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J. Math. Anal. Appl. 311 (2005) 139–146
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Hyers–Ulam stability of linear differential equations of first order, III

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Received 27 October 2004
Available online 30 March 2005
Submitted by Z.S. Athanassov

Abstract

Let X be a complex Banach space and let $I = (a, b)$ be an open interval. In this paper, we will prove the generalized Hyers–Ulam stability of the differential equation $ty'(t) + \alpha y(t) + \beta t^r x_0 = 0$ for the class of continuously differentiable functions $f : I \to X$, where α, β and r are complex constants and x_0 is an element of X. By applying this result, we also prove the Hyers–Ulam stability of the Euler differential equation of second order.

Keywords: Hyers–Ulam stability; Linear differential equation; Euler equation

1. Introduction

In 1940, S.M. Ulam gave a wide ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems [20]. Among those was the question concerning the stability of homomorphisms: let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given any $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$

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[1] This work was supported by 2004 Hongik University Research Fund.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.02.025

for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

In the following year, D.H. Hyers affirmatively answered in his paper [6] the question of Ulam for the case where G_1 and G_2 are Banach spaces. Furthermore, the result of Hyers has been generalized by Th.M. Rassias ([17]; see also [2]). Since then, the stability problems of various functional equations have been investigated by many authors (see [3–5,7–9,18]).

Let X be a normed space and let I be an open interval. Assume that for any function $f : I \to X$ satisfying the differential inequality

$$\left\| a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) + h(t) \right\| \leqslant \varepsilon$$

for all $t \in I$ and for some $\varepsilon \geqslant 0$, there exists a solution $f_0 : I \to X$ of the differential equation

$$a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) + h(t) = 0$$

such that $\|f(t) - f_0(t)\| \leqslant K(\varepsilon)$ for any $t \in I$, where $K(\varepsilon)$ is an expression of ε only. Then, we say that the above differential equation has the Hyers–Ulam stability.

If the above statement is also true when we replace ε and $K(\varepsilon)$ by $\varphi(t)$ and $\Phi(t)$, where $\varphi, \Phi : I \to [0, \infty)$ are functions not dependent on f and f_0 explicitly, then we say that the corresponding differential equation has the generalized Hyers–Ulam stability (or the Hyers–Ulam–Rassias stability).

We may apply these terminologies for other (linear or nonlinear) differential equations. For more detailed definitions of the Hyers–Ulam stability and the generalized Hyers–Ulam stability, we refer the reader to [3,7–9].

C. Alsina and R. Ger were the first authors who investigated the Hyers–Ulam stability of differential equations: they proved in [1] that if a differentiable function $f : I \to \mathbf{R}$ is a solution of the differential inequality $|y'(t) - y(t)| \leqslant \varepsilon$, where I is an open subinterval of $\mathbf{R}$, then there exists a solution $f_0 : I \to \mathbf{R}$ of the differential equation $y'(t) = y(t)$ such that $|f(t) - f_0(t)| \leqslant 3\varepsilon$ for any $t \in I$.

This result of Alsina and Ger has been generalized by S.-E. Takahasi, T. Miura and S. Miyajima: they proved in [19] that the Hyers–Ulam stability holds true for the Banach space valued differential equation $y'(t) = \lambda y(t)$ (see also [12,13]).

Recently, T. Miura, S. Miyajima and S.-E. Takahasi [14] investigated the Hyers–Ulam stability of linear differential equations of nth order, $a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_1 y'(t) + a_0 = 0$, with complex coefficients.

In [15], T. Miura, S. Miyajima and S.-E. Takahasi also proved the Hyers–Ulam stability of linear differential equations of first order, $y'(t) + g(t)y(t) = 0$, where $g(t)$ is a continuous function. Indeed, they dealt with the differential inequality $\|y'(t) + g(t)y(t)\| \leqslant \varepsilon$ for some $\varepsilon > 0$.

Recently, the author [10] proved the Hyers–Ulam stability of differential equations of the form $a(t)y'(t) = y(t)$ as follows: assume that either $a(t) > 0$ for all $t \in I$ or $a(t) < 0$ for all $t \in I$, where $I \subset \mathbf{R}$ is an open interval. If a continuously differentiable function

$f: I \to \mathbf{R}$ satisfies the inequality $|a(t)f'(t) - f(t)| \leqslant \varepsilon$ for all $t \in I$, then there exists a real number c such that

$$\left| f(t) - c \exp\left\{ \int_a^t \frac{du}{a(u)} \right\} \right| \leqslant \varepsilon$$

for any $t \in I$.

The aim of this paper is to investigate the generalized Hyers–Ulam stability of the following nonhomogeneous linear differential equation of first order,

$$ty'(t) + \alpha y(t) + \beta t^r x_0 = 0. \tag{1}$$

We assume that X is a complex Banach space and $I = (a, b)$ is an arbitrary interval with either $0 < a < b \leqslant \infty$ or $-\infty < a < b < 0$. Let α, β and r be complex constants. Suppose $\varphi: I \to [0, \infty)$ is a function with which both $t^{\alpha+r-1}$ and $t^{\alpha-1}\varphi(t)$ are integrable on (a, c) for arbitrary c ($a < c \leqslant b$).

We prove in Theorem 1 that if a continuously differentiable function $f: I \to X$ satisfies the differential inequality

$$\left\| ty'(t) + \alpha y(t) + \beta t^r x_0 \right\| \leqslant \varphi(t) \tag{2}$$

for all $t \in I$, where x_0 is a fixed element of X, then there exists a solution $f_0(t)$ of the differential equation (1) such that

$$\left\| f(t) - f_0(t) \right\| \leqslant |t^{-\alpha}| \left| \int_t^b v^{\alpha-1} \varphi(v)\, dv \right|$$

for all $t \in I$. We also apply this result to the investigation of the Hyers–Ulam stability of the Euler (Cauchy) differential equation.

2. Hyers–Ulam stability of the differential equation (1)

Throughout this section, let $I = (a, b)$ be an open interval with either $0 < a < b \leqslant \infty$ or $-\infty < a < b < 0$.

If we set $x_0 = 1$ in the differential equation (1), then the function

$$y(t) = \begin{cases} \frac{c}{t^\alpha} - \frac{\beta}{\alpha+r} t^r & (\text{for } \alpha + r \neq 0), \\ \frac{c - \beta \ln|t|}{t^\alpha} & (\text{for } \alpha + r = 0), \end{cases}$$

where c is an arbitrary real number, is the general solution of (1) in the class of real-valued functions defined on I.

Following the idea of the paper [11] (see also [1,12,13,16,19]), we will prove the generalized Hyers–Ulam stability of the linear differential equation (1). More precisely, we investigate the solutions of the differential inequality (2) for the class of functions $f: I \to X$.

Theorem 1. *Let X be a complex Banach space and let $I=(a,b)$ be an open interval as above. Assume that a function $\varphi: I \to [0,\infty)$ is given, that α, β, r are complex constants, and that x_0 is a fixed element of X. Furthermore, suppose a continuously differentiable function $f: I \to X$ satisfies the differential inequality* (2) *for all $t \in I$. If both $t^{\alpha+r-1}$ and $t^{\alpha-1}\varphi(t)$ are integrable on (a,c) for any c with $a < c \leqslant b$, then there exists a unique solution $f_0: I \to X$ of the differential equation* (1) *such that*

$$\left\| f(t) - f_0(t) \right\| \leqslant |t^{-\alpha}| \left| \int_t^b v^{\alpha-1} \varphi(v)\, dv \right| \tag{3}$$

for any $t \in I$.

Proof. (a) First, we will prove our theorem for the case of $\alpha + r \neq 0$. Let X^* be the dual space of X, i.e., the set of all continuous linear functionals $\lambda: X \to \mathbf{C}$. For each $\lambda \in X^*$ we associate the function $f_\lambda: I \to \mathbf{C}$ defined by $f_\lambda(t) = \lambda(f(t))$ for all $t \in I$.

For any $\lambda \in X^*$, it holds that $(f_\lambda)'(t) = \lambda(f'(t))$ for every $t \in I$. Hence, it follows from (2) that

$$\begin{aligned} \left| t(f_\lambda)'(t) + \alpha f_\lambda(t) + \lambda(\beta t^r x_0) \right| &= \left| \lambda\big(t f'(t) + \alpha f(t) + \beta t^r x_0\big) \right| \\ &\leqslant \|\lambda\| \left\| t f'(t) + \alpha f(t) + \beta t^r x_0 \right\| \\ &\leqslant \|\lambda\| \varphi(t) \end{aligned} \tag{4}$$

for all $t \in I$.

For simplicity, we use the following notation:

$$z(t) := \left(\frac{t}{a}\right)^\alpha f(t) + \frac{\beta}{(\alpha+r)a^\alpha}\left(t^{\alpha+r} - a^{\alpha+r}\right) x_0$$

for each $t \in I$. By making use of this notation and by (4), we get

$$\begin{aligned} \left|\lambda\big(z(t) - z(s)\big)\right| &= \left| \left(\frac{t}{a}\right)^\alpha f_\lambda(t) - \left(\frac{s}{a}\right)^\alpha f_\lambda(s) + \frac{\beta}{(\alpha+r)a^\alpha}\left(t^{\alpha+r} - s^{\alpha+r}\right)\lambda(x_0) \right| \\ &= \left| \int_s^t \frac{d}{dv}\left[\left(\frac{v}{a}\right)^\alpha f_\lambda(v)\right] dv + \int_s^t \lambda\left(\frac{\beta}{a^\alpha} v^{\alpha+r-1} x_0\right) dv \right| \\ &= \left| \int_s^t \left(\frac{v}{a}\right)^\alpha \left\{ (f_\lambda)'(v) + \frac{\alpha}{v} f_\lambda(v) + \lambda(\beta v^{r-1} x_0) \right\} dv \right| \\ &= \left| \int_s^t \frac{v^{\alpha-1}}{a^\alpha} \left\{ v(f_\lambda)'(v) + \alpha f_\lambda(v) + \lambda(\beta v^r x_0) \right\} dv \right| \\ &\leqslant \|\lambda\| \left| \int_s^t \frac{v^{\alpha-1}}{a^\alpha} \varphi(v)\, dv \right| \end{aligned}$$

for any $s, t \in I$.

Since $\lambda \in X^*$ was selected arbitrarily, we may deduce from the above inequality that

$$\left\| z(t) - z(s) \right\| \leqslant \left| \int_s^t \frac{v^{\alpha-1}}{a^\alpha} \varphi(v)\, dv \right| \tag{5}$$

for all $s, t \in I$. Since $t^{\alpha-1}\varphi(t)$ is assumed to be integrable on (a, c) for any c with $a < c \leqslant b$, we may select $t_0 \in I$, for any given $\varepsilon > 0$, such that $s, t \geqslant t_0$ implies $\|z(t) - z(s)\| < \varepsilon$. That is, $\{z(s)\}_{s \in I}$ is a Cauchy net and hence there exists an $x \in X$ such that $z(s)$ converges to x as $s \to b$, since X is complete.

Hence, by the definition of $z(t)$, we get

$$\begin{aligned} &\left\| f(t) - \left(\frac{a}{t}\right)^\alpha x + \frac{\beta}{\alpha + r} \frac{1}{t^\alpha} (t^{\alpha+r} - a^{\alpha+r}) x_0 \right\| \\ &\quad = \left\| a^\alpha t^{-\alpha} \big(z(t) - x\big) \right\| \\ &\quad \leqslant |a^\alpha t^{-\alpha}| \left\| z(t) - z(s) \right\| + |a^\alpha t^{-\alpha}| \left\| z(s) - x \right\| \end{aligned}$$

for all $s, t \in I$. If we set

$$f_0(t) = \left(\frac{a}{t}\right)^\alpha x - \frac{\beta}{\alpha + r} \frac{1}{t^\alpha} (t^{\alpha+r} - a^{\alpha+r}) x_0 \tag{6}$$

in the last inequality, and if we consider the fact that $z(s) \to x$ as $s \to b$, it then follows from (5) that

$$\left\| f(t) - f_0(t) \right\| \leqslant |t^{-\alpha}| \left| \int_b^t v^{\alpha-1} \varphi(v)\, dv \right|$$

for any $t \in I$, which proves the validity of inequality (3). We can easily verify that f_0 is a solution of the differential equation (1).

Finally, we prove the uniqueness of f_0. If there exists another solution

$$f_1(t) = \left(\frac{a}{t}\right)^\alpha x_1 - \frac{\beta}{\alpha + r} \frac{1}{t^\alpha} (t^{\alpha+r} - a^{\alpha+r}) x_0$$

of the differential equation (1) which satisfies the inequality (3), where x_1 is another element of X, then it follows from (3) that

$$\|x_1 - x\| \leqslant \frac{2}{|a^\alpha|} \left| \int_t^b v^{\alpha-1} \varphi(v)\, dv \right|$$

for each $t \in I$. If we let $t \to b$ in the above inequality, then the integrability hypothesis implies that $x_1 = x$, i.e., there exists a unique solution f_0 of the differential equation (1) which satisfies the inequality (3) because every solution of (1) has the form (6).

(b) We will now prove our theorem for $\alpha + r = 0$. If we set

$$z(t) = \left(\frac{t}{a}\right)^\alpha f(t) + \frac{\beta}{a^\alpha} \big(\ln|t| - \ln|a|\big) x_0,$$

then we get the inequality (5) by using a similar argument as we presented in (a).

Furthermore, we obtain the inequality (3), for any $t \in I$, with

$$f_0(t) = \left(\frac{a}{t}\right)^{\alpha} x - \frac{\beta}{t^{\alpha}}\big(\ln|t| - \ln|a|\big)x_0,$$

which is a solution of the differential equation (1). By applying a similar argument as we used in (a), we can also prove the uniqueness of f_0. □

3. Hyers–Ulam stability of Euler equation

In this section, we will investigate the Hyers–Ulam stability of the second order Euler equation

$$t^2 y''(t) + \alpha t y'(t) + \beta y(t) = 0, \tag{7}$$

which is sometimes called the Cauchy equation.

Let $I = (a, b)$ be an open interval with either $0 < a < b \leqslant \infty$ or $-\infty < a < b < 0$. Assume that α, β are real constants satisfying either $\beta < 0$ or $\beta > 0$, $\alpha < 1$ and $(1-\alpha)^2 - 4\beta > 0$. Set

$$c = \frac{\alpha - 1 - \sqrt{(1-\alpha)^2 - 4\beta}}{2} \quad \text{and} \quad d = \frac{\alpha - 1 + \sqrt{(1-\alpha)^2 - 4\beta}}{2}.$$

We here remark that the function

$$y(t) = \frac{c_1}{t^c} + \frac{c_2}{t^d} \quad (c_1 \text{ and } c_2 \text{ are arbitrary real numbers}) \tag{8}$$

is the general solution of the Euler differential equation (7).

Theorem 2. *If a twice continuously differentiable function $f : I \to \mathbf{R}$ satisfies the differential inequality*

$$\left|t^2 f''(t) + \alpha t f'(t) + \beta f(t)\right| \leqslant \varepsilon \tag{9}$$

for all $t \in I$ and for some $\varepsilon > 0$, then there exists a solution $f_0 : I \to \mathbf{R}$ of the Euler equation (7) *such that*

$$\left|f(t) - f_0(t)\right| \leqslant \frac{\varepsilon}{|\beta|}\left|\left(\frac{b}{t}\right)^{c} - 1\right| \tag{10}$$

for any $t \in I$. In particular, if $I = (a, \infty)$ with an $a > 0$, then

$$\left|f(t) - f_0(t)\right| \leqslant \frac{\varepsilon}{|\beta|} \tag{11}$$

for all $t > a$.

Proof. If we define $g(t) = tf'(t) + cf(t)$ for any $t \in I$, then the inequality (9) yields

$$\left|tg'(t) + dg(t)\right| = \left|t^2 f''(t) + \alpha t f'(t) + \beta f(t)\right| \leqslant \varepsilon$$

for each $t \in I$. Further, if we set $h(t) = -dg(t)$ in the last inequality, then we get

$$\left| -\frac{t}{d}h'(t) - h(t) \right| \leqslant \varepsilon$$

for all $t \in I$.

According to [10, Theorem 3], there exists a real number c_0 such that

$$\left| h(t) - c_0\left(\frac{a}{t}\right)^d \right| \leqslant \varepsilon,$$

i.e.,

$$\left| tf'(t) + cf(t) + \frac{c_0 a^d}{d} t^{-d} \right| \leqslant \frac{\varepsilon}{|d|}$$

for every $t \in I$, where $c < 0$ and $c < d$.

In view of Theorem 1, there exists a solution $f_0 : I \to \mathbf{R}$ of the differential equation,

$$ty'(t) + cy(t) + \frac{c_0 a^d}{d} t^{-d} = 0,$$

such that the inequality (10) holds for any $t \in I$. Indeed, due to (6), there exists a real number c_3 such that the f_0 has the following form

$$f_0(t) = \left(c_3 + \frac{c_0}{d(c-d)} \right) \frac{a^c}{t^c} - \frac{c_0}{d(c-d)} \frac{a^d}{t^d}.$$

In view of (8), $f_0(t)$ is certainly a solution of the Euler equation (7).

In particular, since c is a negative real number, if we assume $I = (a, \infty)$ with an $a > 0$, then the inequality (10) is transformed into the inequality (11). □

Strictly speaking, the Hyers–Ulam stability holds for the second order Euler equation (7) defined on an open interval (a, ∞) with $a > 0$.

References

[1] C. Alsina, R. Ger, On some inequalities and stability results related to the exponential function, J. Inequal. Appl. 2 (1998) 373–380.

[2] Z. Gajda, On stability of additive mappings, Internat. J. Math. Math. Sci. 14 (1991) 431–434.

[3] P. Găvruţă, A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings, J. Math. Anal. Appl. 184 (1994) 431–436.

[4] R. Ger, Superstability is not natural, Rocznik Nauk.-Dydakt. Prace Mat. 159 (1993) 109–123.

[5] R. Ger, P. Šemrl, The stability of the exponential equation, Proc. Amer. Math. Soc. 124 (1996) 779–787.

[6] D.H. Hyers, On the stability of the linear functional equation, Proc. Nat. Acad. Sci. USA 27 (1941) 222–224.

[7] D.H. Hyers, G. Isac, Th.M. Rassias, Stability of Functional Equations in Several Variables, Birkhäuser, Boston, 1998.

[8] D.H. Hyers, Th.M. Rassias, Approximate homomorphisms, Aequationes Math. 44 (1992) 125–153.

[9] S.-M. Jung, Hyers–Ulam–Rassias Stability of Functional Equations in Mathematical Analysis, Hadronic Press, Palm Harbor, 2001.

[10] S.-M. Jung, Hyers–Ulam stability of linear differential equations of first order, Appl. Math. Lett. 17 (2004) 1135–1140.

[11] S.-M. Jung, Hyers–Ulam stability of linear differential equations of first order, II, submitted for publication.
[12] T. Miura, On the Hyers–Ulam stability of a differentiable map, Sci. Math. Jpn. 55 (2002) 17–24.
[13] T. Miura, S.-M. Jung, S.-E. Takahasi, Hyers–Ulam–Rassias stability of the Banach space valued linear differential equations $y' = \lambda y$, J. Korean Math. Soc. 41 (2004) 995–1005.
[14] T. Miura, S. Miyajima, S.-E. Takahasi, Hyers–Ulam stability of linear differential operator with constant coefficients, Math. Nachr. 258 (2003) 90–96.
[15] T. Miura, S. Miyajima, S.-E. Takahasi, A characterization of Hyers–Ulam stability of first order linear differential operators, J. Math. Anal. Appl. 286 (2003) 136–146.
[16] T. Miura, S.-E. Takahasi, H. Choda, On the Hyers–Ulam stability of real continuous function valued differentiable map, Tokyo J. Math. 24 (2001) 467–476.
[17] Th.M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. 72 (1978) 297–300.
[18] Th.M. Rassias, On the stability of functional equations and a problem of Ulam, Acta Appl. Math. 62 (2000) 23–130.
[19] S.-E. Takahasi, T. Miura, S. Miyajima, On the Hyers–Ulam stability of the Banach space-valued differential equation $y' = \lambda y$, Bull. Korean Math. Soc. 39 (2002) 309–315.
[20] S.M. Ulam, Problems in Modern Mathematics, Wiley, 1964.