A class of splines with constrained length

Miguel Antonio Jiménez-Pozo*, Raúl Linares-Gracia

Fac. Physics and Mathematics, Autonomous Univ. Puebla, Apartado Postal J 27, Col. San Manuel, Puebla, Pue. 72571, Mexico

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Abstract

Sometimes one needs to approximate a curve by means of splines that preserve the length of the given curve. This is the case, for instance, of the trajectory of an inclined oil-well, where the length of the path described by the trajectory between any two of its points can be measured by engineers. In a previous paper, using quadratic splines, the first author developed such a model for the petroleum industry and solved the corresponding problem of approximation. But the method employed there does not seem to be appropriate to deal with polynomials of higher degrees that appear when other parameters such as the continuity of the curvature need to be preserved in the models. In this work we introduce another method of focusing on the problem that is independent of the degree of the polynomials and that is simpler. A somewhat surprising result is that despite the quadratic splines, we lose uniqueness of solution in the general case.

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1. Introduction

Inclined oil-wells are designed following the particular conditions of grounds, economic aspects and other factors. A general description can be found in [2]. However, in practice, an inclined oil-well follows a spatial trajectory which also depends on random conditions. Then, a model to approximate the global shape of the well is needed during and after its drilling.

The modeling problem in hand consists in defining a curve whose graph approximates the oil-well trajectory \( \Gamma \) under the following assumptions. Consider to be given a system of rectangular three-dimensional coordinates \((x, y, z)\) whose origin coincides with the origin of the oil-well. Also assume the \(z\)-axis points to the theoretical center of the planet. Let \( P_0 = (0, 0, 0) \) and \( P_1, P_2, \ldots \) be a finite sequence of unknown consecutive points lying in \( \Gamma \) for which, measuring the angles of the vectors tangent to the path with the vectors defined by the gravity and the azimuth, engineers are able to determine the spherical coordinates of these unit tangent vectors. Also the total length of arc \( L_s \) of the path between \( P_s \) and \( P_{s+1} \) can be measured. Such a model can be defined by induction. In fact, once we have determined the coordinates of \( P_1, \ldots, P_s \) and defined the curve between \( P_0 \) and \( P_s \), we proceed to calculate \( P_{s+1} \) and to define the prolongation of the curve between \( P_s \) and \( P_{s+1} \), in such a way that we preserve the known derivatives at

* Corresponding author.
E-mail address: mjimenez@fcfm.buap.mx (M.A. Jiménez-Pozo).
$P_s$ and $P_{s+1}$, as well as the length $L_s$ of this piece of arc. That is, a class of splines with constrained length and free end knots. Since the total length of arc and the values of the derivatives are invariant under translation, the problem is simplified by translating the origin of the system of coordinates to the starting point in each step. This model for the oil-wells leads to the following problem:

To identify whether there exist and to calculate the real numbers $A, B, C, D$ and $z_s > 0$, such that the functions

$$ y(z) = Az^2 + Bz, \quad x(z) = Cz^2 + Dz, \quad z \in [0, z_s], $$  

(1)
satisfy preassigned values for $y'(0)$, $y'(z_s)$, $x'(0)$, $x'(z_s)$, and describe a three-dimensional curve with total length of arc $L := L_s$, given by

$$ L = \int_{z^*}^{z_s} \sqrt{y'(z)^2 + x'(z)^2 + 1} \, dz. $$  

(2)

After that, we evaluate (1) at $z^*$ to completely determine $P_{s+1} - P_s$ and finish the step by translating (1) to the interval $[P_s, P_{s+1}]$.

This problem is solved in [1] as follows: If $y'(z_s) - y'(0) = x'(z_s) - x'(0) = 0$, the piece of arc is linear. That is, $A = C = 0$ and there is not any trouble in obtaining $z_s$, $B$ and $D$. In another case, without loss of generality, suppose $y'(z_s) - y'(0) \neq 0$. Then we find the explicit solution of the problem by making the sophisticated change of variable

$$ v(z) = \frac{(1 + \alpha^2) y'(z) + \alpha \beta}{\sqrt{1 + \alpha^2 + \beta^2}}, $$

into the integral of Eq. (2), where $\alpha = \frac{x'(z_s) - x'(0)}{y'(z_s) - y'(0)}$ and $\beta = x'(0) - \alpha y'(0)$.

However, it seems to be impossible to find an explicit solution to the following problem in spite of its simple appearance:

When engineers design an ideal inclined oil-well, they suppose that the path has a continuous curvature and is contained in a plane. Further this situation leads the problem of calculating $A, B, C$ and $z_s$ such that the function

$$ y(z) = Az^2 + Bz^2 + Cz, \quad z \in [0, z_s], $$  

(3)
satisfies preassigned values of $y'(0)$, $y''(0)$, $y'(z_s)$ and describes a two-dimensional curve with total length of arc $L$. Here we simplified the model by assuming that the final path of the oil-well is indeed contained in a plane. In any case, such problems motivate the theorem of the next section.

For the moment, note that in this application, we must know $y''(0)$ (that is equal by definition to the second derivative at the point $P_s$ in the step determined by the consecutive points $P_s$ and $P_{s+1}$). Further, after obtaining a solution, we must evaluate $y''$ at $z_s$. The resulting value $y''(P_{s+1})$ will be assumed as $y''(0)$ for the next step of induction determined by $P_{s+1}$ and $P_{s+2}$.

The problem given by (3) was already considered by one of the authors in [3]. In that paper, the suggested method introduces a numerical error that is very small due to the particular case treated there. But such an approach should be incorrect in dealing with a rigorous proof of the general theorem given here. Other kinds of cubic splines with constrained length have been also considered in [4]. However, published papers on the present subject seem to be rather rare.

2. Main result

The goal of this work is to solve problems of the type described above by means of the following general result:

**Theorem 1.** For $i = 1, 2, \ldots, m$, fix integers $n_i \geq 1$, $p_i \geq 0$ and $q_i \geq 0$ such that $p_i + q_i = n_i$. Also fix real numbers $b_{i,k}$ and $c_{i,l}$, $1 \leq k := k(i) \leq p_i$, $1 \leq l := l(i) \leq q_i$ and $L > 0$. Then, there exist real numbers $a_{i,j}$, $1 \leq i \leq m$, $1 \leq j := j(i) \leq n_i$ and $z_s > 0$ such that the derivatives of the real-valued algebraic polynomials

$$ y_i(z) = \sum_{j=1}^{n_i} a_{i,j} z^j, \quad z \in [0, z_s], $$  

(4)
satisfy the constraints
\[ y_i^{(k)}(0) = b_{i,k} \quad \text{and} \quad y_i^{(l)}(z_*) = c_{i,l}, \quad i = 1, 2, \ldots, m, \ 1 \leq k \leq p_i, \ 1 \leq l \leq q_i, \tag{5} \]
while the total length of arc of the curve
\[ \Gamma(z_*) := \{(z, y_1(z), y_2(z), \ldots, y_m(z)) : z \in [0, z_*] \} \subset \mathbb{R}^{m+1} \tag{6} \]
is just \( L \).

**Proof.** For the time being, let \( u > 0 \) be a fixed real number. Given \( i, 1 \leq i \leq m, \) set
\[ x_i(t) := \sum_{j=1}^{n_i} h_{i,j} t^j, \quad z := ut, \quad t \in [0, 1], \tag{7} \]
where \( h_{i,j} \) are real numbers to be determined. By substituting \( t = z/u \) into (7), we obtain
\[ x_i(t(z)) := \sum_{j=1}^{n_i} \frac{h_{i,j}}{u^j} z^j. \tag{8} \]

We want to define \( y_i \) through \( x_i \). That is
\[ y_i(z) := x_i(t(z)) \quad \text{with} \quad a_{i,j} := \frac{h_{i,j}}{u^j}, \quad 1 \leq j \leq n_i. \]

Then, we must prove that for any \( u > 0 \), there is a set of values \( h_{i,j} := h_{i,j}(u), \ 1 \leq j \leq n_i, \) such that condition (5) of the theorem is fulfilled. That is
\[ \frac{d^k x_i}{dz^k}(z = 0) = b_{i,k}, \quad 1 \leq k \leq p_i \quad \text{and} \quad \frac{d^l x_i}{dz^l}(z = u) = c_{i,l}, \quad 1 \leq l \leq q_i. \tag{9} \]

Finally, we must choose a value \( u = z_* \) that satisfies the requirement on the length.

We shall treat here the case in which \( p_i \geq 1 \) and \( q_i \geq 1 \). The extreme cases in which one of them is zero (then the other is \( n_i \)) is a simple adaptation of the proof that follows.

Taking derivatives in (8) and evaluating at \( z = 0 \), we obtain
\[ \frac{d^k x_i}{dz^k}(z = 0) = k! \frac{h_{i,k}}{u^k}, \quad 1 \leq k \leq p_i, \]
and from (9), we must define \( h_{i,k} := b_{i,k} u^k \), \( 1 \leq k \leq p_i \). Setting \( g_{i,k} := g_{i,k}(u) := b_{i,k} \frac{u^{k-1}}{k!} \), we are able to express \( h_{i,k} \) as a multiplication by \( u \), i.e.,
\[ h_{i,j} = g_{i,j} u, \quad 1 \leq j \leq p_i. \tag{10} \]

Evaluating now the derivatives of \( x_i \) at \( z = u \), we must have
\[ \frac{d^l x_i}{dz^l}(z = u) = \sum_{s=l}^{n_i} \frac{s!}{(s-l)!} h_{i,s} = c_{i,l}, \quad 1 \leq l \leq q_i. \]
The numbers \( h_{i,s}, 1 \leq j \leq p_i, \) are known from (10). Then set
\[ A_l(u) := \sum_{s=l}^{p_i} \frac{s!}{(s-l)!} g_{i,s}, \quad 1 \leq l \leq p_i \text{ or equal 0 otherwise.} \]

We obtain the \( q_i \times q_i \) system of linear equations
\[ \sum_{s=p_i+1}^{n_i} \frac{s!}{(s-l)!} h_{i,s} = (c_{i,l} u^{l-1} - A_l(u)) u. \]
We will prove later that the determinant of the associated matrix \( M(p_i, q_i) \) of this system of linear equations is different from zero. Then, there exist numbers \( g_{i,s} = g_{i,s}(u), p + 1 \leq s \leq n_i \), that together with (10) lead to the definition

\[
h_{i,j} = g_{i,j}u, \quad 1 \leq j \leq n_i,
\]

and (9) holds.

Note that \( g_{i,j}(u) \) is continuous and uniformly bounded for \( u \in (0, L] \). Define \( \Gamma(u) \) by (6) with \( z \in [0, u] \). We need to find \( u = z*_z \in (0, L] \), for which length \( \Gamma(z*_z) = L \). By a known formula for the total length of arc,

\[
\text{length } \Gamma(u) = uI(u), \quad (11)
\]

where

\[
I(u) = \int_0^1 \left[ \sum_{i=1}^{m} \left[ \sum_{j=1}^{n_i} jg_{i,j}(u) t^{j-1} \right] \right]^2 + 1 \, dt,
\]

is continuous and uniformly bounded in \((0, L] \), due to the properties of each \( g_{i,j} \). Also, observe that \( I(u) \geq 1 \). By (11), length \( \Gamma(u) \) is a continuous function in \((0, L] \), such that length \( \Gamma(L) \geq L \), while length \( \Gamma(u) \rightarrow 0 \) whenever \( u \rightarrow 0 \). Then, by the Bolzano theorem, there exists \( u = z*_z \), which satisfies the assertion of the theorem.

It remains to prove that the determinant of the matrix \( M(p_i, q_i) \) is different from zero. For simplicity, eliminate the subscripts \( i \). For each \( k = 1, \ldots, q \), a factor \( p + k \) appears in the \( k \)th column of \( M(p, q) \). Then \( \text{Det} (M(p, q)) = (p + 1) \cdots (p + q) \text{Det} (N(p, q)) \), where the matrix \( N \) has the form

\[
\begin{bmatrix}
1 & \ldots & 1 & \ldots & 1 \\
p & \ldots & (p + k - 1) & \ldots & (p + q - 1) \\
& \ddots & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots \\
& & & \ddots & \ddots \\
p & \ldots & (p + k - 1) & \ldots & (p + q - 1) \\
p & \ldots & (p - j + 2) & \ldots & (p + q - 1) \\
p & \ldots & (p - j + 2) & \ldots & (p + q - 1) \\
p & \ldots & (p - j + 2) & \ldots & (p + q - 1) \\
p & \ldots & (p - j + 2) & \ldots & (p + q - 1)
\end{bmatrix}
\]

and \( j = 2, 3, \ldots, q \) enumerate the rows and \( k = 1, 2, \ldots, q \) the columns.

Note that the determinant of this matrix coincides with the Wronskian \( W(x^p, \ldots, x^{p+q-1}) \) evaluated at \( x = 1 \). Using the Leibniz formula for the \( r \)-th derivative of the product of two functions

\[
(\phi \psi)^{(r)}(x) = \sum_{s=0}^{r} \binom{r}{s} \phi^{(r-s)} \psi^{(s)}(x)
\]

and usual properties of the determinants, one easily deduces the formula

\[
W(\phi \psi_1, \ldots, \phi \psi_q)(x) = \phi^q(x) W(\psi_1, \ldots, \psi_q)(x).
\]

Applying this last formula with \( \phi(x) := x^p \) and \( \psi_k(x) := x^{k-1}, k = 1, \ldots, q \), and after evaluating at \( x = 0 \), we obtain the result

\[
\text{Det} (M(p, q)) = (p + 1) \cdots (p + q) \text{Det} (N(p, q)) = \prod_{k=1}^{q} k^{q-k}(p + k).
\]

**Remark 2.** Applying numerical methods, for instance, successive bisections of the interval \([0, L]\) with the help of the function \( uI(u) \), we find \( z*_z \) and the remaining coefficients \( h_{i,j} \).

### 3. Uniqueness of solution

In the quadratic case the situation is simple: Consider the particular case of (4), given by

\[
y_i(z) = A_iz^2 + B_iz, \quad z \in [0, z*_z], \ 1 \leq i \leq m.
\]
Following the steps of proof of the theorem, the particular parametric functions given in (7) have the form

\[ x_i(t) = \left( \frac{y_i'(z_*) - y_i'(0)}{2} z_* \right) t^2 + y_i'(0) z_* t, \quad z = z_* t, \quad t \in [0, 1]. \]

Set

\[ E_i^* := \frac{y_i'(z_*) - y_i'(0)}{2}, \quad F_i^* := y_i'(0), \quad 1 \leq i \leq m. \]

It follows from the formula for the total length of arc that

\[ z_* = \frac{L}{\int_0^1 \sqrt{\sum_{i=1}^{m} E_i^* t^2 + \sum_{i=1}^{m} E_i^* F_i^* t + \sum_{i=1}^{m} F_i^* + 1} \, dt}. \]

Therefore the new method is easier than the one employed in [1] and also leads to knowing that there is uniqueness of solution in the quadratic case.

Sometimes we cannot assert the uniqueness of solution for polynomials of higher degrees as the following example shows.

Consider the cubic problem given in (3). Using the same scheme of proof of the theorem, we deduce that the particular parametric function given in (8) has the form

\[ x(t) = \frac{(y'(z_*)) - y'(0)}{3} z_* t^3 + \frac{y''(0)}{2} z_* t^2 + y'(0) z_* t, \quad z = z_* t, \quad t \in [0, 1]. \]

The values \( z_* \) will be the roots in \([0, L]\) of the equation

\[ u I(u) - L = 0, \quad (12) \]

where

\[ I(u) = \int_0^1 \frac{((y'(z_*)) - y'(0)) t^2 + y'(0) + y''(0) u(t(1 - t)))^2 + 1} \, dt. \quad (13) \]

If \( y''(0) = 0 \), then \( I(u) \) is constant and consequently Eq. (12) has only one solution.

Exclude \( y''(0) = 0 \). Fix any set of values \( y'(0), y''(0) \) and \( y'(z_*). \) Observe that \( I(u) \) is a differentiable function at every \( u > 0, I(u) \geq 1, I(u) \rightarrow +\infty \) whenever \( u \rightarrow \infty \) and \( I(u) \) cannot have many local extrema. Then, Eq. (12) has only one solution if \( L \) is sufficiently large. Further, using that \( [u I(u)]' = u I'(u) + I(u) \), we also deduce that there is only one solution for \( L \) sufficiently small. However, in some particular cases it may happen that \( I(u) \) decreases on an interval. Then Eq. (12) may have different solutions for some particular values of \( L \). For instance, if

\[ y'(z_*) = y'(0) = -y''(0) = k, \]

for some constant \( k > 0 \), Eq. (13) becomes

\[ I(u) = k \int_0^1 \sqrt{1 - u(t(1 - t))})^2 + \frac{1}{k} \, dt = k \left( \int_0^1 |1 - u(t(1 - t))| \, dt + \xi_u(k) \right) \]

where \( \xi_u(k) \rightarrow 0 \) whenever \( k \rightarrow \infty \). Evaluating \( u I(u) \) at \( u = 3 \) and \( u = 4 \), we obtain

\[ 4I(4) - 3I(3) = k[-1/6 + (4\xi_4(k) - 3\xi_3(k))] \]

which is negative if \( k \) is large enough. Then there exist \( L > 0 \) and at least three points \( u_1 < u_2 < u_3 \) such that \( u_i I(u_i) - L = 0 \).

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