha}{1-\alpha}|b_n|\,|\phi_n|\right].$$

The last expression is non-negative by hypothesis, and hence by Theorem 2.1, it follows that $f \in S_H^0(\phi, \sigma, \alpha)$. □

The sufficient coefficient conditions for the various classes $S_H^*(\alpha)$, $K_H(\alpha)$, $H(l, \alpha)$, $R_H(\lambda, \alpha)$ and $M_H(l, \lambda, \alpha)$ are all special cases of Theorem 2.2.

Another set of classes of harmonic functions introduced by several authors relates to the analytic univalent classes of uniformly convex functions and parabolic starlike functions. A survey of these functions can be found in [12]. Such subclasses of harmonic functions include the classes $G_H(\alpha)$ and $GK_H(\alpha)$ of Goodman–Rønning-type harmonic functions studied in [13,14] and the classes $RS_H(l, \gamma)$ [11] and $M_H(n, \alpha)$ [5] involving respectively the Salagean-type operator and Ruscheweyh operator. All these classes can again be given a unified treatment by considering the following class of functions.

Definition 2.1. Let σ be a real constant and $\phi(z) = z + \sum_{n=2}^{\infty} \phi_n z^n$ be a given analytic function in U. A harmonic function $f = h + \overline{g} \in S_H^0$ belongs to the class $SP_H^0(\phi, \sigma, \alpha)$ if

$$\Re\left\{(1+e^{i\gamma})\frac{z(h*\phi)'(z)-\sigma\overline{z(g*\phi)'(z)}}{(h*\phi)(z)+\sigma\overline{(g*\phi)(z)}} - e^{i\gamma}\right\} > \alpha, \quad (\gamma \text{ real},\ 0 \le \alpha < 1,\ z \in U). \tag{2.3}$$

Theorem 2.3. *Let $f = h + \overline{g} \in S_H^0$. Then $f \in SP_H^0(\phi, \sigma, \alpha)$ if and only if*

$$(h*\phi)*\left[\frac{z+\frac{(x+1)e^{i\gamma}+x+2\alpha-1}{2-2\alpha}z^2}{(1-z)^2}\right] - \sigma\overline{(g*\phi)}*\left[\frac{\frac{(x+1)e^{i\gamma}+x+\alpha}{1-\alpha}\overline{z}-\frac{(x+1)e^{i\gamma}+x+2\alpha-1}{2-2\alpha}\overline{z}^2}{(1-\overline{z})^2}\right] \ne 0, \quad |x| = 1,\ z \ne 0.$$

Proof. A necessary and sufficient condition for f in $SP_H^0(\phi, \sigma, \alpha)$, with h and g of the form (1.1), is given by (2.3). Since

$$(1+e^{i\gamma})\frac{z(h*\phi)'(z)-\sigma\overline{z(g*\phi)'(z)}}{(h*\phi)(z)+\sigma\overline{(g*\phi)(z)}} - e^{i\gamma} = 1$$

at $z = 0$, condition (2.3) is equivalent to

$$\frac{1}{(1-\alpha)}\left\{(1+e^{i\gamma})\frac{z(h*\phi)'-\sigma\overline{z(g*\phi)'}}{(h*\phi)+\sigma\overline{(g*\phi)}} - e^{i\gamma} - \alpha\right\} \ne \frac{x-1}{x+1}; \quad |x| = 1,\ x \ne -1,\ z \ne 0.$$

This now yields the desired result. □

Proceeding similarly to in Theorem 2.2, the following sufficient coefficient condition for the class $SP_H^0(\phi, \sigma, \alpha)$ is easily derived.

Theorem 2.4. *Let $f = h + \overline{g} \in S_H^0$. Then $f \in SP_H^0(\phi, \sigma, \alpha)$ if*

$$\sum_{n=2}^{\infty} \frac{2n-1-\alpha}{1-\alpha}|a_n|\,|\phi_n| + |\sigma| \sum_{n=2}^{\infty} \frac{2n+1+\alpha}{1-\alpha}|b_n|\,|\phi_n| \le 1.$$

3. The class $TS_H^0(\phi, \sigma, \alpha)$

Several subclasses of analytic functions with negative coefficients have been introduced and studied following the work of Silverman [15]. A unified class of analytic p-valent functions with negative coefficients defined by convolution was introduced in [16] that included many well-known subclasses of analytic functions with negative coefficients as special cases. In this section, we shall devote attention to the subclass $TS_H^0(\phi,\sigma,\alpha)$ of $S_H^0(\phi,\sigma,\alpha)$ consisting of harmonic functions $f = h+\bar{g}$ of the form

$$h(z) = z - \sum_{n=2}^{\infty} a_n z^n, \qquad g(z) = \sigma \sum_{n=2}^{\infty} b_n z^n, \quad a_n \geq 0,\ b_n \geq 0. \tag{3.1}$$

The subclass $TS_H^0(\phi,\sigma,\alpha)$ includes as special cases several subclasses investigated in [4,6–9].

Theorem 3.1. *Let $\phi(z) = z + \sum_{n=2}^{\infty} \phi_n z^n$ with $\phi_n \geq 0$ and f be of the form (3.1). Then $f \in TS_H^0(\phi,\sigma,\alpha)$ if and only if*

$$\sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha} a_n \phi_n + \sigma^2 \sum_{n=2}^{\infty} \frac{n+\alpha}{1-\alpha} b_n \phi_n \leq 1. \tag{3.2}$$

Proof. If f belongs to $TS_H^0(\phi,\sigma,\alpha)$, then (1.2) is equivalent to

$$\Re \left\{ \frac{(1-\alpha)z - \sum_{n=2}^{\infty}(n-\alpha)a_n\phi_n z^n - \sigma^2 \sum_{n=2}^{\infty}(n+\alpha)b_n\phi_n \bar{z}^n}{z - \sum_{n=2}^{\infty} a_n\phi_n z^n + \sigma^2 \sum_{n=2}^{\infty} b_n \phi_n \bar{z}^n} \right\} > 0$$

for $z \in U$. Letting $z \to 1^-$ through real values yields condition (3.2). Conversely, for h and g given by (3.1),

$$\begin{aligned} &\left| (h*\phi) * \left[\frac{z + \frac{x+2\alpha-1}{2-2\alpha} z^2}{(1-z)^2} \right] - \sigma \overline{(g*\phi)} * \left[\frac{\frac{x+\alpha}{1-\alpha}\bar{z} - \frac{x+2\alpha-1}{2-2\alpha}\bar{z}^2}{(1-\bar{z})^2} \right] \right| \\ &\quad > |z| \left[1 - \sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha} |a_n|\,|\phi_n| - \sigma^2 \sum_{n=2}^{\infty} \frac{n+\alpha}{1-\alpha} |b_n|\,|\phi_n| \right] \end{aligned}$$

which is non-negative by hypothesis, thus proving sufficiency of condition (3.2). □

We can obtain as a corollary sharp bounds for $|f(z)|$, for $f \in TS_H^0(\phi,\sigma,\alpha)$.

Corollary 3.1. *Let $\phi(z) = z + \sum_{n=2}^{\infty} \phi_n z^n$ with $\phi_n \geq \phi_2$ $(n \geq 2)$, and $|\sigma| \geq \frac{2-\alpha}{2+\alpha}$. If $f \in TS_H^0(\phi,\sigma,\alpha)$, then for $|z| = r < 1$,*

$$r - \frac{1-\alpha}{(2-\alpha)\phi_2} r^2 \leq |f(z)| \leq r + \frac{1-\alpha}{(2-\alpha)\phi_2} r^2.$$

The result is sharp with equality for $f(z) = z - \frac{1-\alpha}{(2-\alpha)\phi_2} z^2$.

Thus the range of functions in $TS_H^0(\phi,\sigma,\alpha)$ covers the disk $|w| < 1 - (1-\alpha)/[(2-\alpha)\phi_2]$.

It can also be seen that the class $TS_H^0(\phi,\sigma,\alpha)$ is convex.

We now determine the extreme points of the class $TS_H^0(\phi,\sigma,\alpha)$.

Theorem 3.2. *Let*

$$h_1(z) := z, \qquad h_n(z) := z - \frac{1-\alpha}{(n-\alpha)\phi_n} z^n, \quad \text{and} \quad g_n(z) := z + \frac{1-\alpha}{\sigma(n+\alpha)\phi_n} \bar{z}^n, \quad (n = 2, 3, \ldots).$$

A function $f \in TS_H^0(\phi,\sigma,\alpha)$ if and only if f can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} (\lambda_n h_n + \gamma_n g_n),$$

where $\lambda_n \geq 0$, $\gamma_n \geq 0$, $\lambda_1 = 1 - \sum_{n=2}^{\infty}(\lambda_n + \gamma_n)$, and $\gamma_1 = 0$. In particular, the extreme points of $TS_H^0(\phi,\sigma,\alpha)$ are $\{h_n\}$ and $\{g_n\}$.

Proof. Let

$$f(z) = \sum_{n=1}^{\infty} (\lambda_n h_n + \gamma_n g_n) = z - \sum_{n=2}^{\infty} \lambda_n \frac{1-\alpha}{(n-\alpha)\phi_n} z^n + \sigma \sum_{n=2}^{\infty} \gamma_n \frac{1-\alpha}{\sigma^2(n+\alpha)\phi_n} \overline{z^n}.$$

Since

$$\sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha} \lambda_n \frac{1-\alpha}{(n-\alpha)\phi_n} \phi_n + \sigma^2 \sum_{n=2}^{\infty} \frac{n+\alpha}{1-\alpha} \gamma_n \frac{1-\alpha}{\sigma^2 (n+\alpha)\phi_n} \phi_n = \sum_{n=2}^{\infty} (\lambda_n + \gamma_n) = 1 - \lambda_1 \leq 1,$$

it follows from Theorem 3.1 that $f \in TS_H^0(\phi, \sigma, \alpha)$.

Conversely, if $f \in TS_H^0(\phi, \sigma, \alpha)$, then $a_n \leq \frac{1-\alpha}{(n-\alpha)\phi_n}$ and $b_n \leq \frac{1-\alpha}{\sigma^2 (n+\alpha)\phi_n}$. Set

$$\lambda_n = \frac{n-\alpha}{1-\alpha} a_n \phi_n, \qquad \gamma_n = \frac{n+\alpha}{1-\alpha} b_n \phi_n \sigma^2, \qquad \lambda_1 = 1 - \sum_{n=2}^{\infty} (\lambda_n + \gamma_n), \quad \text{and} \quad \gamma_1 = 0.$$

Then

$$\sum_{n=1}^{\infty} (\lambda_n h_n + \gamma_n g_n) = z - \sum_{n=2}^{\infty} a_n z^n + \sigma \sum_{n=2}^{\infty} b_n \bar{z}^n = f(z). \quad \square$$

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References

[1] J. Clunie, T. Sheil-Small, Harmonic univalent functions, Ann. Acad. Sci. Fenn. Ser. AI Math. 9 (1984) 3–25.
[2] T. Sheil-Small, Constants for planar harmonic mappings, J. London Math. Soc. (2) 42 (2) (1990) 237–248.
[3] O.P. Ahuja, J.M. Jahangiri, H. Silverman, Convolutions for special classes of harmonic univalent functions, Appl. Math. Lett. 16 (6) (2003) 905–909.
[4] K. Al-Shaqsi, M. Darus, On harmonic functions defined by derivative operator, J. Inequal. Appl. (2008) 10 pp. Art. ID 263413.
[5] K. Al-Shaqsi, M. Darus, On Goodman–Ronning-type harmonic univalent functions defined by Ruscheweyh operator, Int. Math. Forum 3 (2008) 2161–2174.
[6] J.M. Jahangiri, Coefficient bounds and univalence criteria for harmonic functions with negative coefficients, Ann. Univ. Mariae Curie-Skłodowska Sect. A 52 (2) (1998) 57–66.
[7] J.M. Jahangiri, Harmonic functions starlike in the unit disk, J. Math. Anal. Appl. 235 (1999) 470–477.
[8] J.M. Jahangiri, G. Murugusundaramoorthy, K. Vijaya, Salagean-type harmonic univalent functions, Southwest J. Pure Appl. Math. (2) (2002) 77–82 (electronic).
[9] G. Murugusundaramoorthy, A class of Ruscheweyh-type harmonic univalent functions with varying arguments, Southwest J. Pure Appl. Math. (2) (2003) 90–95 (electronic).
[10] H. Silverman, Harmonic univalent functions with negative coefficients, J. Math. Anal. Appl. 220 (1) (1998) 283–289.
[11] S. Yalçin, M. Öztürk, M. Yamankaradeniz, On the subclass of Salagean-type harmonic univalent functions, JIPAM. J. Inequal. Pure Appl. Math. 8 (2) (2007) 9 pp. Article 54.
[12] F. Rønning, A survey on uniformly convex and uniformly starlike functions, Ann. Univ. Mariae Curie-Skłodowska Sect. A 47 (1993) 123–134.
[13] T. Rosy, B.Adolf Stephen, K.G. Subramanian, J.M. Jahangiri, Goodman–Rønning-type harmonic univalent functions, Kyungpook Math. J. 41 (1) (2001) 45–54.
[14] T. Rosy, B.Adolf Stephen, K.G. Subramanian, J.M. Jahangiri, Goodman-type harmonic convex functions, J. Natur. Geom. 21 (1–2) (2002) 39–50.
[15] H. Silverman, Univalent functions with negative coefficients, Proc. Amer. Math. Soc. 51 (1975) 109–116.
[16] R.M. Ali, M.H. Khan, V. Ravichandran, K.G. Subramanian, A class of multivalent functions with negative coefficients defined by convolution, Bull. Korean Math. Soc. 43 (1) (2006) 179–188.