ELSEVIER

Available online at www.sciencedirect.com
ScienceDirect

Journal of Algebra 316 (2007) 133–146

JOURNAL OF Algebra

www.elsevier.com/locate/jalgebra

On the additive categories of generalized standard almost cyclic coherent Auslander–Reiten components

Piotr Malicki, Andrzej Skowroński *

Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, Chopina 12/18, 87-100 Toruń, Poland

Received 23 May 2006

Available online 9 August 2007

Communicated by Kent R. Fuller

Dedicated to Idun Reiten on the occasion of her 65th birthday

Abstract

We prove geometric and homological properties of modules in the additive categories of generalized standard almost cyclic coherent components in the Auslander–Reiten quivers of finite-dimensional algebras.

Keywords: Representations of algebras

1. Introduction and the main results

Throughout the paper K will denote a fixed algebraically closed field. By an algebra is meant an associative finite-dimensional K-algebra with an identity, which we shall assume (without loss of generality) to be basic. Then such an algebra has a presentation $A \cong KQ_A/I$, where $Q_A = (Q_0, Q_1)$ is the ordinary quiver of A with the set of vertices Q_0 and the set of arrows Q_1 and I is an admissible ideal in the path algebra KQ_A of Q_A (see [1]). If the quiver Q_A has no oriented cycles, the algebra A is said to be *triangular*. We shall denote by mod A the category of finite-dimensional right A-modules and by ind A the full subcategory of mod A formed by all indecomposable modules. By an A-module is always meant an object of mod A. The *Jacobson radical* rad(mod A) of mod A is the ideal of mod A generated by all noninvertible morphisms in ind A. Then the *infinite radical* $\operatorname{rad}^{\infty}(\operatorname{mod} A)$ of mod A is the intersection of

* Corresponding author.
E-mail addresses: pmalicki@mat.uni.torun.pl (P. Malicki), skowron@mat.uni.torun.pl (A. Skowroński).

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2007.07.011

all powers $\mathrm{rad}^i(\mathrm{mod}\,A)$, $i \geqslant 1$, of $\mathrm{rad}(\mathrm{mod}\,A)$. We denote by Γ_A the *Auslander–Reiten quiver* of A, and by τ_A the Auslander–Reiten translation $D\,\mathrm{Tr}$. We shall identify an indecomposable A-module with the vertex of Γ_A corresponding to it. Following [31] a family $\mathcal{C}$ of components of Γ_A is said to be *generalized standard* if $\mathrm{rad}^\infty(X,Y) = 0$ for all modules X and Y from $\mathcal{C}$. We note that different components in a generalized standard family $\mathcal{C}$ are orthogonal, and all but finitely many τ_A-orbits in $\mathcal{C}$ are τ_A-periodic (see [31, (2.3)]). Examples of generalized standard components are preprojective components, preinjective components, connecting components of tilted algebras, and tubes over tame tilted, tubular and canonical algebras [29]. The structure of arbitrary generalized standard components is not yet well understood.

In the representation theory of algebras a prominent role is played by the algebras with a separating family of components in the following sense. A family $\mathcal{C} = (\mathcal{C}_i)_{i\in I}$ of components of the Auslander–Reiten quiver Γ_A of an algebra A is said to be *separating* if the modules in $\mathrm{ind}\,A$ split into three disjoint classes $\mathcal{P}_A$, $\mathcal{C}_A = \mathcal{C}$ and $\mathcal{Q}_A$ such that:

(S1) $\mathcal{C}_A$ is sincere and generalized standard;
(S2) $\mathrm{Hom}_A(\mathcal{Q}_A,\mathcal{P}_A) = 0$, $\mathrm{Hom}_A(\mathcal{Q}_A,\mathcal{C}_A) = 0$, $\mathrm{Hom}_A(\mathcal{C}_A,\mathcal{P}_A) = 0$;
(S3) any morphism from $\mathcal{P}_A$ to $\mathcal{Q}_A$ factors through $\mathrm{add}\,\mathcal{C}_A$.

We then say that $\mathcal{C}_A$ separates $\mathcal{P}_A$ from $\mathcal{Q}_A$ and write $\mathrm{ind}\,A=\mathcal{P}_A \vee \mathcal{C}_A \vee \mathcal{Q}_A$. We also note that then $\mathcal{P}_A$ and $\mathcal{Q}_A$ are uniquely determined by $\mathcal{C}_A$ (see [2, (2.1)] or [29, (3.1)]). Moreover, $\mathcal{C}_A$ is called *sincere* if any simple A-module occurs as a composition factor of a module in $\mathcal{C}_A$. Frequently, we may recover A completely from the shape and categorical behaviour of the separating family $\mathcal{C}_A$ of components of Γ_A. For example, the tilted algebras [14,29], or more generally double tilted algebras [28], are determined by their (separating) connecting components. Further, it was proved in [18] (see also [32]) that the class of algebras with a separating family of stable tubes coincides with the class of concealed canonical algebras. This was extended in [19] to a characterization of all quasitilted algebras of canonical type, for which the Auslander–Reiten quiver admits a separating family of semiregular tubes. Recently the latter has been extended in [23] to a characterization algebras with a separating family of almost cyclic coherent Auslander–Reiten components. Recall that a component Γ of an Auslander–Reiten quiver Γ_A is called *almost cyclic* if all but finitely many modules in Γ lie on oriented cycles contained entirely in Γ. Moreover, a component Γ of Γ_A is said to be *coherent* if the following two conditions are satisfied:

(C1) For each projective module P in Γ there is an infinite sectional path $P = X_1 \to X_2 \to \cdots \to X_i \to X_{i+1} \to X_{i+2} \to \cdots$ (that is, $X_i \neq \tau_A X_{i+2}$ for any $i \geqslant 1$) in Γ.
(C2) For each injective module I in Γ there is an infinite sectional path $\cdots \to Y_{j+2} \to Y_{j+1} \to Y_j \to \cdots \to Y_2 \to Y_1 = I$ (that is, $Y_{j+2} \neq \tau_A Y_j$ for any $j \geqslant 1$) in Γ.

The authors proved in [23, Theorem A] that the Auslander–Reiten quiver Γ_A of an algebra A admits a separating family of almost cyclic coherent components if and only if A is a generalized multicoil enlargement of a (possibly decomposable) concealed canonical algebra C. Moreover, for such an algebra A, we have that A is triangular, $\mathrm{gl\,dim}\,A \leqslant 3$, and $\mathrm{pd}_A\,X \leqslant 2$ or $\mathrm{id}_A\,X \leqslant 2$ for any module X in $\mathrm{ind}\,A$ (see [23, Corollary B and Theorem E]).

One of the aims of the paper is to establish some geometric properties of modules from the additive categories of separating families of almost cyclic coherent Auslander–Reiten components.

Assume $A = KQ/I$ is a triangular algebra. For an A-module X, we denote by $\mathbf{dim}\,X$ the dimension vector of X, being the image of X in the Grothendieck group $K_0(A) = \mathbb{Z}^{Q_0}$ of A. For $\mathbf{d} = (d_i) \in \mathbb{N}^{Q_0}$, we denote by $\operatorname{mod}_A(\mathbf{d})$ the affine variety of A-modules of dimension vector $\mathbf{d}$. Then the algebraic group $G(\mathbf{d}) = \prod_{i \in Q_0} \mathrm{GL}(d_i)$ acts on $\operatorname{mod}_A(\mathbf{d})$ in such a way that the $G(\mathbf{d})$-orbits in $\operatorname{mod}_A(\mathbf{d})$ correspond to the isomorphism classes of A-modules of dimension vector $\mathbf{d}$. We shall identify a point of $\operatorname{mod}_A(\mathbf{d})$ with the A-module of dimension vector $\mathbf{d}$ corresponding to it. For a module M in $\operatorname{mod}_A(\mathbf{d})$ we denote by $\dim_M \operatorname{mod}_A(\mathbf{d})$ the *local dimension* of $\operatorname{mod}_A(\mathbf{d})$ at M, that is the maximal dimension of the irreducible components of $\operatorname{mod}_A(\mathbf{d})$ containing M. Then a module M in $\operatorname{mod}_A(\mathbf{d})$ is said to be *nonsingular* if $\dim_M \operatorname{mod}_A(\mathbf{d})$ coincides with the dimension $\dim_K T_M(\operatorname{mod}_A(\mathbf{d}))$ of the tangent space $T_M(\operatorname{mod}_A(\mathbf{d}))$ of $\operatorname{mod}_A(\mathbf{d})$ at M. We note that if M is a nonsingular module in $\operatorname{mod}_A(\mathbf{d})$, then M belongs to exactly one irreducible component of $\operatorname{mod}_A(\mathbf{d})$. Following [8] the *Tits quadratic form* $q_A : \mathbb{Z}^{Q_0} \to \mathbb{Z}$ is defined by

$$q_A(\mathbf{x}) = \sum_{i \in Q_0} x_i^2 - \sum_{(i \to j) \in Q_1} x_i x_j + \sum_{i,j \in Q_0} r_{ij} x_i x_j,$$

where $\mathbf{x} = (x_i)_{i \in Q_0} \in \mathbb{Z}^{Q_0}$ and r_{ij} is the number of relations from i to j in a minimal admissible set of relations generating the ideal I. Moreover, following [29], we denote by $\chi_A : \mathbb{Z}^{Q_0} \to \mathbb{Z}$ the *Euler quadratic form* such that

$$\chi_A(\mathbf{dim}\,M) = \sum_{i=0}^{\infty} (-1)^i \dim_A \operatorname{Ext}_A^i(M, M)$$

for any A-module M. It is known that q_A and χ_A coincide if $\operatorname{gl\,dim} A \leqslant 2$ (see [8]) but in general they are different (see [27, Section 5]). While the Euler form χ_A reflects the homological behaviour of the module category $\operatorname{mod} A$, the Tits form q_A is related to the geometry of A-modules.

The study of the module varieties is an important and interesting research direction of the modern representation theory of algebras. In particular, the geometry of module varieties with separating families of tubes, or more generally coils, has attracted much attention (see [4–7,26, 27] for some results in this direction).

The following theorem is the first main result of the paper.

Theorem A. *Let A be an algebra with a separating family $\mathcal{C}$ of almost cyclic coherent components in Γ_A, M be a module in* $\operatorname{add}\mathcal{C}$, *and $\mathbf{d} = \mathbf{dim}\,M$. Then the following statements hold*:

(i) *M is a nonsingular point of* $\operatorname{mod}_A(\mathbf{d})$.
(ii) $q_A(\mathbf{d}) \geqslant \chi_A(\mathbf{d}) = \dim_K \operatorname{End}_A(M) - \dim_K \operatorname{Ext}_A^1(M, M) \geqslant 0$.
(iii) $\dim_M \operatorname{mod}_A(\mathbf{d}) = \dim G(\mathbf{d}) - \chi_A(\mathbf{d})$.

We note that for a separating family $\mathcal{C}$ of almost cyclic coherent components we may have (even indecomposable) modules M in $\operatorname{add}\mathcal{C}$ with arbitrary large $\chi_A(\mathbf{dim}\,M)$ (see [27, (5.3)]).

From Drozd's Tame and Wild theorem [10] the class of algebras may be divided into two disjoint classes. One class consists of the tame algebras for which the indecomposable modules occur, in each dimension d, in a finite number of discrete and a finite number of one-parameter families. The second class is formed by the wild algebras whose representation theory comprises the representation theories of all finite-dimensional algebras over K. Hence, a classification of the finite-dimensional modules is only feasible for tame algebras.

As an application of Theorem A and results of [23,26,33] we obtain also the following geometric and homological characterization of the tame algebras with separating families of almost cyclic coherent Auslander–Reiten components.

Theorem B. *Let A be an algebra with a separating family of almost cyclic coherent components in Γ_A. The following conditions are equivalent*:

(i) *A is tame.*
(ii) $\chi_A(\mathbf{dim}\, M) \geqslant 0$ *for any indecomposable A-module M.*
(iii) $\dim G(\mathbf{dim}\, M) \geqslant \dim_M \operatorname{mod}_A(\mathbf{dim}\, M)$ *for any indecomposable A-module M.*
(iv) $\dim_K \operatorname{Ext}^1_A(M, M) \leqslant \dim_K \operatorname{End}_A(M)$ *and* $\operatorname{Ext}^r_A(M, M) = 0$ *for any $r \geqslant 2$ and any indecomposable A-module M.*

The following result on arbitrary generalized standard almost cyclic coherent Auslander–Reiten components is the third main result of the paper.

Theorem C. *Let A be an algebra, $\mathcal{C}$ a generalized standard almost cyclic coherent component of Γ_A, and M a module in* $\operatorname{add}\mathcal{C}$. *Then* $\dim_K \operatorname{Ext}^1_A(M, M) \leqslant \dim_K \operatorname{End}_A(M)$.

We stress that the class of algebras with generalized standard almost cyclic coherent Auslander–Reiten components is large (see Proposition 2.9 and the following comments).

For basic background from the representation theory we refer to the books [1,3,29] and on the geometric methods in representation theory to [11,16] (see also [13,30] for basic algebraic geometry).

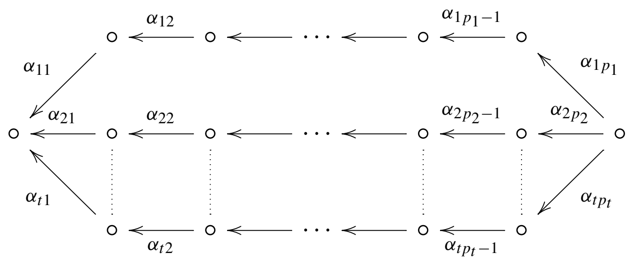
2. Generalized standard almost cyclic coherent components

Let A be an algebra and $\mathcal{C}$ a family of components in Γ_A. Following [9] the family $\mathcal{C}$ is said to be *standard* if the full subcategory of mod A formed by the modules from $\mathcal{C}$ is equivalent to the mesh category $K(\mathcal{C})$ of $\mathcal{C}$. It is known [20] that every standard family of components of Γ_A is generalized standard but the converse implication is not true in general. A component in Γ_A of the form $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$, $r \geqslant 1$, is said to be a *stable tube of rank* r. Therefore, a stable tube of rank r in Γ_A is an infinite component consisting of τ_A-periodic indecomposable A-modules having period r. A stable tube of rank 1 is said to be *homogeneous*. Moreover, a stable tube $\mathcal{T}$ of Γ_A admits a distinguished τ_A-orbit, called the *mouth* of $\mathcal{T}$, consisting of modules having exactly one direct predecessor and exactly one direct successor in $\mathcal{T}$. An indecomposable A-module X with $\operatorname{End}_A(X) \cong K$ is said to be a *brick*. We have the following characterization of standard stable tubes established in [34, (1.3)].

Lemma 2.1. *Let A be an algebra and $\mathcal{T}$ a stable tube of Γ_A. The following conditions are equivalent*:

(i) *$\mathcal{T}$ is standard.*
(ii) *The mouth of $\mathcal{T}$ consists of pairwise orthogonal bricks.*
(iii) $\operatorname{rad}^\infty(Z, Z) = 0$ *for all modules Z in $\mathcal{T}$.*
(iv) *$\mathcal{T}$ is generalized standard.*

An important role in our considerations is played by the concealed canonical algebras. We exhibit first the class of canonical algebras introduced by Ringel [29, (3.7)]. Let $t \geqslant 2$ be a positive integer, $\mathbf{p} = (p_1, \dots, p_t)$ be a t-tuple of positive integers, and $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_t)$ be a t-tuple of pairwise different elements of $\mathbb{P}_1(K) = K \cup \{\infty\}$, normalized such that $\lambda_1 = \infty$, $\lambda_2 = 0$, $\lambda_3 = 1$. Denote by $\Delta(\mathbf{p})$ the quiver of the form



For $t = 2$, put $\Lambda(\mathbf{p}, \boldsymbol{\lambda}) = K\Delta(\mathbf{p})$. For $t \geqslant 3$, assume that $\mathbf{p}$ consists of integers $p_i \geqslant 2$, consider the ideal $I(\mathbf{p}, \boldsymbol{\lambda})$ in $K\Delta(\mathbf{p})$ generated by the elements

$$\alpha_{ip_i} \dots \alpha_{i2}\alpha_{i1} + \alpha_{2p_2} \dots \alpha_{22}\alpha_{21} + \lambda_i \alpha_{1p_1} \dots \alpha_{12}\alpha_{11}, \quad i = 3, \dots, t,$$

and the bound quiver algebra $\Lambda(\mathbf{p}, \boldsymbol{\lambda}) = K\Delta(\mathbf{p})/I(\mathbf{p}, \boldsymbol{\lambda})$. Then $\Lambda(\mathbf{p}, \boldsymbol{\lambda})$ is called the *canonical algebra* of type $(\mathbf{p}, \boldsymbol{\lambda})$, $\mathbf{p}$ the weight sequence of $\Lambda(\mathbf{p}, \boldsymbol{\lambda})$, and $\boldsymbol{\lambda}$ the parameter sequence of $\Lambda(\mathbf{p}, \boldsymbol{\lambda})$. It has been shown in [29, (3.7)] that Auslander–Reiten quiver Γ_Λ of $\Lambda = \Lambda(\mathbf{p}, \boldsymbol{\lambda})$ admits a canonical separating family $\mathcal{T}_\Lambda = (\mathcal{T}_\Lambda^\lambda)_{\lambda \in \mathbb{P}_1(K)}$ of pairwise orthogonal standard stable tubes of tubular type $\mathbf{p}$, and hence $\operatorname{ind} \Lambda = \mathcal{P}_\Lambda \vee \mathcal{T}_\Lambda \vee \mathcal{Q}_\Lambda$ for the corresponding subcategories $\mathcal{P}_\Lambda$ and $\mathcal{Q}_\Lambda$ of $\operatorname{ind} \Lambda$. Following [17], by a *concealed canonical algebra* (of type $(\mathbf{p}, \boldsymbol{\lambda})$) we mean an algebra of the form $C = \operatorname{End}_\Lambda(T)$, where T is a tilting module from the additive category $\operatorname{add} \mathcal{P}_\Lambda$ of $\mathcal{P}_\Lambda$ and $\Lambda = \Lambda(\mathbf{p}, \boldsymbol{\lambda})$. Then again we have a decomposition

$$\operatorname{ind} C = \mathcal{P}_C \vee \mathcal{T}_C \vee \mathcal{Q}_C,$$

where $\mathcal{T}_C = \operatorname{Hom}_\Lambda(T, \mathcal{T}_\Lambda)$ is the family $\mathcal{T}_C^\lambda = \operatorname{Hom}_\Lambda(T, \mathcal{T}_\Lambda^\lambda)$, $\lambda \in \mathbb{P}_1(K)$, of pairwise orthogonal standard stable tubes of tubular type $\mathbf{p}$, separating $\mathcal{P}_C$ from $\mathcal{Q}_C$. The representation type of a concealed canonical algebra C of type $(\mathbf{p}, \boldsymbol{\lambda})$ is controlled by the *genus* $g(C)$ of C defined as

$$g(C) = 1 + \frac{1}{2}\left((t-2)p - \sum_{i=1}^{t} \frac{p}{p_i}\right),$$

where $p = \text{l.c.m.}(p_1, \dots, p_t)$. It has been shown in [18, (7.1)] that a concealed canonical algebra C is tame if and only if $g(C) \leqslant 1$. We also note that if $g(C) \neq 1$ then $\mathcal{T}_C$ is the unique separating family of stable tubes in Γ_C.

The following theorem proved in [18] (see also [32]) shows importance of concealed canonical algebras.

Theorem 2.2. *Let A be a connected algebra. Then Γ_A admits a separating family of stable tubes if and only if A is a concealed canonical algebra.*

The indecomposable modules in the canonical family $\mathcal{T}_\Lambda$ of stable tubes of a canonical algebra $\Lambda = \Lambda(\mathbf{p}, \boldsymbol{\lambda})$ have a rather simple structure: the dimension vectors of all indecomposable modules forming the mouth of the tubes in $\mathcal{T}_\Lambda$ have coordinates 0 or 1. The following consequence of [15, Theorem 3] shows that there are many concealed canonical algebras C with a separating family of stable tubes in Γ_C consisting of complicated indecomposable modules.

Proposition 2.3. *Let $\Lambda = \Lambda(\mathbf{p}, \boldsymbol{\lambda})$ be a canonical algebra of wild type and m be a positive integer. Then there exist infinitely many pairwise nonisomorphic concealed canonical algebras C of the form $C = \operatorname{End}_\Lambda(T)$, for a tilting module T in $\operatorname{add} \mathcal{P}_\Lambda$, such that the coordinates of the dimension vectors of all indecomposable C-modules in $\mathcal{T}_C$ are at least m.*

The authors proved in [22, Theorem A] that a component Γ of the Auslander–Reiten quiver Γ_A of an algebra A is almost cyclic and coherent if and only if Γ is generalized multicoil, that is, can be obtained (as a translation quiver) from a finite family of stable tubes by a sequence of admissible operations. In [23, Section 3] we introduced admissible operations (ad 1)–(ad 5) and their duals (ad 1*)–(ad 5*) on an arbitrary algebra and a generalized standard family Γ of infinite components of Γ_A. Let now B be a (not necessarily connected) algebra and $\mathcal{T}$ a (generalized) standard family of stable tubes in Γ_B. Following [23] by a *generalized multicoil enlargement* of B, using modules from $\mathcal{T}$, we mean an algebra A obtained from B by an iteration of admissible operations of types (ad 1)–(ad 5) and (ad 1*)–(ad 5*) performed either on stable tubes of $\mathcal{T}$ or on generalized multicoils obtained from stable tubes of $\mathcal{T}$ by means of operations done so far. Then Γ_A admits a (generalized) standard family $\mathcal{C}$ of generalized multicoils obtained from the family $\mathcal{T}$ of stable tubes of Γ_B by a sequence of admissible operations of types (ad 1)–(ad 5) and (ad 1*)–(ad 5*) corresponding to the admissible operations leading from B to A (compare [23, (3.7)]). In particular, the class of tubular extensions (respectively, tubular coextensions) of B using modules from $\mathcal{T}$ in the sense of [29, (4.7)] coincides with the class of generalized multicoil enlargements of B involving only admissible operations of type (ad 1) (respectively, (ad 1*)). From now on by a concealed canonical algebra we mean a finite product of connected concealed canonical algebras.

The following theorem, proved in [23, Theorem A], will be crucial for our further considerations.

Theorem 2.4. *Let A be an algebra. The following statements are equivalent*:

(i) *Γ_A admits a separating family of almost cyclic coherent components.*
(ii) *A is a generalized multicoil enlargement of a concealed canonical algebra C.*

We also note that the class of tubular extension (respectively, tubular coextension) of concealed canonical algebras coincides with the class of algebras having a separating family of ray tubes (respectively, coray tubes) in their Auslander–Reiten quiver (see [17,19]). Moreover, these algebras are quasitilted algebras of canonical type.

We recall also the following theorem on the structure of the module category of a generalized multicoil enlargement of a concealed canonical algebra proved in [23, Theorems C and F].

Theorem 2.5. *Let A be a generalized multicoil enlargement of a concealed canonical algebra C, $\mathcal{C}_A$ the associated separating family of generalized multicoils, and* $\operatorname{ind} A = \mathcal{P}_A \vee \mathcal{C}_A \vee \mathcal{Q}_A$. *Then the following statements hold*:

(i) *There is a unique factor algebra* A_{l} *of* A *which is a tubular coextension of* C *such that* $\operatorname{ind} A_{\mathrm{l}} = \mathcal{P}_{A_{\mathrm{l}}} \vee \mathcal{T}_{A_{\mathrm{l}}} \vee \mathcal{Q}_{A_{\mathrm{l}}}$ *for a separating family* $\mathcal{T}_{A_{\mathrm{l}}}$ *of coray tubes in* $\Gamma_{A_{\mathrm{l}}}$, *and* $\mathcal{P}_A = \mathcal{P}_{A_{\mathrm{l}}}$.
(ii) *There is a unique factor algebra* A_r *of* A *which is a tubular extension of* C *such that* $\operatorname{ind} A_{\mathrm{r}} = \mathcal{P}_{A_{\mathrm{r}}} \vee \mathcal{T}_{A_{\mathrm{r}}} \vee \mathcal{Q}_{A_{\mathrm{r}}}$, *for a separating family* $\mathcal{T}_{A_{\mathrm{r}}}$ *of ray tubes in* $\Gamma_{A_{\mathrm{r}}}$, *and* $\mathcal{Q}_A = \mathcal{Q}_{A_{\mathrm{r}}}$.
(iii) *A is tame if and only if* A_{l} *and* A_{r} *are tame.*

In the above notation, the algebras A_{l} and A_{r} are called the *left* and *right quasitilted algebras* of A. Moreover, the algebras A_{l} and A_{r} are tame if and only if A_{l} and A_{r} are products of tilted algebras of Euclidean type or tubular algebras. Recall that an algebra Λ is called *quasitilted* if $\operatorname{gl\,dim} \Lambda \leqslant 2$ and for any indecomposable Λ-module X we have $\operatorname{pd}_\Lambda X \leqslant 1$ or $\operatorname{id}_\Lambda X \leqslant 1$ [12].

The following proposition describes homological properties of modules in the additive categories of separating families of almost cyclic coherent Auslander–Reiten components.

Proposition 2.6. *Let* A *be an algebra with a separating family* $\mathcal{C}_A$ *of almost cyclic coherent components in* Γ_A *and* M *be a module in* $\operatorname{add} \mathcal{C}_A$. *Then the following statements hold*:

(i) $\operatorname{pd}_A M \leqslant 2$ *and* $\operatorname{id}_A M \leqslant 2$.
(ii) $\operatorname{Ext}^r_A(M, M) = 0$ *for* $r \geqslant 2$.

Proof. The statement (i) is proved in [23, Theorem E]. Moreover, (i) implies $\operatorname{Ext}^r_A(M, M) = 0$ for $r \geqslant 3$. We prove that also $\operatorname{Ext}^2_A(M, M) = 0$. Let $\operatorname{ind} A = \mathcal{P}_A \vee \mathcal{C}_A \vee \mathcal{Q}_A$. Consider the projective cover $\pi : P(M) \to M$ of M in $\operatorname{mod} A$ and $\Omega(M) = \operatorname{Ker} \pi$. Then we have an exact sequence

$$0 \to \Omega(M) \to P(M) \to M \to 0$$

and consequently $\operatorname{Ext}^2_A(M, M) \cong \operatorname{Ext}^1_A(\Omega(M), M)$. Further, in the proof of [23, Theorem E], we showed that $\Omega(M) = M_1 \oplus M_2$, where M_1 is a projective module and M_2 is a module in $\operatorname{add} \mathcal{P}_A$. Moreover, we have $\operatorname{Hom}_A(\mathcal{C}_A, \mathcal{P}_A) = 0$, because $\mathcal{C}_A$ separates $\mathcal{P}_A$ from $\mathcal{Q}_A$. Applying now the Auslander–Reiten formula, we obtain

$$\operatorname{Ext}^1_A\big(\Omega(M), M\big) \cong D\overline{\operatorname{Hom}}_A\big(M, \tau_A \Omega(M)\big) \cong D\overline{\operatorname{Hom}}_A(M, \tau_A M_2) = 0.$$

Therefore, we obtain $\operatorname{Ext}^2_A(M, M) = 0$, and (ii) holds. □

We need also the following extension of Lemma 2.1.

Proposition 2.7. *Let* A *be an algebra and* $\mathcal{C}$ *an almost cyclic coherent component of* Γ_A. *The following two statements are equivalent*:

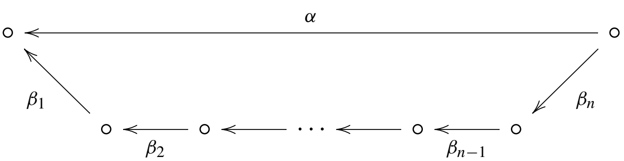
(i) $\mathcal{C}$ *is standard.*
(ii) $\mathcal{C}$ *is generalized standard.*

Proof. The implication (i) ⇒ (ii) follows from the general result proved in [20]. Assume that $\mathcal{C}$ is generalized standard in $\operatorname{mod} A$. Denote by $\operatorname{ann} \mathcal{C}$ the annihilator of $\mathcal{C}$ in A, that is, the intersection of the annihilators $\operatorname{ann} M$ of all modules M in $\mathcal{C}$. Then $\operatorname{ann} \mathcal{C}$ is a two-sided ideal of A and $\mathcal{C}$ is a component of the Auslander–Reiten quiver Γ_B of $B = A / \operatorname{ann} \mathcal{C}$. Moreover, $\mathcal{C}$ is generalized standard in $\operatorname{mod} B$. Because $\mathcal{C}$ is an almost cyclic and coherent component of Γ_B, applying [22, Theorem A], we conclude that, as a translation quiver, $\mathcal{C}$ is a generalized multicoil, and hence can be obtained from a finite family $\mathcal{T}$ of stable tubes by an iteration of admissible operations of types (ad 1)–(ad 5) and (ad 1*)–(ad 5*). Let $\mathcal{T}^*$ be the family of indecomposable B-modules in $\mathcal{C}$ corresponding to the vertices of the family $\mathcal{T}$. We note that $\mathcal{T}^*$ is usually not a translation subquiver of $\mathcal{C}$. Let $J = \operatorname{ann}\mathcal{T}^*$ be the annihilator of the family $\mathcal{T}^*$ in B and $C = B/J$. Because $\mathcal{C}$ is a generalized standard generalized multicoil, we obtain that $\mathcal{T}^*$ is a family of stable tubes in Γ_C, and isomorphic to $\mathcal{T}$ as a translation quiver. Further, $\mathcal{T}^*$ is a generalized standard family of stable tubes in Γ_C, and is sincere. Moreover, different tubes of $\mathcal{T}^*$ are orthogonal in mod C. Applying now Lemma 2.1 we obtain that $\mathcal{T}^*$ is a finite family of pairwise orthogonal standard stable tubes of Γ_C. By construction, B is a generalized multicoil enlargement of C, using modules from $\mathcal{T}^*$ and the admissible operations of types (ad 1)–(ad 5) and (ad 1*)–(ad 5*), corresponding to the admissible operations leading from $\mathcal{T}$ to $\mathcal{C}$. Finally, applying [29, (4.7)(1)], [2, (2.5), (2.6)] and [21, (2.1)], we conclude that $\mathcal{C}$ is a standard almost cyclic and coherent component of Γ_B, and hence of Γ_A. □

The following proposition will be also applied.

Proposition 2.8. *Let R be an algebra and Γ be an almost cyclic coherent component of Γ_R. Then there exists a tame algebra A with a separating family $\mathcal{C}_A$ of almost cyclic coherent components such that the translation quiver Γ is isomorphic to a component of $\mathcal{C}_A$.*

Proof. For a positive integer n, denote by $\Lambda(n)$ the path algebra $K\Delta(n)$ of the quiver $\Delta(n)$ of the form



Observe that $\Lambda(n)$ is the canonical algebra $\Lambda(\mathbf{p})$ with $\mathbf{p} = (1, n)$. Moreover, the separating family $\mathcal{T}_{\Lambda(n)}$ of stable tubes of $\Gamma_{\Lambda(n)}$ consists of a stable tube $\mathcal{T}(n)$ of rank n and a family of homogeneous tubes indexed by the elements of K.

It follows from [22, Theorem A] that the translation quiver Γ can be obtained from a finite family $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ of stable tubes by an iteration of admissible operations of types (ad 1)–(ad 5) and (ad 1*)–(ad 5*), described in [22, Section 2]. Let $r_1, r_2, \ldots, r_m$ be the ranks of the stable tubes $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$, respectively. Consider the algebra

$$\Lambda = \Lambda(r_1) \times \Lambda(r_2) \times \cdots \times \Lambda(r_m).$$

Clearly, then Γ_Λ admits a separating family $\mathcal{T}_\Lambda$ of stable tubes containing the stable tubes $\mathcal{T}(r_1)$, $\mathcal{T}(r_2), \ldots, \mathcal{T}(r_m)$ of ranks $r_1, r_2, \ldots, r_m$. Take now the generalized multicoil enlargement A of the concealed canonical algebra Λ using modules from the tubes $\mathcal{T}(r_1), \mathcal{T}(r_2), \ldots, \mathcal{T}(r_m)$ of $\mathcal{T}_\Lambda$ and the admissible operations of types (ad 1)–(ad 5) and (ad 1*)–(ad 5*) corresponding the admissible operations leading from the stable tubes $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$, isomorphic to the tubes $\mathcal{T}(r_1), \mathcal{T}(r_2), \ldots, \mathcal{T}(r_m)$, to the translation quiver Γ. Then by Theorem 2.4, the Auslander–Reiten quiver Γ_A admits a separating family $\mathcal{C}_A$ of components consisting of an almost cyclic coherent component (generalized multicoil) $\mathcal{C}$, isomorphic to Γ as a translation quiver, and homogeneous tubes. Moreover, the left quasitilted part A_l of A is a product of tubular coextensions of the algebras $\Lambda(r_1), \Lambda(r_2), \ldots, \Lambda(r_m)$ using only modules from the stable tubes $\mathcal{T}(r_1), \mathcal{T}(r_2), \ldots, \mathcal{T}(r_m)$, and consequently is a product of tilted algebras of Euclidean types $\widetilde{\mathbb{A}}_{s_1}, \widetilde{\mathbb{A}}_{s_2}, \ldots, \widetilde{\mathbb{A}}_{s_m}$ with all indecomposable projective modules in the preprojective components. Similarly, the right quasitilted part A_r of A is a product of tilted algebras of Euclidean types $\widetilde{\mathbb{A}}_{t_1}, \widetilde{\mathbb{A}}_{t_2}, \ldots, \widetilde{\mathbb{A}}_{t_m}$ with all indecomposable injective modules in the preinjective components. Therefore, applying Theorem 2.4 and [29, (4.9)], we obtain that A is tame. □

Our next aim is to show that the class of algebras A having (generalized) standard almost cyclic coherent components in Γ_A is very large. We recall that, by [23, Theorem E], any algebra A with a separating family of almost cyclic coherent components in Γ is of global dimension at most 3. The second named author introduced in [34] a wide class of generalized canonical algebras, containing the class of all canonical algebras. This allows to construct complicated algebras of arbitrary high global dimension whose Auslander–Reiten quivers contain an infinite family of generalized standard stable tubes. The final result of this section is an application of this idea.

Proposition 2.9. *Let R be an algebra and Γ be an almost cyclic coherent component of Γ_R, and let B be an algebra. Then there exists an algebra A such that the following statements hold*:

(i) *B is a factor algebra of A.*
(ii) *Γ_A admits a generalized standard almost cyclic coherent component $\mathcal{C}$ such that $\mathcal{C} \cong \Gamma$ as translation quivers.*
(iii) *Every simple B-module occurs as a composition factor of infinitely many indecomposable modules in $\mathcal{C}$.*

Proof. Applying again [22, Theorem A], we conclude that the translation quiver Γ can be obtained from a finite family $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ of stable tubes by an iteration of admissible operations of types (ad 1)–(ad 5) and (ad 1*)–(ad 5*). Let $r_1, r_2, \ldots, r_m$ be the ranks of the tubes $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$. Combining Theorem 2.1 and Corollary 2.5 in [34], we obtain that there is a generalized canonical algebra C such that the following statements hold:

(1) B is a factor algebra of C.
(2) Γ_C admits a (generalized) standard family $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_m$ of stable tubes of ranks $r_1, r_2, \ldots, r_m$, respectively.
(3) Every simple B-module occurs as a composition factor of infinitely many indecomposable modules of each of the tubes $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_m$.

Let A be the generalized multicoil enlargement of C using modules from the tubes $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_m$ and the admissible operations of types (ad 1)–(ad 5) and (ad 1*)–(ad 5*) corresponding to the admissible operations leading from the family of stable tubes $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ to the translation quiver Γ. Then Γ_A admits a generalized standard component $\mathcal{C}$, isomorphic to Γ as a translation quivers, and containing all indecomposable C-modules from the tubes $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_m$. Hence, every simple B-module occurs as a composition factor of infinitely many indecomposable modules in $\mathcal{C}$. Moreover, C is a factor algebra of A, and hence B is a factor algebra of A. □

A wider class of algebras for which the Auslander–Reiten quiver admits (generalized) standard almost cyclic coherent components is formed by the generalized multicoil enlargements of concealed generalized canonical algebras introduced in [24].

We end this section with the following general result proved in [35, Theorem 1].

Proposition 2.10. *Let A be an algebra and $\mathcal{C}$ a generalized standard family of components of Γ_A. Then* $\operatorname{add}\mathcal{C}$ *is closed under extensions.*

3. Geometry of module varieties

In this section we collect geometric facts needed in the proofs of the main theorems of the paper.

Let $A = KQ/I$ be a triangular algebra. For $\mathbf{d} \in \mathbb{N}^{Q_0}$, we denote by $\underline{\underline{\mathrm{mod}}}_A(\mathbf{d})$ the scheme of A-modules with dimension vector $\mathbf{d}$. Then the set $\underline{\underline{\mathrm{mod}}}_A(\mathbf{d})(K)$ of rational points of $\underline{\underline{\mathrm{mod}}}_A(\mathbf{d})$ forms the affine module variety $\mathrm{mod}_A(\mathbf{d})$. The group scheme $\underline{\underline{G}}(\mathbf{d}) = \prod_{i \in Q_0} \underline{\underline{\mathrm{GL}}}_{d_i}$ acts also on $\underline{\underline{\mathrm{mod}}}_A(\mathbf{d})$ such that the orbits of the group $G(\mathbf{d}) = \underline{\underline{G}}(\mathbf{d})(K)$ of rational points in the affine variety $\mathrm{mod}_A(\mathbf{d})$ form the isoclasses of A-modules of dimension vector $\mathbf{d}$. For a module X in $\mathrm{mod}_A(\mathbf{d})$, we denote by $T_X(\underline{\underline{\mathrm{mod}}}_A(\mathbf{d}))$ the tangent space to the scheme $\underline{\underline{\mathrm{mod}}}_A(\mathbf{d})$ at X and by $T_X(\underline{\underline{\mathcal{O}}}(X))$ the tangent space to the $\underline{\underline{G}}(\mathbf{d})$-orbit $\underline{\underline{\mathcal{O}}}(X)$ of X at X. Then we have the following useful result proved by Voigt in [36] (see also [11, (1.1)]).

Proposition 3.1. *For a module $X \in \mathrm{mod}_A(\mathbf{d})$, we have an isomorphism of K-vector spaces*

$$\mathrm{Ext}^1_A(X, X) \cong T_X\big(\underline{\underline{\mathrm{mod}}}_A(\mathbf{d})\big)/T_X\big(\underline{\underline{\mathcal{O}}}(X)\big).$$

Let X be a module in $\mathrm{mod}_A(\mathbf{d})$. The local dimension $\dim_X \underline{\underline{\mathrm{mod}}}_A(\mathbf{d})$ (respectively, $\dim_X \mathrm{mod}_A(\mathbf{d})$) is the maximal dimension of the irreducible components of $\underline{\underline{\mathrm{mod}}}_A(\mathbf{d})$ (respectively, $\mathrm{mod}_A(\mathbf{d})$) containing X. It is known that $\dim_X \underline{\underline{\mathrm{mod}}}_A(\mathbf{d}) = \dim_X \mathrm{mod}_A(\mathbf{d})$. The module X is said to be a nonsingular point of the scheme $\underline{\underline{\mathrm{mod}}}_A(\mathbf{d})$ (respectively, the variety $\mathrm{mod}_A(\mathbf{d})$) if $\dim_X \underline{\underline{\mathrm{mod}}}_A(\mathbf{d}) = \dim_K T_X(\underline{\underline{\mathrm{mod}}}_A(\mathbf{d}))$ (respectively, $\dim_X \mathrm{mod}_A(\mathbf{d}) = \dim_K T_X(\mathrm{mod}_A(\mathbf{d}))$). If X is a nonsingular point of $\underline{\underline{\mathrm{mod}}}_A(\mathbf{d})$ then X is a nonsingular point of $\mathrm{mod}_A(\mathbf{d})$ and belongs to exactly one irreducible component of $\mathrm{mod}_A(\mathbf{d})$ [30, (II.2.6)]. The following result is well known (see [26, (2.2)] for a proof).

Proposition 3.2. *Let X be a module in $\mathrm{mod}_A(\mathbf{d})$ with $\mathrm{Ext}^2_A(X, X) = 0$. Then X is a nonsingular point of $\underline{\underline{\mathrm{mod}}}_A(\mathbf{d})$.*

For a module X in $\mathrm{mod}_A(\mathbf{d})$, denote by $\mathcal{O}(X)$ the $G(\mathbf{d})$-orbit of X in $\mathrm{mod}_A(\mathbf{d})$. Then X is a nonsingular point of $\mathcal{O}(X)$, and hence we have $\dim \mathcal{O}(X) = \dim_K T_X(\mathcal{O}(X))$. We also note that $\dim \underline{\underline{G}}(\mathbf{d}) = \dim G(\mathbf{d})$, $\dim \underline{\underline{\mathcal{O}}}(X) = \dim \mathcal{O}(X)$ and $\dim_K T_X(\underline{\underline{\mathcal{O}}}(X)) = \dim_K T_X\mathcal{O}(X)$. Moreover, we have also the following fact (see [16]).

Lemma 3.3. *Let X be a module in $\mathrm{mod}_A(\mathbf{d})$. Then*

$$\dim_K \mathrm{End}_A(X) = \dim G(\mathbf{d}) - \dim \mathcal{O}(X).$$

Proposition 3.4. *Let X be an A-module, $\mathbf{d} = \mathbf{dim}\, X$, and assume that X is a nonsingular point of $\underline{\underline{\mathrm{mod}}}_A(\mathbf{d})$. Then*

$$\dim G(\mathbf{d}) - \dim_X \ \mathrm{mod}_A(\mathbf{d}) = \dim_K \mathrm{End}_A(X) - \dim_K \mathrm{Ext}^1_A(X, X).$$

Proof. We have $\dim_X \underline{\underline{\mathrm{mod}}}_A(\mathbf{d}) = \dim_K T_X(\underline{\underline{\mathrm{mod}}}_A(\mathbf{d}))$, because X is nonsingular in $\underline{\underline{\mathrm{mod}}}_A(\mathbf{d})$. Then applying Proposition 3.1 and Lemma 3.3, we obtain the equalities

$$\begin{aligned}
&\dim G(\mathbf{d}) - \dim_X \ \mathrm{mod}_A(\mathbf{d}) \\
&\quad = \dim_K \mathrm{End}_A(X) + \dim \mathcal{O}(X) - \dim_X \ \mathrm{mod}_A(\mathbf{d}) \\
&\quad = \dim_K \mathrm{End}_A(X) + \dim \underline{\underline{\mathcal{O}}}(X) - \dim_X \underline{\underline{\mathrm{mod}}}_A(\mathbf{d}) \\
&\quad = \dim_K \mathrm{End}_A(X) + \dim_K T_X\big(\underline{\underline{\mathcal{O}}}(X)\big) - \dim_K T_X\big(\underline{\underline{\mathrm{mod}}}_A(\mathbf{d})\big) \\
&\quad = \dim_K \mathrm{End}_A(X) - \dim_K \mathrm{Ext}^1_A(X, X). \qquad \square
\end{aligned}$$

We end this section with the following theorem proved by de la Peña [25, (1.2)].

Theorem 3.5. *Let $A = KQ/I$ be a tame triangular algebra. Then, for any $\mathbf{d} \in \mathbb{N}^{Q_0}$, we have*

$$\dim G(\mathbf{d}) - \dim \mathrm{mod}_A(\mathbf{d}) \geqslant 0.$$

4. Proof of Theorem C

Let A be an algebra, $\mathcal{C}$ be a generalized standard almost cyclic coherent component of Γ_A and M be a module in $\mathrm{add}\,\mathcal{C}$. Then it follows from Proposition 2.7 that $\mathcal{C}$ is a standard component of Γ_A, and consequently $\mathrm{add}\,\mathcal{C}$ is isomorphic, as a K-category, to the additive category $\mathrm{add}\, K(\mathcal{C})$ of the mesh category $K(\mathcal{C})$ of $\mathcal{C}$.

It follows from Proposition 2.8 that there exists a tame algebra B with a separating family $\mathcal{C}_B$ of almost cyclic coherent components in Γ_B such that the translation quiver $\mathcal{C}$ is isomorphic to a component Γ of $\mathcal{C}_B$. Since $\mathcal{C}_B$ is a generalized standard family of almost cyclic coherent components, applying again Proposition 2.8, we conclude that $\mathcal{C}_B$ is a family of pairwise orthogonal standard components, and hence Γ is a standard component of Γ_B. In particular, $\mathrm{add}\,\Gamma$ is isomorphic to $\mathrm{add}\, K(\Gamma)$ as a K-category. Observe also that the K-categories $\mathrm{add}\, K(\mathcal{C})$ and $\mathrm{add}\, K(\Gamma)$ are isomorphic, because $\mathcal{C} \cong \Gamma$ as translation quivers. It follows from Proposition 2.10 that $\mathrm{add}\,\mathcal{C}$ is closed under extensions in $\mathrm{mod}\ A$ and $\mathrm{add}\,\Gamma$ is closed under extensions in $\mathrm{mod}\ B$. Therefore, there exists a module $N \in \mathrm{add}\,\Gamma$ such that $\dim_K \mathrm{Ext}^1_A(M, M) = \dim_K \mathrm{Ext}^1_B(N, N)$ and $\dim_K \mathrm{End}_A(M) = \dim_K \mathrm{End}_B(N)$. Moreover, it follows from Proposition 2.6 that $\mathrm{Ext}^2_B(N, N) = 0$. In particular, if $\mathbf{d} = \mathbf{dim}\, N$, then N is a nonsingular module of the scheme $\underline{\underline{\mathrm{mod}}}_B(\mathbf{d})$. Applying Proposition 3.4, we obtain

$$\dim_K \mathrm{End}_B(N) - \dim_K \mathrm{Ext}^1_B(N, N) = \dim G(\mathbf{d}) - \dim_N \mathrm{mod}_B(\mathbf{d}).$$

Further, B is a triangular tame algebra, and hence, applying Theorem 3.5, we obtain

$$\dim G(\mathbf{d}) - \dim_N \mathrm{mod}_B(\mathbf{d}) \geqslant \dim G(\mathbf{d}) - \dim \mathrm{mod}_B(\mathbf{d}) \geqslant 0.$$

Summing up, we have $\dim_K \operatorname{Ext}^1_B(N,N) \leqslant \dim_K \operatorname{End}_B(N)$, and hence the required inequality $\dim_K \operatorname{Ext}^1_A(M,M) \leqslant \dim_K \operatorname{End}_A(M)$ holds.

5. Proof of Theorem A

Let A be an algebra with a separating family $\mathcal{C} = \mathcal{C}_A$ of almost cyclic coherent components in Γ_A, M be a module in $\operatorname{add}\mathcal{C}$, and $\mathbf{d} = \mathbf{dim}\, M$. Since, by [23, Corollary B], A is a triangular algebra, the Tits form q_A and the Euler form χ_A are well defined. It follows from Proposition 2.6 that $\operatorname{Ext}^r_A(M,M) = 0$ for $r \geqslant 2$, and hence

$$\chi_A(\mathbf{d}) = \chi_A(\mathbf{dim}\, M) = \dim_K \operatorname{End}_A(M) - \dim_K \operatorname{Ext}^1_A(M,M).$$

Because $\mathcal{C}$ is a separating family of components of Γ_A, it is a generalized standard family of components of Γ_A, and hence $\mathcal{C}$ consists of pairwise orthogonal generalized standard almost cyclic coherent components. Therefore, applying Theorem C, we conclude that $\dim_K \operatorname{End}_A(M) - \dim_K \operatorname{Ext}^1_A(M,M) \geqslant 0$. Further, by Proposition 3.2, $\operatorname{Ext}^2_A(M,M) = 0$ implies that M is a nonsingular module of $\underline{\underline{\operatorname{mod}}}_A(\mathbf{d})$, and so M is also a nonsingular module of $\operatorname{mod}_A(\mathbf{d})$. Moreover, by Proposition 3.4, we then obtain $\dim_M \operatorname{mod}_A(\mathbf{d}) = \dim G(\mathbf{d}) - \chi_A(\mathbf{d})$. Finally, let $A = KQ/I$, where $Q = (Q_0, Q_1)$ is the quiver of A and I is an admissible ideal of KQ. The module variety $\operatorname{mod}_A(\mathbf{d})$ is a closed subset of the affine space $\prod_{(i\to j)\in Q_1} K^{d_i d_j}$. Then, applying Krull's Generalized Principal Ideal theorem, we get the inequality

$$\dim_M \operatorname{mod}_A(\mathbf{d}) \geqslant \sum_{(i\to j)\in Q_1} d_i d_j - \sum_{i,j\in Q_0} r_{ij} d_i d_j$$

where r_{ij} is the number of relations from i to j in a minimal admissible set of relations generating the ideal I. Since $\dim G(\mathbf{d}) = \sum_{i\in Q_0} d_i^2$, we then obtain $q_A(\mathbf{d}) \geqslant \dim G(\mathbf{d}) - \dim_M \operatorname{mod}_A(\mathbf{d}) = \chi_A(\mathbf{d})$.

6. Proof of Theorem B

Let A be an algebra with a separating family $\mathcal{C}_A$ of almost cyclic coherent components of Γ_A, and $\operatorname{ind} A = \mathcal{P}_A \vee \mathcal{C}_A \vee \mathcal{Q}_A$. Then, by Theorem 2.5, there are the left quasitilted factor algebra A_{l} of A and the right quasitilted factor algebra A_{r} of A such that $\Gamma_{A_{\mathrm{l}}}$ admits a separating family $\mathcal{T}_{A_{\mathrm{l}}}$ of coray tubes, $\Gamma_{A_{\mathrm{r}}}$ admits a separating family $\mathcal{T}_{A_{\mathrm{r}}}$ of ray tubes, and $\operatorname{ind} A_{\mathrm{l}} = \mathcal{P}_{A_{\mathrm{l}}} \vee \mathcal{T}_{A_{\mathrm{l}}} \vee \mathcal{Q}_{A_{\mathrm{l}}}$, $\operatorname{ind} A_{\mathrm{r}} = \mathcal{P}_{A_{\mathrm{r}}} \vee \mathcal{T}_{A_{\mathrm{r}}} \vee \mathcal{Q}_{A_{\mathrm{r}}}$, and $\mathcal{P}_A = \mathcal{P}_{A_{\mathrm{l}}}$, $\mathcal{Q}_A = \mathcal{Q}_{A_{\mathrm{r}}}$. Moreover, A is tame if and only if A_{l} and A_{r} are tame. Recall also from [12] that an algebra Λ is quasitilted if and only if $\operatorname{gl\,dim} \Lambda \leqslant 2$ and for any indecomposable Λ-module X we have $\operatorname{pd}_\Lambda X \leqslant 1$ or $\operatorname{id}_\Lambda X \leqslant 1$. Therefore, applying Proposition 2.6, we conclude that $\operatorname{Ext}^r_A(M,M) = 0$ for any indecomposable A-module M and $r \geqslant 2$. Then, by Propositions 3.2 and 3.4, for any indecomposable A-module M we have

$$\begin{aligned}\chi_A(\mathbf{dim}\, M) &= \dim_K \operatorname{End}_A(M) - \dim_K \operatorname{Ext}^1_A(M,M)\\ &= \dim G(\mathbf{dim}\, M) - \dim_M \operatorname{mod}_A(\mathbf{dim}\, M).\end{aligned}$$

Moreover, by Theorem A, we have $\chi_A(\mathbf{dim}\, M) \geqslant 0$ for any indecomposable module M in $\mathcal{C}_A$. Finally, it follows from [33, Theorem A] that a quasitilted algebra Λ is tame if and only if

$\dim_K \operatorname{Ext}^1_\Lambda(X, X) \leqslant \dim_K \operatorname{End}_\Lambda(X)$ for any module X in $\operatorname{ind} \Lambda$. Therefore, the required equivalences of (i)–(iv) hold.

Acknowledgments

The both authors gratefully acknowledge support from the Polish Scientific Grant KBN No. 1 P03A 018 27.

References

[1] I. Assem, D. Simson, A. Skowroński, Elements of the Representation Theory of Associative Algebras 1: Techniques of Representation Theory, London Math. Soc. Stud. Texts, vol. 65, Cambridge Univ. Press, Cambridge, 2006.
[2] I. Assem, A. Skowroński, B. Tomé, Coil enlargements of algebras, Tsukuba J. Math. 19 (1995) 453–479.
[3] M. Auslander, I. Reiten, S.O. Smalø, Representation Theory of Artin Algebras, Cambridge Stud. in Adv. Math., vol. 36, Cambridge Univ. Press, Cambridge, 1995.
[4] M. Barot, J. Schröer, Module varieties over canonical algebras, J. Algebra 246 (2001) 175–192.
[5] G. Bobiński, Geometry of regular modules over canonical algebras, Trans. Amer. Math. Soc., in press.
[6] G. Bobiński, A. Skowroński, Geometry of modules over tame quasi-tilted algebras, Colloq. Math. 79 (1999) 85–118.
[7] G. Bobiński, A. Skowroński, Geometry of periodic modules over tame concealed and tubular algebras, Algebr. Represent. Theory 5 (2002) 187–200.
[8] K. Bongartz, Algebras and quadratic forms, J. London. Math. Soc. 28 (1983) 461–469.
[9] K. Bongartz, P. Gabriel, Covering spaces in representation theory, Invent. Math. 65 (1982) 331–378.
[10] Y.A. Drozd, Tame and wild matrix problems, in: Representation Theory II, in: Lecture Notes in Math., vol. 832, Springer-Verlag, Berlin, 1980, pp. 242–258.
[11] P. Gabriel, Finite representation type is open, in: Representations of Algebras, in: Lecture Notes in Math., vol. 488, Springer-Verlag, Berlin, 1975, pp. 132–155.
[12] D. Happel, I. Reiten, S.O. Smalø, Tilting in abelian categories and quasitilted algebras, Mem. Amer. Math. Soc. 575 (1996).
[13] R. Hartshorn, Introduction to Algebraic Geometry, Grad. Texts in Math., vol. 52, Springer-Verlag, Berlin, 1977.
[14] O. Kerner, Tilting wild algebras, J. London. Math. Soc. 39 (1989) 29–47.
[15] O. Kerner, A. Skowroński, Quasitilted one-point extensions of wild hereditary algebras, J. Algebra 244 (2001) 785–827.
[16] H. Kraft, Geometric methods in representation theory, in: Representations of Algebras, in: Lecture Notes in Math., vol. 944, Springer-Verlag, Berlin, 1981, pp. 180–258.
[17] H. Lenzing, H. Meltzer, Tilting sheaves and concealed-canonical algebras, in: Representations of Algebras, in: CMS Conf. Proc., vol. 18, Amer. Math. Soc., Providence, RI, 1996, pp. 455–473.
[18] H. Lenzing, J.A. de la Peña, Concealed-canonical algebras and separating tubular families, Proc. London Math. Soc. 78 (1999) 513–540.
[19] H. Lenzing, A. Skowroński, Quasi-tilted algebras of canonical type, Colloq. Math. 71 (1996) 161–181.
[20] S. Liu, Infinite radicals in standard Auslander–Reiten components, J. Algebra 166 (1994) 245–254.
[21] P. Malicki, Generalized coil enlargements of algebras, Colloq. Math. 76 (1998) 57–83.
[22] P. Malicki, A. Skowroński, Almost cyclic coherent components of an Auslander–Reiten quiver, J. Algebra 229 (2000) 695–749.
[23] P. Malicki, A. Skowroński, Algebras with separating almost cyclic coherent Auslander–Reiten components, J. Algebra 291 (2005) 208–237.
[24] P. Malicki, A. Skowroński, Concealed generalized canonical algebras and standard stable tubes, J. Math. Soc. Japan 59 (2007) 521–539.
[25] J.A. de la Peña, On the dimension of the module-varieties of tame and wild algebras, Comm. Algebra 19 (1991) 1795–1807.
[26] J.A. de la Peña, A. Skowroński, Geometric and homological characterizations of polynomial growth strongly simply connected algebras, Invent. Math. 126 (1996) 287–296.
[27] J.A. de la Peña, A. Skowroński, The Tits and Euler forms of a tame algebra, Math. Ann. 315 (1999) 37–59.
[28] I. Reiten, A. Skowroński, Characterizations of algebras with small homological dimensions, Adv. Math. 179 (2003) 122–154.

[29] C.M. Ringel, Tame Algebras and Integral Quadratic Forms, Lecture Notes in Math., vol. 1099, Springer-Verlag, Berlin, 1984.
[30] I.R. Shafarevich, Basic Algebraic Geometry, Grad. Texts in Math., vol. 213, Springer-Verlag, Berlin, 1977.
[31] A. Skowroński, Generalized standard Auslander–Reiten components, J. Math. Soc. Japan 46 (1994) 517–543.
[32] A. Skowroński, On omnipresent tubular families of modules, in: Representations of Algebras, in: CMS Conf. Proc., vol. 18, Amer. Math. Soc., Providence, RI, 1996, pp. 641–657.
[33] A. Skowroński, Tame quasi-tilted algebras, J. Algebra 203 (1998) 470–480.
[34] A. Skowroński, Generalized canonical algebras and standard stable tubes, Colloq. Math. 90 (2001) 77–93.
[35] A. Skowroński, G. Zwara, Degeneration-like orders on the additive categories of generalized standard Auslander–Reiten components, Arch. Math. 74 (2000) 11–21.
[36] D. Voigt, Induzierte Darstellungen in der Theorie der endlichen, algebraichen Gruppen, Lecture Notes in Math., vol. 592, Springer-Verlag, Berlin, 1977.