JOURNAL OF DIFFERENTIAL EQUATIONS **140**, 365–377 (1997)
ARTICLE NO. DE973313

Global Regularity of Solution for General Degenerate Parabolic Equations in 1-D

W. Jäger

Institute of Applied Mathematics, Heidelberg University, Heidelberg, Germany

and

Yunguang Lu

Young Scientist Laboratory of Mathematical Physics, Wuhan Institute of Physics and Mathematics, Academia Sinica, People's Republic of China; and

Received November 5, 1996

This paper considers the Cauchy problem for the general degenerate parabolic equations (1.1) with initial data (1.2). In the critical condition meas$\{u: g(u)=0\}=0$ we obtain the regular estimate $G(u)\in C^{(1)}$, where $G(u)=\int_0^u g(s)\,ds$. A new maximum principle is introduced to obtain the estimate and is applied to some special equations such as prous media equation, an infiltration equation to obtain the optimal estimate $|(u^{m-1})_x|\leqslant M$. Finally an interesting equation related to the Broadwell model (where $g(u)$ has two zero points) is studied and a uniquely regular solution $u\in C^{(1)}$ is obtained. Moreover the estimates $u_x\leqslant\rho(f(u)-u^2)/g(u)$ and $\rho\geqslant\inf_x\rho_0(x)/(1+4t(\inf_x\rho_0(x)))$ are proved for the solution of the Navier–Stokes equations corresponding to the Broadwell model. © 1997 Academic Press

1. INTRODUCTION

In this paper, we consider the Cauchy problem for the degenerate parabolic equations

$$u_t+F(u)_x+H(u)=G(u)_{xx} \tag{1.1}$$

with initial data

$$u_{|t=0}=u_0(x), \tag{1.2}$$

* Current address.

0022-0396/97 $25.00

where $F \in C^2$, $H \in C^1$, $g(u) = G'(u) \geqslant 0$, and $g \in C^2$. The existence and uniqueness of weak solutions of (1.1), (1.2) have been well studied [2, 4, 7, 8, 16, 17]. However, the regularity of weak solutions, as we know has only been studied in the case of $g(u)$ having one zero point. It is well known that if there is an interval on which $g(u) = 0$ and $F(u)$ is a nonliner function, then in general, the shock waves will appear in the solution of the Cauchy problem (1.1), (1.2) even if the initial data are smooth and small [17]. An open problem is the regularity of solutions if there is no interval on which $g(u) = 0$. In this paper, we study the regularity of solution for general $g(u)$ satisfying $\text{meas}\{u\colon g(u) = 0\} = 0$.

As with the porous media equation, in general we cannot expect the Cauchy problem (1.1), (1.2) to have a classical solution throughout $R_T = R \times (0, T]$, since the equation (1.1) is parabolic degenerate at the points u_i $(i = 1, 2, \ldots n)$, $g(u_i) = 0$. Thus we introduce the following standard definition of weak solution for (1.1), (1.2).

Definition 1. A function $u(x, t)$ defined on $\bar{R}_T = R \times [0, T]$ will be called a weak solution of the Cauchy problem (1.1), (1.2) if

(i) u is bounded, continuous in $\bar{R}_T$;

(ii) $G(u)$ has a bounded generalized derivative with respect to x in $\bar{R}_T$;

(iii) u satisfies the identity

$$\iint_{R_T} \phi_x[F(u)_x - F(u)] - \phi_t u + \phi H(u)\, dx\, dt = \int_{-\infty}^{\infty} \phi(x, 0)\, u_0(x)\, dx \quad (1.3)$$

for all $\phi \in C_0^1(\bar{R}_T)$ which vanish for large $|x|$ and for $t = T$.

This paper is constructed as follows: In Section 2 we first consider the existence of viscosity solutions for the equation

$$u_t + F(u)_x + H(u) = ((g(u) + \varepsilon)\, u_x)_x \quad (1.4)$$

with the initial data

$$u_{|t=0} = u_0^\varepsilon(x) = u_0 * J^\varepsilon = \int_R u_0(x - y)\, J^\varepsilon(y)\, dy, \quad (1.5)$$

where J^ε is a mollifier and ε is a positive constant.

Throughout this paper, we assume that $u_0(x) \in W^{1,\infty}$. Then

$$u_0^\varepsilon(x) \in C^\infty, \qquad |u^\varepsilon(x)| \leqslant M, \qquad |u_{0,x}^\varepsilon(x)| \leqslant M,$$
$$\left|\frac{d^i u_0^\varepsilon(x)}{dx^i}\right| \leqslant M(\varepsilon), \qquad i = 2, 3, \ldots, \quad (1.6)$$

where the positive constant M is independent of ε and $M(\varepsilon)$ depends on ε.

The local existence of solution for the Gauchy problem (1.4), (1.5) in $\bar{R}_\tau = R \times [0, \tau]$ (for a small $\tau > 0$) is standard if $u_0^\varepsilon(x) \in C^{(2+\alpha)}(R)$ since the equation (1.4) is strictly parabolic for any fixed $\varepsilon > 0$ (see [9, 10, 12, 15]), where the function space $C^{(2+\alpha)}$ is defined as follows:

DEFINITION 2. For any $P_1(t', x')$, $P_2(t'', x'') \in R_T$, let

$$d(P_1, P_2) = |x' - x'| + |t' - t''|^{1/2};$$

$$|u|_0 = \sup_{R_T} |u(x, t)|, \qquad |u|_\alpha = |u|_0 + \sup_{P_1, P_2 \in R_T} \frac{|u(P_1) - u(P_2)|}{d(P_1, P_2)^\alpha},$$

$$0 < \alpha \leqslant 1;$$

$$|u|_{1+\alpha} = |u|_0 + |u_x|_\alpha, \qquad |u|_{2+\alpha} = |u|_{1+\alpha} + |u_x|_{1+\alpha} + |u_t|_\alpha, \tag{1.7}$$

where $u(x, t) \in C^{(q)}(R_T)(q = 0, \alpha, 1+\alpha, 2+\alpha)$ means that $|u|_q$ is finite and $u(x) \in C^{(q)}(R)$ means that $|u|_q$ is finite with $t_1 = t_2$.

To extend the local solution to the entire upper-half space, in Section 2, give a new method for using the maximum principle to estimate u_x^ε. This method is well used in [14] to get the positive lower bound of the density for the isentropic gas dynamics system with viscosity. Differing from the classical method to compare the solution in positive time with initial data, we make a transformation and set up a relation between u_x^ε and $g(u^\varepsilon) + \varepsilon$ and then obtain the bound of $(G(u^\varepsilon) + \varepsilon u^\varepsilon)_x$ uniformly with respect to ε. If we multiply $g(u^\varepsilon) + \varepsilon$ to (1.4), we obtain an equation about $G(u^\varepsilon) + \varepsilon u^\varepsilon$ with uniformly bounded coefficients. Gilding's method ([6]) gives the uniform boundedness of $G(u^\varepsilon) + \varepsilon u^\varepsilon$ in $C^{(1)}$. Under the assumption of $\text{meas}\{u: g(u) = 0\} = 0$, we obtain a subsequence u^{ε_k} such that u^{ε_k} goes to u a.e. and so the regularity of $G(u) \in C^{(1)}$. In Section 3, we apply the method given in Section 2 to some special equations, such as porous media equation and an infiltration equation. The optimal estimate $|(u^{m-1})_r| \leqslant M$ (see [1, 8]) is obtained. In Section 4, we consider the Navier–Stokes equations corresponding to the Broadwell model. For a special case of density $\rho = \text{const}$, a unique regular solution $u \in C^{(1)}$ is obtained. The a-priori estimates $u_x \leqslant \rho(f(u) - u^2)/g(u)$ and $\inf_x \rho_0(x)/(1 + 4t(\inf_x \rho_0(x))) \leqslant \rho$ are proved for the solution of the Navier–Stokes equations.

2. REGULARITY OF SOLUTION

To get the existence of the solution for the Cauchy problem (1.1), (1.2), we first study the viscous solutions for the Cauchy problem (1.4), (1.5). The following local solution lemma is standard and can be found in [9, 10, 12, 15].

LEMMA 1. *Let* $u_0(x)\in C^{2+\alpha}$ *for* $\alpha\in(0,1)$, $F(u)$, $H(u)$ *and* $g(u)$ *be smooth functions. Then for any fixed* $\varepsilon>0$, *there exists a smooth solution, for the Cauchy problem* (1.4), (1.5) *in some* $R_\tau=R\times(0,\tau]$, *which satisfies*

$$|u_\varepsilon|_{2+\alpha}\leqslant 2M,$$

where M *is the bound of* $|u_0|_{2+\alpha}$.

To extend the local solution in Lemma 1 to R_T for arbitrary $T>0$, we need the estimates in $|u^\varepsilon|_{2+\alpha}$. But the main difficulty is to obtain the estimate of u_x^ε since other estimates in $|u^\varepsilon|_{2+\alpha}$ can be obtained by using Itaya's method [10] which is based on the bound of u_x^ε and the estimates of the fundamental solution for a linear parabolic equation.

LEMMA 2. *Let the conditions in Lemma* 1 *be satisfied and* $H'(u)=h(u)\geqslant 0$. *Then the solutions of the Cauchy problem* (1.4), (1.5) *satisfy*

$$|u|_0\leqslant |u_0(x)|_0, \tag{2.1}$$

where for simplicity the superscript ε *in* u *is omitted. Moreover, if*

$$F(u)=F_1(u)+F_2(u), \tag{2.2}$$

where

$$F_1''(u)\geqslant 0, \qquad f_2(u)=\frac{F_2(u)}{g(u)+\varepsilon}\geqslant 0, \qquad h(u)\,f_2(u)-f_2'(u)\,H(u)\geqslant 0, \tag{2.3}$$

then

$$u_x\leqslant f_2(u) \tag{2.4}$$

in the case where the intial data satisfy (2.4). *Similarly if*

$$F(u)=\bar F_1(u)+\bar F_2(u), \tag{2.5}$$

where

$$\bar F_1''(u)\leqslant 0, \qquad \bar f_2(u)=\frac{\bar F_2(u)}{g(u)+\varepsilon}\leqslant 0, \qquad h(u)\,\bar f_2(u)-\bar f_2'(u)\,H(u)\leqslant 0, \tag{2.6}$$

then

$$u_x\geqslant \bar f_2(u) \tag{2.7}$$

in the case where the initial data satisfy (2.7).

Proof. The first conclusion (2.1) is trivial. To prove (2.4), we rewrite (1.4) as follows:

$$u_t + F_1(u)_x + H(u) = ((g(u)+\varepsilon)(u_x - f_2(u)))_x. \tag{2.8}$$

Letting $v = u_x - f_2(u)$ and differenting (2.8) with respect to x, we have

$$\begin{aligned} &v_t + f_2'(u)((g(u)+\varepsilon)v)_x + F_1'(u)\, v_x + h(u)v \\ &\quad + F_1''(u)\, u_x^2 + h(u)\, f_2(u) - H(u)\, f_2'(u) = ((g(u)+\varepsilon)v)_{xx}. \end{aligned} \tag{2.9}$$

Thus the condition (2.3) in Lemma 2 and (2.9) give the following inequality

$$v_t + a(x,t)\, v_x + b(x,t)v \leqslant (g(u)+\varepsilon)\, v_{xx}, \tag{2.10}$$

where a, b are functions of u, u_x, u_{xx}. Therefore the maximum principle [9, 14] applied to (2.10) give the estimate $v \leqslant 0$ if $v_0(x) \leqslant 0$. So the estimate (2.4) is proved. Similarly we can obtain the proof of (2.7).

For the case of $H(u) = 0$, we can always find a suitable large constant C such that $F(u) - C < 0$, $F(u) + C > 0$ since u is bounded and $F(u)$ is continuous. Thus we choose $F_1, F_2, \bar{F}_1, \bar{F}_2$ in Lemma 2 as $F_1 = -C$, $F_2(u) = F(u) + C$, $\bar{F}_1 = C$, $\bar{F}_2(u) = F(u) - C$ respectively. Moreover since $u^\varepsilon_{0,x}$ is bounded from (1.6), $(g(u^\varepsilon_0)+\varepsilon)\,|u^\varepsilon_{0,x}|$ is bounded. Therefore the initial data always satisfy (2.4), (2.7) for large constant C. So we obtain

$$|u_x| \leqslant \frac{\max(|F_2|_0, |\bar{F}_2|_0)}{g(u)+\varepsilon} \leqslant \frac{M}{g(u)+\varepsilon} \leqslant M(\varepsilon). \tag{2.11}$$

The left estimates in $|u|_{2+\alpha}$ can be obtained by using Itaya's method [10]. So the global existence and uniqueness of solution for the Cauchy problem (1.4), (1.5) is obtained.

THEOREM 3. *Let* $u_0(x) \in C^{2+\alpha}$ *for* $\alpha \in (0,1)$, *where* $|u_0(x)|_{2+\alpha}$ *may depend on* ε. *Let* $H(u) = 0$, $g(u) \in C^2$. *Then for any fixed* $\varepsilon > 0$, *there exists a unique smooth solution, for the Cauchy problem* (1.4), (1.5) *in* $R_T = R \times (0, T]$, *which satisfies*

$$|u^\varepsilon|_{2+\alpha} \leqslant M(\varepsilon). \tag{2.12}$$

Moreover if $|u^\varepsilon_0|_1 \leqslant M$, *then*

$$|G(u^\varepsilon) + \varepsilon u^\varepsilon|_1 \leqslant M. \tag{2.13}$$

Proof. The existence and uniqueness of solution is clear from the a-priori estimates $|u^\varepsilon|_{2+\alpha}$. From the estimate (2.11), $z^\varepsilon = G(u^\varepsilon) + \varepsilon u^\varepsilon$ is Hölder continuous with respect to x with Hölder exponent 1.

Multiplying $g(u^\varepsilon)+\varepsilon$ by (1.4), we have

$$(z^\varepsilon)_t + F'(u^\varepsilon)(z^\varepsilon)_x = (g(u^\varepsilon)+\varepsilon)(z^\varepsilon)_{xx}. \tag{2.14}$$

Since $F'(u^\varepsilon)$ and $g(u^\varepsilon+\varepsilon$ are bounded independent of ε, z^ε is Hölder continuous with respect to t with Hölder exponent $\frac{1}{2}$ by using Gilding's results in [6]. Theorem 3 is proved.

Since z^ε is uniformly bounded in $C^{(1)}$, there exists a subsequence z^{ε_k} converges to a function $z(x,t)\in C^{(1)}$ a.e. on any compact set in $R\times[0,T]$. So $G(u^{\varepsilon_k})=\int_0^{u^{\varepsilon_k}} g(s)\,ds$ converges to $z(x,t)$. If $\text{meas}\{u: g(u)=0\}=0$, then $\int_0^u g(s)\,ds=z(x,t)$ uniquely defines a value $u(x,t)$ for any given $z(x,t)$. Thus $u^{\varepsilon_k}\to u(x,t)$ a.e. on any compact set in $R\times[0,T]$. Letting $\varepsilon\downarrow 0$ in (1.4), we obtain the following main conclusion in this section.

THEOREM 4. *If* $H(u)=0$, $F(u)\in C^2$, $g(u)\in C^2$, $\text{meas}\{u: g(u)=0\}=0$ *and* $|u_0|_1\leqslant M$, *then there exists a unique weak solution* $u(x,t)$, *for the Cauchy problem* (1.1), (1.2), *which satisfies the regular estimate*

$$|G(u)|_1\leqslant M. \tag{2.15}$$

3. APPLICATION IN THEORY OF INFILTRATION

In this section, we apply Lemma 2 in Section 2 to some special functions $F(u)=c_1u^n$, $H(u)=c_2u^h$ and $G(u)=u^m$, where c_1, c_2, $n\geqslant 1$, $h\geqslant 1$, and $m>1$ are all constants.

For simplicity, we let $0\leqslant u_0u_0(x)\leqslant M$. Instead of adding a viscous constant ε to the equation (1.1), we consider the Cauchy problem (1.1) with the initial data

$$u_{|t=0}=u_0^\varepsilon(x)=\left[\int_{-\infty}^{\infty}((u_0(x-y))^{m-1}+\varepsilon^{m-1})\,G^\varepsilon(y)\,dy\right]^{1/(m-1)}, \tag{3.1}$$

where G^ε is a mollifier and ε is a positive constant.

When $c_1=c_2=0$, (1.1) is the porous media equation. When $c_1<0$, $c_2=0$, (1.1) is an equation derived from the theory of infiltration [8]. In both cases, the optimal estimate is $|(u^{m-1})_x|\leqslant M$ from the previous results in [1, 8]. In this section we use the conclusion in Lemma 2 to obtain the same estimate.

LEMMA 5. *If* $c_2\geqslant 0$, $h\geqslant 1$, $n\geqslant 1$, $m>1$ *and if the solution of the Cauchy problem* (1.1), (3.1) *exists* (*Lemma* 1) *in* $R_T=R\times(0,T]$ *and satisfies*

$$0<u^\varepsilon\leqslant M(\varepsilon), \qquad |u^\varepsilon|_{2+\alpha}\leqslant M(\varepsilon), \tag{3.2}$$

then

$$\varepsilon e^{-M_1 t} \leqslant u^\varepsilon \leqslant M + \varepsilon, \tag{3.3}$$

where $M_1 = c_2(M+\varepsilon)^{h-1}$.

Proof. Since $\varepsilon \leqslant u_0^\varepsilon \leqslant M+\varepsilon$, we have from (1.1) that

$$u_t + F(u)_x \leqslant G(u)_{xx}. \tag{3.4}$$

So the upper bound $u^\varepsilon \leqslant M+\varepsilon$ is obtained by using the maximum principle to (3.4). To obtain the lower bound in (3.3), we let $v = e^{M_1 t}u - \varepsilon$. Then we have from (1.1) that

$$\begin{cases} v_t + c_1 n u^{n-1} v_x + (c_2 u^{h-1} - M_1) v \geqslant (m u^{m-1} v_x)_x \\ u_{|t=0} = u_0^\varepsilon(x) - \varepsilon \geqslant 0. \end{cases} \tag{3.5}$$

From (3.5) we have $v \geqslant 0$. This means $u^\varepsilon \geqslant e^{-M_1 t}\varepsilon$. Lemma 5 is proved.

The lower bound of u^ε in (3.3) ensures that the equation (1.1) is strictly parabolic in R_T for any finite time T and for any fixed ε. So all the analyses about the parabolic equation given in Section 2 are valid.

LEMMA 6. *If the conditions in Lemma 5 are satisfied, and moreover* $h > 1$ *and* $|(u_0^{m-1})_x| \leqslant M$, *then*

$$|((u^\varepsilon)^{m-1})_x| \leqslant M_2 \tag{3.6}$$

for a positive constant M_2 *independent of* ε.

Proof. We choose

$$\begin{cases} F_1(u) = -Lu, & F_2(u) = c_1 u^n + Lu \\ \bar{F}_1(u) = Lu, & \bar{F}_2(u) = c_1 u^n - Lu \end{cases} \tag{3.7}$$

then $F_1''(u) = \bar{F}_1''(u) = 0$, $F_2(u) \geqslant 0$, $\bar{F}_2(u) \leqslant 0$ for a sufficiently large, positive constant L. Moreover

$$\begin{aligned} h(u)\, f_2(u) - f_2'(u)\, H(u) &= c_2 h u^{h-1} \frac{c_1 u^n + Lu}{m u^{m-1}} - c_2 u^k \left(\frac{c_1 n u^{n-1} + L}{m u^{m-1}} - \frac{(m-1)(c_1 u^n + Lu)}{m u^m} \right) \\ &= c_2 \left[\frac{(h-1)\, L u^h + (h-n)\, c_1 u^{n+h-1}}{m u^{m-1}} + \frac{(m-1)\, u^n (c_1 u^n + Lu)}{m u^m} \right]. \end{aligned} \tag{3.8}$$

Since $c_2 \geqslant 0$, $h > 1$, and $m > 1$, we can always choose L large enough so that $(h-1)L + (h-n)c_1 u^{n-1} > 0$. So the right hand side of (3.8) is non-negative. Similarly we obtain that

$$h(u)\bar{f}_2(u) - \bar{f}_2'(u)H(u) \leqslant 0. \tag{3.9}$$

Thus the conditions (2.3) and (2.6) in Lemma 2 are satisfied. If $|(u_0^{m-1})_x| \leqslant M$, we have from (3.1) that $|((u_0^\varepsilon)^{m-1})_x| \leqslant M$. Then for a sufficiently large L, the initial data (3.1) satisfy the estimates (2.4) and (2.7). So we have from Lemma 2 that

$$|u_x^\varepsilon| \leqslant \frac{Lu^\varepsilon + |c_1|(u^\varepsilon)^n}{m(u^\varepsilon)^{m-1}} \leqslant \frac{M_2}{m(u^\varepsilon)^{m-2}} \tag{3.10}$$

for a positive constant M_2. Lemma 6 is proved.

Therefore we obtain the following main theorem in this section.

THEOREM 7. *Let* $F(u) = c_1 u^n$, $H(u) = c_2 u^h$, $G(u) = u^m$ *and* $c_2 \geqslant 0$, $h > 1$, $m > 1$, $n \geqslant 1$. *If* $0 \leqslant u_0(x) \leqslant M$, $|(u_0^{m-1})_x| \leqslant M$, *then the Cauchy problem* (1.1), (1.2) *has a weak solution satisfying the regular estimate*

$$|u^{m-1}(x,t)|_1 \leqslant M. \tag{3.11}$$

4. APPLICATION IN NAVIER–STOKES EQUATIONS

The Navier–Stokes equations corresponding the Broadwell model may be written [3]

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ u_t + (f'(u) - u)u_x + (f(u) - u^2)\dfrac{u_x}{\rho} = \dfrac{1}{\rho}(g(u)u_x)_x, \end{cases} \tag{4.1}$$

for $(x,t) \in R \times R^+$, where ρ and u denote the density and the velocoty of the moving particles.

The first equation in (4.1) is hyperbolic and the second is of degenerate parabolic type because

$$g(u) = \frac{2(1 - f(u)}{(1+3u^2)^{3/2}}, \qquad f(u) = \frac{1}{3}(2(1+3u^2)^{1/2} - 1). \tag{4.2}$$

At points $(x,t) \in R_T = R \times (0,T]$, where $|u| < 1$, i.e., $g(u) > 0$, the second equation in (4.1) is parabolic, but at points where $|u| = 1$, i.e., $g(u) = 0$, it is not. When $g(u)$ is a constant, the system (4.1) is similar to the compressible Navier–Stokes equations of the isentropic gas.

Let $\rho = 1$. Then we obtain the following degenerate parabolic equation from the second equation in (4.1):

$$u_t + (f'(u) - u) u_x = (g(u)\, u_x)_x. \tag{4.3}$$

The interest of the equation (4.3) is that $g(u)$ has two zero points at $u = 1$ and $u = -1$, respectively.

We consider the Cauchy problem

$$\begin{cases} u_t + F(u)_x = (g(u)\, u_x)_x, \\ u_{|t=0} = u_0^\varepsilon(x) = \int_{-\infty}^{\infty} (1-\varepsilon)\, u_0(x-y)\, G^\varepsilon(y)\, dy, \end{cases} \tag{4.4}$$

where $F(u) = f(u) - (u^2/2)$.

LEMMA 8. *If* $|u_0(x)| \leqslant 1$, $-\frac{1}{2} \leqslant u_{0,x}(x) \leqslant \frac{1}{4}$, *then the solutions* u^ε *of the Cauchy problem* (4.4) *satisfy*

$$|u^\varepsilon| \leqslant 1 - \varepsilon, \qquad -\tfrac{1}{2}(1 + 3(u^\varepsilon)^2)^{3/2} \leqslant u_x^\varepsilon \leqslant \tfrac{1}{4}(1 + 3(u^\varepsilon)^2)^{3/2}. \tag{4.5}$$

Proof. Since $|u_0(x)| \leqslant 1$, then $|u_0^\varepsilon(x)| \leqslant 1 - \varepsilon$. So applying the maximum principle to (4.4) gives $|u^\varepsilon| \leqslant 1 - \varepsilon$.

To prove the second part in (4.5), we choose

$$F_1 = \frac{u^2}{2}, \qquad F_2 = f(u) - u^2, \qquad \bar{F}_1 = 1 - \frac{u^2}{2}, \qquad \bar{F}_2 = f(u) - 1. \tag{4.6}$$

By simple calculations,

$$F_2 = \tfrac{1}{3}(1 + 3u^2)^{1/2}\, (2 - (1 + 3u^2)^{1/2}) \tag{4.7}$$

and

$$\bar{F}_2 = \tfrac{2}{3}((1 + 3u^2)^{1/2} - 2). \tag{4.8}$$

So $F_2 = 0$ if $|u| = 1$, $F_2 > 0$ if $|u| < 1$ and $\bar{F}_2 = 0$ if $|u| = 1$, $\bar{F}_2 > 0$ if $|u| < 1$. Moreover

$$g(u) = \frac{4(2 - (1 + 3u^2)^{1/2})}{3(1 + 3u^2)^{3/2}} \tag{4.9}$$

and

$$\frac{F_2}{g} = \frac{1}{4}(1+3u^2)^2, \qquad \frac{\bar{F}_2}{g} = -\frac{1}{2}(1+3u^2)^{3/2}. \tag{4.10}$$

Since $-\frac{1}{2} \leqslant u^{\varepsilon}_{0,x} \leqslant \frac{1}{4}$ from the conditions in Lemma 8, the initial data satisfy the second estimate in (4.5). So the conclution in Lemma 2 gives the proof of the second part in (4.5). Lemma 8 is proved.

From Lemma 8, we have the following theorem:

THEOREM 9. *If* $|u_0(x)| \leqslant 1$, $-\frac{1}{2} \leqslant u_{0,x}(x) \leqslant \frac{1}{4}$, *then the Cauchy problem* (4.3) *with the initial data* $u_0(x)$ *has a unique weak solution* u *satisfying*

$$u \in C^{(1)}, \qquad |u| \leqslant 1. \tag{4.11}$$

Remark. The uniqueness in Theorem 9 may be obtained by using the method in the proof of Theorem 1 in [8] which is based on the following

LEMMA 10. *Let* $F(u) = f(u) - (u^2/2)$, $G(u) = \int_{-1}^{u} g(s)\,ds$, *and* f, g *be given by* (4.2). *If* $|u_2| \leqslant 1$, $|u_1| \leqslant 1$, *then there exists a constant* A *which depends only on* f *and* g, *such that*

$$(F(u_2) - F(u_1))^2 \leqslant A(G(u_2) - G(u_1))(u_2 - u_1). \tag{4.12}$$

Proof. If $u_1 = u_2$, (4.12) is satisfied. Thus without loss of generality, we let $u_1 < u_2$.

Define

$$K(u) = \frac{(F(u) - F(u_1))^2}{G(u) - G(u_1)}. \tag{4.13}$$

Then $K(u)$ is continuous for $u \in (u_1, 1]$ and

$$\lim_{u \to u_1^+} K(u) = \lim_{u \to u_1^+} \frac{2(F(u_2) - F(u_1))(f'(u) - u)}{g(u)} = 0 \tag{4.14}$$

since

$$\frac{f'(u) - u}{g(u)} = \frac{3}{4} u(1 + 3u^2)$$

is bounded for $u \in [-1, 1]$. Moreover

$$K'(u) = \frac{2(F(u) - F(u_1))(f'(u) - u)}{G(u) - G(u_1)} - \left(\frac{F(u) - F(u_1)}{G(u) - G(u_1)}\right)^2 g(u) \tag{4.15}$$

is continue for $u \in (u_1, 1]$ and

$$\lim_{u \to u_1^+} K'(u) = \left(\lim_{u \to u_1^+} \frac{2(f'(u) - u)}{g(u)}\right)(f'(u_1) - u_1) - \left(\lim_{u \to u_1^+} \frac{f'(u) - u}{g(u)}\right)^2 g(u_1) \tag{4.16}$$

exists and is bounded independent of u_1. Thus we can obtain the proof of Lemma 10 by using the Tayler expansion to function $K(u)$ at $u = u_1$.

Finally we end this section by the following lemma.

LEMMA 11. *If* $|u_0(x)| \leqslant 1$, $u_{0,x} \leqslant \rho_0(x)(f(u_0) - u_0^2)/g(u_0)$, $\inf_x \rho_0(x) > 0$, *then we have the a-priori estimates in form for the solution* (ρ, u) *of* (4.1) *with initial data* $(\rho_0(x), u_0(x))$ *as follows*:

$$|u| \leqslant 1, \qquad u_x \leqslant \frac{\rho(f(u) - u^2)}{g(u)}, \qquad \rho \geqslant \frac{\inf_x \rho_0(x)}{1 + 4t(\inf_x \rho_0(x))}. \tag{4.17}$$

Proof. We can add a small positive constant to $g(u)$ such that the second equation in (4.1) is strictly parabolic. But for simplicity, we omit the process and give the proof in form.

Since $f(u) - u^2$ has two zero points at $u = 1$ and $u = -1$, applying the maximum principle to the second equation in (4.1) gives $u \leqslant 1$ and $u \geqslant -1$ respectively.

To estimate the second inequality in (4.17), we rewrite the second equation in (4.1) as follows:

$$u_t + uu_x + \frac{1}{\rho}((f(u) - u^2)\,\rho)_x = \frac{1}{\rho}(g(u)\,u_x)_x. \tag{4.18}$$

Then

$$u_t + uu_x = \frac{1}{\rho}\left(g(u)\left(u_x - \frac{\rho(f(u) - u^2)}{g(u)}\right)\right)_x. \tag{4.19}$$

Differenting (4.19) with respect to x and letting

$$v = u_x - \frac{\rho(f(u) - u^2)}{g(u)}, \qquad f_2(u) = \frac{f(u) - u^2}{g(u)} \tag{4.20}$$

we have

$$(u_x)_t + u_x^2 + uu_{xx} = \frac{1}{\rho}(g(u)v)_{xx} + \left(\frac{1}{\rho}\right)_x (g(u)v)_x. \tag{4.21}$$

Let the right hand side in (4.21) be I. We have from (4.21)

$$v_t + f_2(u)\,\rho_t + f_2'(u)\,\rho u_t + u_x^2 + uv_x + u(f_2(u)\,\rho_x + f_2'(u)\,\rho u_x) = I. \tag{4.22}$$

Substituting the first equation in (4.1) and (4.19) into (4.22), we have

$$v_t + f_2(u)\,\rho v + v^2 + uv_x = \frac{1}{\rho}(g(u)v)_{xx} + \left(\left(\frac{1}{\rho}\right)_x - f_2'(u)\right)(g(u)v)_x. \tag{4.23}$$

Applying the maximum principle to (4.23) gives $v \leqslant 0$ since $v_0(x) \leqslant 0$ from the conditions in Lemma 11.

Thus

$$u_x \leqslant \frac{\rho(f(u) - u^2)}{g(u)} = \frac{\rho}{4}(1 + 3u^2)^2 \leqslant 4\rho. \tag{4.24}$$

Substituting $u_x \leqslant 4\rho$ into the first equation in (4.1), we have

$$\rho_t + 4\rho^2 + u\rho_x \geqslant 0. \tag{4.25}$$

Thus

$$\left(\frac{1}{\rho} - 4t\right)_t + u\left(\frac{1}{\rho} - 4t\right)_x \leqslant 0. \tag{4.26}$$

Applying the maximum principle to (4.26) gives

$$\frac{1}{\rho} - 4t \leqslant \sup_x \frac{1}{\rho_0(x)} = \frac{1}{\inf_x \rho_0(x)}. \tag{4.27}$$

Therefore

$$\rho \geqslant \frac{\inf_x \rho_0(x)}{1 + 4t(\inf_x \rho_0(x))}. \tag{4.28}$$

Lemma 11 is proved.

ACKNOWLEDGMENTS

Yunguang Lu was supported in part by a Humboldt fellowship, by SFB 359 at the University of Heidelberg, and by a President Fund in the Chinese Academy of Sciences. This paper was finished during Yunguang Lu's stay in SISSA in Italy. He is very grateful to Prof. Albert Bressan and Prof. C. Klingenberg for their kind hospitality.

REFERENCES

1. D. G. Aronson, Regularity properties of flows through porous media, *SIAM J. Appl. Math.* **17** (1969), 461–467.
2. H. Brezis and M. G. Crandall, Uniqueness of solution of the initial value problem for $u_t - \Delta G(u) = 0$, *J. Pure Appl. Math.* **58** (1979), 153–163.
3. R. E. Caflisch, Navier–Stokes and Boltzmann shock profiles for a model of gas dynamics, *Comm. Pure Appl. Math.* **32** (1979), 521–554.
4. J. I. Diaz and R. Kersner, On a nonlinear degenerate parabolic equation in infiltration or evaporation through a porous medium, *J. Differential Equations* **69** (1987), 368–403.
5. E. DiBenedetto, "Degenerate Parabolic Equations," Springer-Verlag, Berlin/New York, 1993.
6. B. H. Gilding, Hölder continuity of solutions of parabolic equations, *J. London Math. Soc.* **13** (1976), 103–106.
7. B. H. Gilding, Improved theory for a nonlinear degenerate parabolic equation, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **XVI** (1989), 165–224.
8. B. H. Gilding and L. A. Peletier, The Gauchy problem for an equation in the theory of infiltration, *Arch. Rat. Mech. Anal.* **16** (1976), 127–140.
9. A. Friedman, "Partial Differential Equations of Parabolic Type," Prentice–Hall, Englewood Cliffs, NJ, 1964.
10. N. Itaya, On the temporally global problem of the generalized Burgers' equation, *J. Math. Kyoto University* **14** (1974), 129–177.
11. W. Jäger and Y. G. Lu, Hölder continuity of solutions of degenerate parabolic equations, to appear.
12. O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Ural'ceva, "Linear and Quasilinear Equations of Parabolic Type," Translation of Mathematical Monographs, Vol. 23, Amer. Math. Soc., Providence, RI, 1968.
13. Y. G. Lu, Global Hölder continuous solution of isentropic gas dynamics, *Proc. Roy. Soc. Edinburgh A* **123** (1993), 231–238.
14. Y. G. Lu, An explicit lower bound of viscosity solutions to isentropic gas dynamics and to Euler equation, Preprint 95-18 (SFB 359), Heidelberg University, 1995.
15. J. Nash, *Bull. Soc. Math. France* **90** (1962), 487–497.
16. O. A. Oleinik, A. S. Kalashnikov, and Yui-lin Chzou, The Cauchy problem and boundary problems for equations of the type of unsteady filtration, *Izv. Akad. Nauk SSR Ser. Math.* **22** (1958), 667–704.
17. J. A. Smoller, "Shock Waves and Reaction-Diffusion Equations," Springer-Verlag, New York/Heidelberg/Berlin, 1982.
18. A. I. Vol'pert and S. I. Hudjaev, Cauchy problem for degenerate second order quasilinear parabolic equations, *Math. USSR Sb.* **7**, No. 3 (1969), 365–387.