Measures of fuzzy compactness in $L$-fuzzy topological spaces

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**Abstract**

In this paper, the concept of fuzzy compactness degrees is presented in $L$-fuzzy topological spaces with the help of implication operator. Some properties of fuzzy compactness degrees are researched.

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1. Introduction

Since Chang [1] introduced the concept of compactness to $[0,1]$-topological space, many researchers have tried successfully to generalize the compactness theory of general topology to fuzzy setting (see [2–11]).

In an $L$-topology, open sets were fuzzy, but the topology comprising those open sets was a crisp subset of $L^X$. Fuzzification of openness was first initiated by Höhle [12] in 1980 and later developed to $L$-subsets of $L^X$ independently by Kubiak [13] and Šostak [14] in 1985. In 1991, Ying [15] studied Höhle's topology and called it fuzzifying topology.

There have been some works about fuzzy compactness in $L$-fuzzy topological spaces (see [16–24]). In the sense of [16–18,20–22], an $L$-fuzzy set is either fuzzy compact or not. Considering the measure of fuzzy compactness, the notion of compactness degrees was presented in $L$-fuzzy topological spaces [23,24] and furthermore it was investigated in [19, 25–27].

The aim of this paper is to present a new notion of fuzzy compactness degrees in $L$-fuzzy topological spaces with the help of implication operator “$\rightarrow$”. Some properties of fuzzy compactness degrees are researched.

2. Preliminaries

Throughout this paper, $(L, \vee, \wedge, \cdot)$ is a complete DeMorgan algebra (i.e., complete lattice with order-reversing involution) [28]. The smallest element and the largest element in $L$ are denoted by $\bot$ and $\top$, respectively.

We say that $a$ is wedge below $b$ in $L$, denoted by $a \prec b$, if for every subset $D \subseteq L$, $\bigvee \{ a \in L : a \prec b \}$ implies $d \geq a$ for some $d \in D$.

A complete lattice $L$ is completely distributive if and only if $b = \bigvee \{ a \in L : a \prec b \}$ for each $b \in L$. For any $b \in L$, define $\beta(b) = \{ a \in L : a \prec b \}$. Some properties of $\beta$ can be found in [29,30].

In a complete DeMorgan frame $L$, there exists a binary operation $\rightarrow$. Explicitly the implication is given by $a \rightarrow b = \bigvee \{ c \in L : a \land c \leq b \}$.
It is easy to check the following properties of $\mapsto$.

1. $(a \mapsto b) \geq c \iff a \land c \leq b$;
2. $a \mapsto b = \top \iff a \leq b$;
3. $a \mapsto (\land_i b_i) = \land_i (a \mapsto b_i)$;
4. $(\lor_i a_i) \mapsto b = \land_i (a_i \mapsto b)$.

We interpret $[a \leq b]$ as the degree to which $a \leq b$, i.e., $[a \leq b] = a \mapsto b$.

**Definition 2.1 ([12,31,14]).** An $L$-fuzzy topology on a set $X$ is a map $\tau : L^X \to L$ such that

1. $\tau(\top) = \tau(\bot) = \top$;
2. $\forall U, V \in L^X, \tau(U \lor V) \geq \tau(U) \land \tau(V)$;
3. $\forall U_j \in L^X, j \in J, \tau(\lor_{j \in J} U_j) \geq \land_{j \in J} \tau(U_j)$.

$\tau(U)$ can be interpreted as degree to which $U \in L^X$ is an open set; $\tau^*(U) = \tau(U')$ is called the degree of closedness of $U, \forall U \subseteq L^X, \tau(U') = \land_{A \in U} \tau(A)$ will be called the degree of openness of $U$. The pair $(X, \tau)$ is called an $L$-fuzzy topological space.

A map $f : (X, \tau) \to (Y, \delta)$ is called continuous with respect to $L$-fuzzy topologies $\tau$ and $\delta$ if $\tau(f^{-1}(U)) \geq \delta(U)$ holds for all $U \in L^Y$, where $f^{-1}(U)$ is defined by $f^{-1}(U)(x) = U(f(x)), x \in X$.

For a subfamily $\Phi \subseteq L^X, 2^{(\Phi)}$ denotes the set of all finite subfamilies of $\Phi$.

**Definition 2.2 ([23,24]).** An $L$-fuzzy inclusion on $X$ is a mapping $\tilde{\subset} : L^X \times L^X \to L$ defined by the equality $\tilde{\subset}(A, B) = \land_{x \in X} (A(x) \lor B(x))$.

In what follows, we shall write $[A \tilde{\subset} B]$ instead of $\tilde{\subset}(A, B)$.

**Definition 2.3 ([9]).** Let $(X, \tau)$ be an $L$-fuzzy topological space, $a \in L \setminus \{\bot\}$ and $G \in L^X$. A subfamily $U$ in $L^X$ is called a $Q_a$-cover of $G$ if $a \leq [G \tilde{\subset} \lor U]$.

**Lemma 2.4 ([28,9]).** Let $f : X \to Y$ be a set map. The fuzzy powerset operators $f^{-1}_L : L^X \to L^Y$ and $f^{-1}_L : L^Y \to L^X$ define by $f^{-1}_{L}(a)(y) = \lor\{a(x) : f(x) = y\}, f^{-1}_{L}(b) = b \circ f$. Then for any $\Phi \subseteq L^Y$, we have that

\[
\land_{y \in Y} \left( f^{-1}_L(G^y) \lor \lor_{B \in \Phi} B(y) \right) = \land_{x \in X} \left( (G^x) \lor \lor_{B \in \Phi} f^{-1}_L(B)(x) \right).
\]

3. Measures of fuzzy compactness

In order to generalize the notion of fuzzy compactness to $L$-fuzzy topological spaces, let us recall fuzzy compactness in $L$-topology, firstly.

Let $(X, \tau)$ be an $L$-topological space. $G \in L^X$ is fuzzy compact if and only if for every family $U \subseteq \tau$, it follows that $[G \tilde{\subset} \lor U] \leq \lor_{\forall \in \tau(U)} [G \tilde{\subset} \lor \forall]([9])$.

For $U \subseteq \tau$ we define $\chi_U = \top$, where $\chi_U(U) = \land_{A \in U} \chi_U(A), \chi_U(\top) = \bot, \forall \in \tau, \chi_U(\forall) = \bot, \forall \not\in \tau$ and

\[
[G \tilde{\subset} \lor U] \leq \lor_{\forall \in \tau(U)} [G \tilde{\subset} \lor \forall] \quad \forall \in \tau(U)
\]

Then $G \in L^X$ is fuzzy compact if and only if for every family $U \subseteq L^X$, it follows that $\tau(U) \subseteq [G \tilde{\subset} \lor U] \leq \lor_{\forall \in \tau(U)} [G \tilde{\subset} \lor \forall]$. Therefore we can naturally introduce the notion of fuzzy compactness degrees as follows:

**Definition 3.1.** Let $\tau : L^X \to L$ be a map. $\forall G \in L^X$, let

\[
\cd_{\tau}(G) = \land_{U \subseteq L^X} \left( \tau(U) \leq [G \tilde{\subset} \lor U] \leq \lor_{\forall \in \tau(U)} [G \tilde{\subset} \lor \forall] \right)
\]

\[
= \land_{U \subseteq L^X} \left( \tau(U) \mapsto [G \tilde{\subset} \lor U] \mapsto \lor_{\forall \in \tau(U)} [G \tilde{\subset} \lor \forall] \right)
\]

\[
= \land_{U \subseteq L^X} \left( \land_{A \in U} \tau(A) \mapsto \land_{x \in X} \left( (G^x) \lor \lor_{A \in U} A(x) \right) \mapsto \lor_{\forall \in \tau(U)} \land_{x \in X} \left( (G^x) \lor \lor_{A \in \forall} A(x) \right) \right)
\]

If $(X, \tau)$ be an $L$-fuzzy topological space, Then $\cd_{\tau}(G)$ is called the fuzzy compactness degree of $G$ with respect to $\tau$. Obviously, $G$ is fuzzy compact in $L$-topological space $\tau$ if and only if $\cd_{\tau}(G) = \top$. 
The following lemma is obvious by the properties of operation $\mapsto$.

**Lemma 3.2.** Let $(X, \tau)$ be an $L$-fuzzy topological space and $G \in L^X$. Then \( \text{cd}_z(G) \geq a \) if and only if for any $\mathcal{U} \subseteq L^X$,

\[
\tau(\mathcal{U}) \land \left[ \text{G} \land \mathcal{U} \right] \land a \leq \bigvee_{v \in 2^U} \left[ \text{G} \land v \right].
\]

By **Lemma 3.2** we can easily obtain the following result.

**Theorem 3.3.** Let $(X, \tau)$ be an $L$-fuzzy topological space and $G \in L^X$. Then

\[
\text{cd}_z(G) = \bigvee \left\{ a \in L : \tau(\mathcal{U}) \land \left[ \text{G} \land \mathcal{U} \right] \land a \leq \bigvee_{v \in 2^U} \left[ \text{G} \land v \right], \forall \mathcal{U} \subseteq L^X \right\}.
\]

**Remark 3.4.** In an $I$-topological space $(X, \mathcal{T})$, Šostak defined respectively the compactness spectrum $C(G)$ and the compactness degree $c(G)$ of a fuzzy subset $G$ as follows:

\[
C(G) = \left\{ b \in [0, 1] : b \leq \left[ \text{G} \land \mathcal{U} \right] \Rightarrow b \leq \bigvee_{v \in 2^U} \left[ \text{G} \land v \right], \forall \mathcal{U} \subseteq \mathcal{T} \right\},
\]

\[
c(G) = \inf([0, 1] \setminus C(G)) (\inf \emptyset = 1).
\]

By **Theorem 3.3** we know that $c(G) \leq \text{cd}_z(G)$ in $I$-topological space $(X, \mathcal{T})$. But in general, $c(G) \neq \text{cd}_z(G)$. This can be seen from the following example.

Let $X = [a, b]$ be a closed interval and let $\mathcal{T} = \{ \top \} \cup \{ 0.5 \land \chi_x : x \in \delta \}$, where $\delta$ denotes the natural topology on $X$. It is easy to check that $c(0.5 \land \chi_x) = 0.5 < 1 = \text{cd}_z(0.5 \land \chi_x)$. Moreover $c(\top) = 0.5 < 1 = \text{cd}_z(\top)$.

In order to write simple, “for any $\mathcal{U} \subseteq L^X$” will be omitted in the following description under the circumstances without causing confusion.

**Theorem 3.5.** Let $(X, \tau)$ be an $L$-fuzzy topological space. Then $\forall G, H \in L^X$, $\text{cd}_z(G \lor H) \geq \text{cd}_z(G) \land \text{cd}_z(H)$.

**Proof.** By **Theorem 3.3** we have

\[
\text{cd}_z(G \lor H) = \bigvee \left\{ a \in L : \tau(\mathcal{U}) \land \left[ (G \lor H) \land \mathcal{U} \right] \land a \leq \bigvee_{v \in 2^U} \left[ (G \lor H) \land v \right], \forall \mathcal{U} \subseteq L^X \right\}
\]

\[
= \bigvee \left\{ a \in L : \tau(\mathcal{U}) \land \left[ G \land \mathcal{U} \right] \land \left[ H \land \mathcal{U} \right] \land a \leq \bigvee_{v \in 2^U} \left[ G \land v \right] \land \left[ H \land v \right], \forall \mathcal{U} \subseteq L^X \right\}
\]

\[
\geq \bigvee \left\{ a \in L : \tau(\mathcal{U}) \land \left[ G \land \mathcal{U} \right] \land a \leq \bigvee_{v \in 2^U} \left[ G \land v \right], \forall \mathcal{U} \subseteq L^X \right\}
\]

\[
= \text{cd}_z(G) \land \text{cd}_z(H).
\]

**Theorem 3.6.** Let $(X, \tau)$ be an $L$-fuzzy topological space. Then $\forall G, H \in L^X$, $\text{cd}_z(G \land H) \geq \text{cd}_z(G) \land \tau^*(H)$.

**Proof.** By **Theorem 3.3** we have

\[
\text{cd}_z(G \land H) = \bigvee \left\{ a \in L : \tau(\mathcal{U}) \land \left[ (G \land H) \land \mathcal{U} \right] \land a \leq \bigvee_{v \in 2^U} \left[ (G \land H) \land v \right], \forall \mathcal{U} \subseteq L^X \right\}
\]

\[
= \bigvee \left\{ a \in L : \tau(\mathcal{U}) \land \left[ GH \land \mathcal{U} \right] \land a \leq \bigvee_{v \in 2^U} \left[ GH \land v \right], \forall \mathcal{U} \subseteq L^X \right\}
\]

\[
\geq \bigvee \left\{ a \land \tau^*(H) : \tau(\mathcal{U}) \land \left[ G \land \mathcal{U} \right] \land a \leq \bigvee_{v \in 2^U} \left[ G \land v \right], \forall \mathcal{U} \subseteq L^X \right\}
\]

\[
= \text{cd}_z(G) \land \tau^*(H).
\]

**Corollary 3.7.** Let $(X, \tau)$ be an $L$-fuzzy topological spaces. Then $\forall G \in L^X$, $\text{cd}_z(G) \geq \text{cd}_z(\top) \land \tau^*(G)$. 

Theorem 3.8. Let \( \tau_1, \tau_2 : L^X \rightarrow L \) be two maps and satisfy \( \tau_2 \leq \tau_1 \). Then \( \forall G \in L^X, cd_{\tau_1}(G) \leq cd_{\tau_2}(G) \).

Proof. It is straightforward by Theorem 3.3. \( \square \)

Corollary 3.9. Let \((X, \tau_1), (X, \tau_2)\) be two \(L\)-fuzzy topological spaces and satisfy \( \tau_2 \leq \tau_1 \). Then \( \forall G \in L^X, cd_{\tau_1}(G) \leq cd_{\tau_2}(G) \).

Corollary 3.10. Let \((X, \tau)\) be an \(L\)-fuzzy topological spaces and let \(B\) be a base or subbase \([32–34]\) of \(\tau\). Then \( \forall G \in L^X, cd_{\tau}(G) \leq \frac{1}{2} \).

Theorem 3.11. Let \( f : X \rightarrow Y \) be a set map, \( \tau_1 \) be an \(L\)-fuzzy topology on \(X\), \( \tau_2 \) be an \(L\)-fuzzy topology on \(Y\), and \( f : (X, \tau_1) \rightarrow (Y, \tau_2) \) be continuous. Then \( cd_{\tau_2}(f^{-1}(G)) \geq cd_{\tau_1}(G) \).

Proof. This can be proved from the following inequality.

\[
\begin{align*}
\text{cd}_{\tau_2}(f^{-1}_L(G)) &= \bigvee \left\{ a \in L : \tau_2(\mathcal{U}) \land \left[ f^{-1}_L(G) \subset \bigvee \mathcal{V} \right] \land a \leq \bigvee_{\mathcal{V} \in \mathcal{L}(U)} \left[ f^{-1}_L(G) \subset \bigvee \mathcal{V} \right], \forall \mathcal{U} \subseteq L^X \right\} \\
&\geq \bigvee \left\{ a \in L : \tau_1(f^{-1}_L(\mathcal{U})) \land \left[ G \subset \bigvee f^{-1}_L(\mathcal{U}) \right] \land a \leq \bigvee_{\mathcal{V} \in \mathcal{L}(U)} \left[ G \subset \bigvee f^{-1}_L(\mathcal{V}) \right], \forall \mathcal{U} \subseteq L^X \right\} \\
&\geq \bigvee \left\{ a \in L : \tau_1(\mathcal{R}) \land \left[ G \subset \bigvee \mathcal{R} \right] \land a \leq \bigvee_{\mathcal{V} \in \mathcal{L}(U)} \left[ G \subset \bigvee \mathcal{V} \right], \forall \mathcal{R} \subseteq L^X \right\} \\
&= \text{cd}_{\tau_1}(G). \quad \square
\end{align*}
\]

4. The generalized Tychonoff theorem

In this section, we suppose that \(L\) be completely distributive.

Let \( \tau : L^X \rightarrow L \) be a map and \( a \in L \). We define \( \tau_a = \{ A \in L^X : \tau(A) \geq a \} \). If \((X, \tau)\) be an \(L\)-fuzzy topological space, then \( \forall a \in L, \tau_a \) is a \(L\)-topology.

Lemma 4.1. Let \((X, \tau)\) be an \(L\)-fuzzy topological space and \(G \in L^X\). Then \( cd_{\tau}(G) \geq a \) if and only if \( \forall b \leq a, \forall r \in \beta(b) \), each \(Q_b\)-cover \(\mathcal{U} \subseteq \tau_b \) of \(G\) has a finite subfamily \(\mathcal{V}\) which is a \(Q_r\)-cover of \(G\).

Proof. (Necessity) Suppose \( cd_{\tau}(G) \geq a \). Then \( \forall \mathcal{U} \subseteq L^X, \tau(\mathcal{U}) \land \left[ G \subset \bigvee \mathcal{V} \right] \land a \leq \bigvee_{\mathcal{V} \in \mathcal{L}(U)} \left[ G \subset \bigvee \mathcal{V} \right] \) by Lemma 3.2.

\( \forall b \leq a, \forall r \in \beta(b) \), if \( \forall \mathcal{U} \subseteq \tau_b \) is a \(Q_b\)-cover of \(G\), then \( b \leq \tau(\mathcal{U}) \) and \( b \leq \bigvee_{\mathcal{V} \in \mathcal{L}(U)} \left[ G \subset \bigvee \mathcal{V} \right] \). Hence \( b \leq \tau(\mathcal{U}) \land \left[ G \subset \bigvee \mathcal{V} \right] \land a \).

We have that \( r \in \beta(b) \) if and only if \( \beta(b) \) is a \(Q_r\)-cover of \(G\).

(Sufficiency) Suppose \( \forall b \leq a, \forall r \in \beta(b) \), each \(Q_b\)-cover \(\mathcal{U} \subseteq \tau_b \) of \(G\) has a finite subfamily \(\mathcal{V}\) which is a \(Q_r\)-cover of \(G\).

\( \forall \mathcal{U} \subseteq L^X, \forall r \in M(L), \forall f \in \beta(r) \), if \( \forall b \leq \tau(\mathcal{U}) \land \left[ G \subset \bigvee \mathcal{V} \right] \land a \), then there exists \( b \in \beta(\tau(\mathcal{U}) \land \left[ G \subset \bigvee \mathcal{V} \right] \land a) \) such \( r = \beta(b) \). If \( r \leq b \), \( \mathcal{U} \) has a finite subfamily \(\mathcal{V}\) which is a \(Q_r\)-cover of \(G\).

We have that \( r \leq a, \tau(\mathcal{U}) \geq b \) and \( b \leq \bigvee_{\mathcal{V} \in \mathcal{L}(U)} \left[ G \subset \bigvee \mathcal{V} \right] \), i.e., \( \forall \mathcal{U} \subseteq \tau_b \) is a \(Q_b\)-cover of \(G\). Hence \( \mathcal{U} \) has a finite subfamily \(\mathcal{V}\) which is a \(Q_r\)-cover of \(G\). We can obtain that \( r \leq \bigvee_{\mathcal{V} \in \mathcal{L}(U)} \left[ G \subset \bigvee \mathcal{V} \right] \). Therefore \( \tau(\mathcal{U}) \land \left[ G \subset \bigvee \mathcal{V} \right] \land a \leq \bigvee_{\mathcal{V} \in \mathcal{L}(U)} \left[ G \subset \bigvee \mathcal{V} \right] \). Thus \( cd_{\tau}(G) \geq a \). \( \square \)

Lemma 4.2. Let \((X, \tau)\) be an \(L\)-fuzzy topological space and let \(\phi\) be a subbase of \(\tau\). Then

\[
\text{cd}_{\tau}(G) = \bigvee \left\{ a \in L : \phi(\mathcal{U}) \land \left[ G \subset \bigvee \mathcal{V} \right] \land a \leq \bigvee_{\mathcal{V} \in \mathcal{L}(U)} \left[ G \subset \bigvee \mathcal{V} \right] \right\}.
\]

Proof. \( cd_{\tau}(G) \leq \bigvee \left\{ a \in L : \phi(\mathcal{U}) \land \left[ G \subset \bigvee \mathcal{V} \right] \land a \leq \bigvee_{\mathcal{V} \in \mathcal{L}(U)} \left[ G \subset \bigvee \mathcal{V} \right] \right\} \) is obvious by Corollary 3.10.

We only need to prove that

\[
\text{cd}_{\tau}(G) \geq \bigvee \left\{ a \in L : \phi(\mathcal{U}) \land \left[ G \subset \bigvee \mathcal{V} \right] \land a \leq \bigvee_{\mathcal{V} \in \mathcal{L}(U)} \left[ G \subset \bigvee \mathcal{V} \right] \right\}
\]

Suppose \( \forall \mathcal{U} \subseteq L^X, \phi(\mathcal{U}) \land \left[ G \subset \bigvee \mathcal{V} \right] \land a \leq \bigvee_{\mathcal{V} \in \mathcal{L}(U)} \left[ G \subset \bigvee \mathcal{V} \right] \).

\( \forall b \leq a, \forall r \in \beta(b) \), if \( \forall \mathcal{U} \subseteq \phi(\mathcal{U}) \) is a \(Q_b\)-cover of \(G\), then \( \phi(\mathcal{U}) \geq b \) and \( b \leq \bigvee_{\mathcal{V} \in \mathcal{L}(U)} \left[ G \subset \bigvee \mathcal{V} \right] \). We have that \( r \in \beta(\mathcal{U}) \land \left[ G \subset \bigvee \mathcal{V} \right] \land a \) is a \(Q_r\)-cover of \(G\).

Now, we need to prove that \( \forall b \leq a, \forall r \in \beta(b) \), each \(Q_b\)-cover \(\mathcal{U} \subseteq \tau_b \) of \(G\) has a finite subfamily \(\mathcal{V}\) which is a \(Q_r\)-cover of \(G\).
Suppose that there exist \( b \leq a, r \in \beta(b) \) and \( U \subseteq \tau_b \) such that \( U \) is a \( Q_b \)-cover of \( G \), but \( U \) does not have any subfamily is \( Q_r \)-cover of \( G \). Let
\[
\Gamma = \{ P : U \subseteq P \subseteq \tau_b, P \) is a \( Q_b \)-cover of \( G \) and \( P \) does not have any subfamily is \( Q_r \)-cover of \( G \}\)

Then \((\Gamma, \subseteq)\) is a nonempty partially ordered set and each chain has an upper bound, hence by Zorn’s Lemma, \( \Gamma \) has a maximal element \( \Omega \). Now we prove that \( \Omega \) satisfies the following conditions:

(i) \( \Omega \subseteq \tau_b, \Omega \) is a \( Q_b \)-cover of \( G \), but \( \Omega \) does not have any subfamily is \( Q_r \)-cover of \( G \);

(ii) \( B \in L^X, \tau(B) \geq b, \) if \( C \subseteq \Omega \) and \( C \geq B, \) then \( B \in \Omega \);

(iii) \( B, C \in L^X, \tau(B) \geq b, \tau(C) \geq b, \) if \( B \cap C \in \Omega \), then \( B \in \Omega \) or \( C \in \Omega \).

We only verify (iii). If \( B \not\in \Omega \) and \( C \not\in \Omega \), then \( B \cup C \not\in \Gamma \) and \( C \cup B \not\in \Gamma \). This implies that there exist finite subfamily \( V_1 \cup V_2 \subseteq \Omega \) such that \( V_1 \subseteq \{ B \} \) and \( V_2 \subseteq \{ C \} \) are \( Q_b \)-cover of \( G \), i.e., \( r \leq \left[ G \cap \left( V_1 \cup \{ B \} \right) \right] \) and \( r \leq \left[ G \cap \left( V_2 \cup \{ C \} \right) \right] \). We can observe that \( W = V_1 \cup V_2 \). Then \( r \leq \left[ G \cap \left( W \cup \{ B \} \right) \right] \) and \( r \leq \left[ G \cap \left( W \cup \{ C \} \right) \right] \). Thus \( r \leq \left[ G \cap \left( W \cup \{ B \land C \} \right) \right] \). We can obtain that \( \Omega \) has a finite subfamily \( W \) which is \( Q_b \)-cover of \( G \) by \( B \land C \in \Omega \). This contradicts (i), (iii) is proved.

From (ii) and (iii), it is immediate that if \( D \in \Omega, P_1, P_2, \ldots, P_n \in \tau_b \) and \( D \supseteq P_1 \cup P_2 \cup \cdots \cup P_n \), then there exists \( i(1 \leq i \leq n) \) such that \( P_i \in \Omega \).

Since \( r \in \beta(b) \), there exists \( s \in \beta(B) \) and \( r \in \beta(s) \). Let \( \Psi = \{ A \subseteq \Omega : \phi(A) \geq s \} \). If \( \Psi \) is a \( Q_b \)-cover of \( G \), then \( \Psi \) has a finite subfamily is \( Q_r \)-cover of \( G \), this contradicts (i). Therefore we have \( b \not\subseteq \left[ G \cap \Psi \right] \). This implies that there exists an \( x \in X \), such that for any \( A \subseteq \Psi, r \not\subseteq \left( G \cap \Psi \right) \times A(x) \).

By (i), we know that \( b \leq \tau(\Omega) \) and \( b \subseteq \left( G \cap \Omega \right) \times x \). Hence there exists \( D \in \Omega \) such that \( r \subseteq \left( G \cap \Omega \right) \times D(x) \).

Now we prove that \( \forall \in \beta(b), \phi_\in \{ B \subseteq L^X, \phi(B) \geq t \} \) is a subbase of \( \tau_b \).

\( \forall \in \beta(b), \forall \in \tau_b \), we have that
\[
t \in \beta(b) \subseteq \beta(\tau(B)) = \beta \left( \bigvee_{\lambda \in A} V_{\lambda} \bigwedge_{\lambda \in A} \bigvee_{\beta \in \Lambda} \phi(W_{\lambda, \beta}^B) \right) \subseteq \bigcup_{\lambda \in A} \bigwedge_{\lambda \in A} \bigvee_{\beta \in \Lambda} \beta(\phi(W_{\lambda, \beta}^B)).
\]

Hence there exist \( \{ V_{\lambda} \}_{\lambda \in A} \) such that
\[
(\mathcal{B}_1) \quad V_{\lambda} = B;
\]
\[
(\mathcal{B}_2) \quad \text{for each } \lambda \in A, \text{there exists } \{ W_{\lambda, \beta}^B \}_{\beta \in \Lambda} \text{ satisfying } \left( \bigwedge_{\lambda \in A} \bigvee_{\beta \in \Lambda} W_{\lambda, \beta}^B = V_{\lambda} \right);
\]
\[
(\mathcal{B}_3) \quad \text{for each } \beta \in \Lambda, t \in \beta(\phi(W_{\lambda, \beta}^B)),
\]

i.e., there exists \( \left( W_{\lambda, \beta}^B \right)_{\lambda \in A, \beta \in \Lambda} \subseteq \phi_\in, \) such that \( V_{\lambda} = \left( W_{\lambda, \beta}^B \right)_{\lambda \in A, \beta \in \Lambda} \).

Let \( D = \bigvee_{\lambda \in A} \Lambda_\lambda, A_\lambda, \text{ where } A_\lambda \in \phi_\in. \) Then there exists \( i \in I \) such that
\[
r \in \beta \left( G \cap \bigwedge_{j \in I} A_j(x) \right) \subseteq \bigcap_{j \in I} (G \cap A_j(x)).
\]

This implies that \( r \subseteq \left( G \cap A_j(x) \right) \) for each \( j \in I \). By \( D \supseteq \bigwedge_{j \in I} A_j \) we know that there is \( j \in I \) such that \( A_j \in \Omega \), this contradicts \( r \not\subseteq \left( G \cap A_j(x) \right) \).

By Lemma 4.1 the proof is obtained.

**Theorem 4.3.** Let \( (X, \tau) \) be the product \( L \)-fuzzy topological space of \( \{ (X_j, \tau_j) \}_{j \in J} \). \( \forall G = \bigwedge_{j \in J} G_j \in L^{\prod \tau_j}, \text{cd}_\tau(G) \geq \bigwedge_{j \in J} \text{cd}_{\tau_j}(G_j) \), where \( G_j \in L^{\tau_j} \) for any \( j \in J \).

**Proof.** Let \( \phi : L^X \to L, \phi(A) = \bigvee_{j \in J} \bigvee_{p_j^{-1}(A) = \lambda} \phi_j(A) \) for any \( A \subseteq L^X \) be subbase of \( \tau \) and \( G = \bigwedge_{j \in J} G_j \in L^{\prod \tau_j} \).

In order to prove \( \text{cd}_\tau(G) \geq \bigwedge_{j \in J} \text{cd}_{\tau_j}(G_j) \), let \( \bigwedge_{j \in J} \text{cd}_{\tau_j}(G_j) = a \). Then for any \( j \in J, \text{cd}_{\tau_j}(G_j) \geq a. \) By Lemma 4.2, we only need to prove that
\[
\forall U \subseteq L^X, \phi(U) \subseteq \left[ G \cap \bigwedge_{j \in J} \phi(U_j) \right] \land a \leq \bigvee_{\nu \in 2^{\nu_0} \left[ G \cap \bigwedge_{j \in J} \phi(U_j) \right]} \quad (1)
\]
\[
\forall U \subseteq L^X, \forall b \in \beta(\phi(U)) \subseteq \left[ G \cap \bigwedge_{j \in J} \phi(U_j) \right] \land a, \text{ we have that } b \in \beta(\phi(U)), b \in \beta \left( \left[ G \cap \bigwedge_{j \in J} \phi(U_j) \right] \right) \text{ and } b \in \beta(a). \forall A \in U, \text{ there exists } j \in J \text{ and } A_j \in L^{\tau_j} \text{ such that } b \in \beta(\tau_j(A_j)) \text{ by } b \in \beta(\phi(U)) = \beta \left( \bigwedge_{A \subseteq U} \bigvee_{j \in J} \bigwedge_{p_j^{-1}(A) = \lambda} \phi_j(A_j) \right). \text{ Let } U_j = \{ A_j \in L^{\tau_j} : p_j^{-1}(A_j) = A_j \subseteq U_j \} \text{ and } B_j = \{ p_j^{-1}(A_j) : A_j \subseteq L^{\tau_j}, p_j^{-1}(A_j) = A_j, \forall A_j \subseteq U_j, i \in I \subseteq J \}. \text{ We can obtain that } U = \bigcup_{j \in J} B_j, \text{ and } U_i \subseteq I \subseteq j, \tau_i(U_i) \subseteq \left[ G \cap \bigwedge_{j \in J} \phi(U_j) \right] \land a \leq \bigvee_{\nu \in 2^{\nu_0} \left[ G \cap \bigwedge_{j \in J} \phi(U_j) \right]} \left[ G \cap \bigwedge_{j \in J} \phi(U_j) \right].
\]
At the same time, for any \( x \in X \), we have
\[
b \in \beta \left( G'(x) \lor \bigvee_{A \in \mathcal{U}} A(x) \right) = \beta \left( \bigvee_{j \in J} G'_j(x_j) \lor \bigvee_{i \in I} \bigwedge_{A \in \mathcal{B}_i} A(x) \right)
\]
by \( b \in \beta \left( \left[ G \subseteq \bigvee \mathcal{U} \right] \right) \).

(1) If \( b \in \beta \left( \bigvee_{j \in J} G'_j(x_j) \right) \) for any \( x \in X \), then obviously \( b \leq \bigvee_{x \in X} \left[ G \subseteq \bigvee \mathcal{U} \right] \). This shows that inequality (1) is true.

(2) Suppose that \( b \notin \beta \left( \bigvee_{j \in J} G'_j(x_j) \right) \) for some \( x = \{x_j\}_{j \in J} \in X \). Then \( b \notin \beta \left( G'_j(x_j) \right) \) for any \( j \in J \). Now we prove that there exists \( k \in I \) such that \( b \notin \beta \left( G_k(y_k) \lor \bigvee \mathcal{U}_k(y_k) \right) \) for any \( y_k \in X_k \). If \( \forall i \in I \), there exists \( y_i \in X_i \) such that \( b \notin \beta \left( G'_i(y_i) \lor \bigvee_{B \in \mathcal{B}_i} B(y_i) \right) \). Let \( z = \{z_j\}_{j \in J} \) such that \( z_j = y_j \) when \( j \in I \), \( z_j = x_j \) otherwise. By the equality
\[
G'(z) = \left( \bigvee_{j \in J} G'_j(y_j) \right) \lor \left( \bigvee_{j \in J} G'_j(x_j) \right)
\]
we obtain that \( b \notin \beta(G'(z)) \).

Moreover for any \( i \in I \), by the following fact
\[
b \notin \beta \left( \bigvee_{B \in \mathcal{U}_i} B(y_i) \right) = \beta \left( \bigvee_{B \in \mathcal{U}_i} \bigwedge_{A \in \mathcal{B}_i} A(z) \right) = \beta \left( \bigvee_{A \in \mathcal{B}_i} A(z) \right),
\]
we have \( b \notin \bigcup_{i \in I} \beta \left( \bigwedge_{A \in \mathcal{B}_i} A(z) \right) = \beta \left( \bigvee_{i \in I} \bigwedge_{A \in \mathcal{B}_i} A(z) \right) \). This implies \( b \notin \beta \left( \bigvee_{A \in \mathcal{U}} A(z) \right) \). This yields a contradiction. Thus we obtain the proof that there exists \( k \in I \) such that \( b \notin \beta \left( G_k(y_k) \lor \bigvee \mathcal{U}_k(y_k) \right) \) for any \( y_k \in X_k \). This shows \( b \leq \beta \left( G_k \subseteq \bigvee \mathcal{U}_k \right) \). Thus \( b \leq \tau_k(\mathcal{U}_k) \land \left[ \beta \left( G_k \subseteq \bigvee \mathcal{U}_k \right) \right] \land a \). We can obtain
\[
b \leq \bigvee_{\mathcal{D}_k \in \beta(\mathcal{D}_k)} \left[ \bigwedge_{\mathcal{D}_k \in \beta(\mathcal{D}_k)} \left( G_k \subseteq \bigvee \mathcal{D}_k \right) \right] = \bigvee_{\mathcal{D}_k \in \beta(\mathcal{D}_k)} \left[ \bigwedge_{\mathcal{D}_k \in \beta(\mathcal{D}_k)} \left( G_k \lor \bigvee \mathcal{D}_k \right) \right] (y_k)
\]
\[
= \bigvee_{\mathcal{D}_k \in \beta(\mathcal{D}_k)} \left[ \bigwedge_{\mathcal{D}_k \in \beta(\mathcal{D}_k)} \left( P_k^-(G_k) \lor \bigvee_{D \in \mathcal{D}_k} P_k^-(D) \right) \right] (y)
\]
\[
\leq \bigvee_{\mathcal{D}_k \in \beta(\mathcal{D}_k)} \left[ \bigwedge_{\mathcal{D}_k \in \beta(\mathcal{D}_k)} \left( G \lor \bigvee_{D \in \mathcal{D}_k} P_k^-(D) \right) \right] (y)
\]
\[
\leq \bigvee_{\mathcal{D}_k \in \beta(\mathcal{D}_k)} \left[ \bigwedge_{\mathcal{D}_k \in \beta(\mathcal{D}_k)} \left( G \lor \bigvee_{D \in \mathcal{D}_k} \mathcal{V}_k \right) \right] (y)
\]
\[
\leq \bigvee_{\mathcal{D}_k \in \beta(\mathcal{D}_k)} \left[ \mathcal{G} \subseteq \bigvee \mathcal{V} \right].
\]
Thus we complete the proof of (1). Therefore \( cd_\tau(G) \geq a = \bigwedge_{j \in J} \mathcal{C}_j(G) \). \( \Box \)

The following example is from [29]. It shows that \( cd_\tau(G) = \bigwedge_{j \in J} \mathcal{C}_j(G) \) is not true in Theorem 4.3.

**Example 4.4.** Take \( Y = \{a_i \mid i \in \mathbb{N}\} \). For every \( i \in \mathbb{N} \), take
\[
X_i = Y \quad \text{and} \quad \delta_i = \left[ 0, 1 \right]^{X_i}, \quad A_i(a_j) = \left\{ \begin{array}{ll}
1, & j = 1; \\
1/i, & j \geq 2.
\end{array} \right.
\]
Then for every \( i \in \mathbb{N} \), \( A_i \) is clearly not fuzzy compact since \( \left[ 0, 1 \right]^{X_i} \), \( \delta_i \) is discrete. But from Example 10.3.4 in [29] we know that \( A = \prod_{i \in \mathbb{N}} A_i \) is fuzzy compact in \( \prod_{i \in \mathbb{N}} (X_i, \delta_i) \), hence it is also fuzzy compact. Thus we have that
\[
1 = cd_{\prod_{i \in \mathbb{N}} \delta_i}(A) = \bigwedge_{i \in \mathbb{N}} cd_{\delta_i}(A_i) = 0.
\]
By Theorems 3.11 and 4.3 we can obtain the following corollary.

**Corollary 4.5.** Let \( (X, \tau) \) be the product \( L \)-fuzzy topological space of \( \{(X_j, \tau_j)\}_{j \in J} \). Then \( cd_\tau(\prod) = \bigwedge_{j \in J} cd_{\tau_j}(\prod_j) \), where \( \prod_j \) is the largest element in \( L^X \).
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References