Enumeration of 2-connected Loopless 4-regular Maps on the Plane

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In this paper rooted (near-) 4-regular maps on the plane are counted with respect to the root-valency, the number of edges, the number of inner faces, the number of non-root vertex loops, the number of non-root vertex blocks, and the number of multi-edges. As special cases, formulae of several types of rooted 4-regular maps such as 2-connected 4-regular planar maps, rooted 2-connected (connected) 4-regular planar maps without loops are also presented. Several known results on 4-regular maps and trees of Tutte are also concluded. Finally, asymptotic formulae for the numbers of those types of maps are given.

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1. INTRODUCTION

We begin with some definitions. Terms mentioned without definition may be found in [3, 9, 19, 22].

— Graphs here are connected and may have loops or multi-edges (or parallel edges as some people called it, sometimes we also use this conception in convenient). A planar map is a graph $G$ drawn on the sphere $S_0$ such that edges intersect at vertices and each component of $S_0 - G$ is a disc called face. Generally, we may define a map on higher surfaces.

— A map is rooted if an edge, a vertex on the edge and a direction along one side of the edge are all distinguished. All maps here are rooted unless special statements are given.

— A graph (map) is $k$-connected if it needs at least $k$ vertices to separate the graph (map) [3]. One may see that this definition is slightly different from that given by Tutte [18]. For instance, a 2-connected graph (map) may have loops which have been excluded by Tutte.

— A (rooted) near-4-regular map is such one having all the vertices 4-valent except possibly the rooted one. It is clear that a near-4-regular map is Eulerian. A map is called near-simple if no loops or multi-edges are permitted except possibly only two parallel edges containing the root-vertex.

4-regular maps are very important for applications in many fields such as rectilinear embedding in VLSI, the Gaussian crossing problem in graph theory, the knot problem in topology and the enumerations of some other types of maps [9–12]. Rooted (near-) 4-regular maps (or their duals: quadrangulations) have been investigated by many scholars. We list them (as far as we know) as follows:

(1) rooted bicubic maps [20];
(2) rooted trees [21];
(3) rooted quadrangulations [5];
(4) rooted c-nets via quadrangulations [13];
(5) rooted one-faced maps on surfaces [23, pp. 212, 213];
(6) rooted 4-regular planar maps [9, pp. 159–166];
(7) rooted near-4-regular planar Eulerian trials [16];
(8) rooted loopless 4-regular maps on the projective plane, torus and the Klein bottle [14–17].

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We expect that several more classes of 4-regular maps could be added to this list. This is main aim of this paper.

**Remark.** Here we regard planar trees, or more generally, maps with one face on surfaces (some people also called them monopoles) as special kinds of near-4-regular maps. A near-4-regular Eulerian trial is one with all its vertices 4-valent except possibly two: one is the root-vertex while the other is a candidate for identifying with the former. One may see that this is a concept quite involved in nonplanar maps. For a survey one may refer to [8] or extensively [2].

## 2. A General Functional Equation for Planar Maps

In this section we shall set up a general equation with up to six more parameters for rooted near-4-regular maps on the sphere which will imply several new results for some classes of maps unhandled before and conclude several known results cited in the list above. But first we should give some more definitions on maps.

Let $U$ be the set of all the rooted near-4-regular maps on the plane and its enumerating functions be as

$$f(x, y, z, t, w, q) = \sum_{M \in U} x^{2m(M)} y^{e(M)} z^{n(M)} t^{\alpha(M)} w^{\beta(M)} q^{\gamma(M)}, \quad (1)$$

where the variables $x$, $y$, $z$, $t$, $w$ and $q$ mark, respectively, the root-valency, the number of edges, the number of inner faces, the number of nonroot-vertex-loops, the number of cut-vertices other than the rooted one and the number of multi-edges. One should pay attention to our definition of multi-edges. That is, if there are two or more parallel edges connecting two vertices, then $q$ counts only once.

The set $U$ should be partitioned into three parts as

$$U = U_{0} + U_{1} + U_{2},$$

where $U_{0}$ has only one element, the vertex map, and

$$U_{1} = \{M | e_{r}(M) \text{ is a loop} \}.$$  

**Lemma 1.** Let $U_{(1)} = \{M - e_{r}(M) | M \in U_{1} \}$. Then $U_{(1)} = U \ominus U$, where ‘\(\ominus\)’ is the $1v$-production of the sets of maps defined in [9, pp. 88, 89].

**Proof.** For a map $M \in U$, the root-edge $e_{r}(M)$ is a loop. The inner and outer regions determined by $e_{r}(M)$ are, respectively, two elements of $U$. Since this procedure is reversible, $U_{(1)} = U \ominus U$.  \(\Box\)

By the above lemma, the enumerating function of $U_{1}$ is

$$f_{1} = x^{2}yzf^{2}. \quad (2)$$

**Lemma 2.** Let $U_{(2)} = \{M \bullet e_{r}(M) | M \in U_{2} \}$. Then $U_{(2)} = U - U_{(2)} - U_{0}$, where $U_{(2)}$ is the set of those in $U$ with their root-valencies 2.

**Proof.** Notice that shrinking of the root-edge of a map in $U_{2}$ will result in an element in $U$ with its root-valency not less than 4. Thus, $U_{(2)}$ is a subset of $U_{0} - U_{(2)} - U_{0}$.

On the other hand, splitting the root-vertex of a map in $U - U_{(2)} - U_{0}$ will lead to a map in $U_{2}$. Hence, $U - U_{(2)} - U_{0} \subseteq U$.  \(\Box\)
By Lemma 2, the enumerating function of $U(2)$ is
\[ f(2) = f - x^2 F_2 - 1, \]
where $F_2$ is the enumerating function of $U(2)$.

The set $U(2)$ may be further parted into several more parts as
\[ U(2) = U^1(2) + U^2(2) + U^3(2), \]
with
\[
U^1(2) = \{ M | e_{r}(M) \text{ is a loop with its inner face of valency } 1 \}; \\
U^2(2) = \{ M | e_{P_{r}}(M) \text{ is a loop and it bounds a face of valency } 1 \},
\]
in which $P$ is the rotation of $M$ at $v_r$.

It is clear that maps of $U^1(2)$ and $U^2(2)$ have the structures as depicted in Figure 1, where there remains a map in $U$ after deleting the corresponding loops.

Since splitting a vertex into two may also increase a pair of multi-edges, we have to divide $U^1(2)$ into two parts as
\[ U^1(2) = U^{11}(2) + U^{12}(2), \]
in which elements in $U^{11}(2)$ have the structures as shown in Figure 2, where what left after deleting a pair of corresponding loops is a pair of maps in $U$.

What follows from the maps with the structures in Figures 1 and 2 are
\[
f^{11}(2) = x^2yz(f - 1), \quad f^{12}(2) = x^4y^2z^2f^2, \\
f^{12}(2) = x^2yz(f - 1) - x^4y^2z^2f^2.
\]

Hence, the contribution of $U^{11}(2)$ to $U$ is
\[ f^1_{2} = \frac{ytq}{x^2} x^4y^2z^2f^2 + \frac{yt}{x^2} (x^2yz(f - 1) - x^4y^2z^2f^2). \]

Similarly, the set $U^{22}(2)$ may be parted as follows:
\[ U^{22}(2) = U^{21}(2) + U^{22}(2), \]
where elements in $U^{21}(2)$ have the structures as shown in Figure 3.
FIGURE 2. Maps in $U_{(2)}^{11}$ which will result in a pair of parallel edges after splitting the root-vertex.

FIGURE 3. Maps in $U_{(2)}^{21}$ which will result in a pair of parallel edges after splitting the root-vertex.

Hence, the contribution of $U_{2}^{2}$ to $U$ is

$$f_{2}^{2} = \frac{yl}{x^{2}}x^{4}y^{2}z^{2}f^{2} + \frac{yl}{x^{2}}(x^{2}yz(f - 1) - x^{4}y^{2}z^{2}f^{2}). \quad (5)$$

Next, we concentrate on the calculation of $f_{2}^{3}$, the contribution of $U_{2}^{3}$ to $U$.

Since splitting of the root-vertex of those in $U_{(2)}^{3}$ may increase a cut-vertex which is not the rooted one and no more loops beyond the root-vertex are increased, the set $U_{(2)}^{3}$ has to be divided into three parts as

$$U_{(2)}^{3} = U_{(2)}^{31} + U_{(2)}^{32} + U_{(2)}^{33}, \quad (6)$$

in which

$$M \in U_{(2)}^{31} \iff M \in U_{(A)} + U_{(B)},$$

where those of $U_{(A)}$ and $U_{(B)}$ are composed of a map in $U_{(2)} - L$ and another in $U - U_{0}$ as depicted in Figure 4 (here $L$ denotes the loop map), i.e.,

**Lemma 3.** $U_{(A)}$ and $U_{(B)}$ are, respectively, the composition of $U_{(2)} - L$ and $U - U_{0}$.

Thus, the contribution of $U_{(A)}$ and $U_{(B)}$ are the same as

$$f_{(A)} = f_{(B)} = x^{2}(F_{2} - yz)(f - 1). \quad (7)$$
FIGURE 4. Two types of maps in $\mathcal{U}_{(2)}^{(1)}$ which will increase a nonroot-cut-vertex after splitting the root-vertex.

FIGURE 5. Maps in $\mathcal{U}_{(A)}^{1} + \mathcal{U}_{(B)}^{1}$ which will result in a pair of parallel edges after splitting the root-vertex.

It is clear that splitting the root-vertex of those in $\mathcal{U}_{(A)}$ may also result in a pair of parallel edges. So, it should be parted according to

$$\mathcal{U}_{(A)} = \mathcal{U}_{(A)}^{1} + \mathcal{U}_{(A)}^{2},$$

where maps in $\mathcal{U}_{(A)}^{1}$ consist of a loop, a map in $\mathcal{U}(2) - \mathcal{U}_{0}$ and a pair of maps in $\mathcal{U}$ as shown in the Figure 5.

So, the contributions of $\mathcal{U}_{(A)}^{1}$ and $\mathcal{U}_{(A)}^{2}$ are, respectively,

$$f_{(A)}^{1} = x^4(F_2 - yz)yzf^2,$$
$$f_{(A)}^{2} = x^2(F_2 - yz)(f - 1) - x^4yz(F_2 - yz)f^2.$$

Further, the enumerating function of $\mathcal{U}_{A}$ is

$$f_{A} = \frac{ywq}{x^2}(x^4(F_2 - yz)yzf^2) + \frac{yw}{x^2}(x^2(F_2 - yz)(f - 1) - x^4yz(F_2 - yz)f^2). \tag{8}$$

The same things happen to $\mathcal{U}_{B}$, i.e., $f_{B} = f_{A}$. Consequently, we have that

$$f_{(2)}^{31} = 2x^2(F_2 - yz)(f - 1),$$
$$f_{(2)}^{32} = 2qywyz(F_2 - yz)f^2 + 2yw((F_2 - yz)(f - 1) - x^2yz(F_2 - yz)f^2).$$

We now turn to $\mathcal{U}_{(2)}^{32}$ defined in (6). By the definition of near-4-regular maps, the following properties are easy for one to test and useful in our arguments to come.

Fact 1. For a map $M \in \mathcal{U}_{(2)}^{32}$, splitting the root-vertex $v_r(M)$ will not increase the number of nonroot-vertex loops or the number of nonroot-vertex cut-vertices but may increase the number of multi-edges.
FIGURE 6. Splitting the root-vertex of a map in $U_{32}^{(2)}$ will result in a group of at least two parallel edges and not increase the cut-vertices.

Fact 2. For any map $M \in U_{32}^{(2)}$, the root-valency of $M$ is at least 6.

Fact 3. For an Eulerian graph (map), its edge-connectivity is an even number. Hence, a map $M \in U_{32}^{(2)}$ if and only if it has a structure as depicted in Figure 6, where the three shadowed regions determined by three nested loops are those of root-valencies 2 while other four are the elements in $U$.

Let $R_1$, $R_2$ and $R_3$ are, respectively, the subsets of $U_{32}^{(2)}$ such that

$R_1 = \{ M | e_r \text{ is a loop} \}$,
$R_2 = \{ M | e_p \text{ is a loop} \}$,
$R_3 = \{ M | e_{p^2} \text{ is a loop} \}$,

where $P$ is the rotation of $M$ at the root-vertex. Then we have that

$U_{32}^{(2)} = \bigcup_{i=1}^{3} R_i$.

For a subset $A$ of $U$, we denote $|A|$ as the contribution of $A$ to $U$. Then by the principle of inclusion–exclusion, the contribution of $U_{32}^{(2)}$ to $U$ is

$|R_1| + |R_2| + |R_3| - \sum_{1 \leq i < j \leq 3} \left| R_i \cap R_j \right| + \left| R_1 \cap R_2 \cap R_3 \right|
= 3x^6yzF_2^2f^4 - 3x^6y^2zF_2f^4 + x^6y^3z^3f^4
= x^6yz(3F_2^2 - 3yzF_2 + (yz)^2)f^4,$

i.e.,

$f_{32}^2 = \frac{yz}{x^2}(x^6yz(3F_2^2 - 3yzF_2 + (yz)^2)f^4)
= qyz^2x^4(3F_2^2 - 3yzF_2 + (yz)^2)f^4.$ (9)
On the other hand, relations (3)—(5) imply that

\[ f_2^3 = f_2^{31} + f_2^{32} + f_2^{33} \]
\[ = 2w(q-1)x^2yz(F_2 - yz)f^2 + 2y(w-1)(F_2 - yz)(f - 1) \]
\[ + (q - 1)x^4y^2z(3F_2^2 - 3yzF_2 + (yz)^2)f^4 \]
\[ + \frac{y}{x^2}(f - 1 - x^2F_2 - 2x^2yz(f - 1)). \]  

(10)

Combining this with (2)—(5) yields

\[ f = 1 + x^2yzf^2 + 2(t-1)y^2z(f - 1) + 2(t-1)x^2y^3z f^2 \]
\[ + 2w(q-1)x^2yz(F_2 - yz)f^2 + 2y(w-1)(F_2 - yz)(f - 1) \]
\[ + (q - 1)x^4y^2z(3F_2^2 - 3yzF_2 + (yz)^2)f^4 + \frac{y}{x^2}(f - 1 - x^2F_2). \]

After rearranging the items in the above equation, we have our first main results.

**Theorem A.** The enumerating function defined in (1) satisfies the following equation:

\[ af^4 + bf^2 + cf + d = 0, \]  

(11)

in which the coefficients of \( f^0 \), \( f \) and \( f^2 \) can be expressed as

\[ a = (q - 1)x^6y^2z(3F_2^2 - 3yzF_2 + (yz)^2), \]
\[ b = (x^4yz + 2(q - 1)y^2z x^4 + 2w(q-1)x^4y^2z(F_2 - yz), \]
\[ c = y - x^2 + 2(t - 1)x^2y^2z + 2x^2y(w-1)(F_2 - yz), \]
\[ d = x^2 - y - x^2yF_2 - 2(t-1)x^2y^2z - 2x^2y(w-1)(F_2 - yz). \]

**Remark.** The equation in Theorem A is really vast and complicated and seems impossible for one to extract a complete function from (11). But it contains up to six more parameters which makes it possible for us to solve (11) completely or determines the corresponding 4-regular maps with special values. This is exactly what we shall do in the next section.

### 3. Calculations

In this section we shall deal with various types of rooted (near-) 4-regular on the plane and all the arguments are under the restriction \( q = 1 \) since only in this case can we solve the corresponding equation(s).

Let \( q = 1 \) (i.e., ignoring the effects of multi-edges). Then the equation in Theorem A becomes

\[ x^4yzf^2 + (y - x^2 + 2(t-1)y^2z x^2 + 2x^2y(w-1)(F_2 - yz)f \]
\[ + x^2 - y - x^2yF_2 - 2(t-1)x^2y^2z - 2x^2y(w-1)(F_2 - yz) = 0. \]  

(12)
Let \( x = x^2 \) and \( \triangle \) be the discriminant of (12). Then we have
\[
\Delta = (y - x + 2(t - 1)x^2z + 2xy(w - 1)F)^2
- 4x^2yz(x - 2xy^2z(t - 1) - 2xy(w - 1)F - y - xy(F + yz)),
\]
in which \( F = F_2 - yz \).

In fact, \( q = 1 \) will turn \( \triangle \) to become a polynomial of \( x \) with degree not great than 3 which makes it possible for one to use the famous double-root methods developed by Brown [4] to solve the corresponding equation and completely determine the exact expression of the enumerating functions.

Now we are allowed to write
\[
\Delta = (y + ux)^2(1 - vx).
\]

Then by comparing the coefficients of \( x, x^2 \) and \( x^3 \) we have the following system of equations:
\[
\begin{align*}
2yu - y^2v &= 2y(-1 + 2y^2z(t - 1) + 2y(w - 1)F); \\
u^2 - 2yuv &= (-1 + 2y^2z(t - 1) + 2y(w - 1)F)^2 + 4y^2z; \\
u^2v &= 4yz[1 - 2y^2z(t - 1) - y^2z - ywF].
\end{align*}
\]

Let \( \theta = u + 1 - 2y^2z(t - 1) - 2y(w - 1)F \). Then \( v = \frac{2\theta}{y} \) and
\[
\theta = \frac{4y^2z}{2 - 4y^2z(t - 1) - 4y(w - 1)F - 3\theta}.
\]

We may extract the function \( f \) from the corresponding equation and expand it into a power series, i.e.,
\[
2yzf = \sum_{m \geq 1} \alpha_m \left\{ 1 - 2y^2z - 2y(w - 1)F - \frac{3m}{m + 1} \theta \right\} \frac{2^m \theta^m x^{2m-2}}{y^m},
\]
where
\[
\alpha_m = \frac{(2m - 2)!}{2^{2m-1}m!(m - 1)!}.
\]

Applying Lagrangian inversion (for a reference one may see [7]) for (14) and \( \theta^m \) one may find that
\[
f = 1 + \sum_{m \geq 2} \sum_{n \geq m} C(m, n, l)[yz(t - 1) + (w - 1)F]^l y^{2n+l-m-1}z^{n-1}x^{2(m-1)}
\]
where
\[
C(m, n, l) = \frac{2^l 3^{n-m}(m - 1)}{(2n - m)(2n - m - 1)} \binom{2m - 2}{m - 1} \binom{2n + l - m - 2}{l} \binom{2n - m}{n}.
\]

Set \( t = w = 1 \). Then we have the following known result:

**Corollary 1** (Liu [9, P. 165]). The enumerating function of rooted planar near-4-regular maps is
\[
1 + \sum_{m \geq 2} \sum_{n \geq m} \frac{3^{n-m}(2m - 2)!(2n - m - 2)!}{(m - 1)!(m - 2)!(n - m)!n!} y^{2n-m-1}z^{n-1}x^{2(m-1)}.
\]
Since maps on surfaces must obey Euler formula, a classical formula for rooted trees may also be concluded.

**Corollary 2 (Tutte [21]).** The enumerating function of rooted trees is

\[ \frac{\partial (yf(x, y, y^{-1}, 1, 1))}{\partial y}\bigg|_{y=0} = \sum_{m \geq 1} \frac{1}{m} \left( \frac{2m - 2}{m - 1} \right) x^{2m-2}. \]

Furthermore, we have the following:

**Corollary 3 ([115]).** The enumerating function of those in \( U \) with exactly \( k \) nonroot-vertex loops is

\[
\frac{1}{k!} \left. \frac{\partial^k f(x, y, z, t, 1, 1)}{\partial t^k} \right|_{t=0} = \sum_{m \geq 2} \sum_{\substack{l \geq k \geq n \geq m \geq 2 \quad \text{n.m} \quad \text{y.m}}} \frac{(-1)^l \cdot 2^l \cdot 3^n - m}{n} \left( \frac{2m - 2}{m - 1} \right) \left( \frac{l + n - 1}{l} \right) \left( \frac{2n + l - m - 1}{n - m} \right) \times \left( \frac{l}{k} \right) \left( \frac{1 + m - 1}{2n + l - m - 1} + 2y^2z(l + 1) \right)^{2(n+l)-m-1} x^{2(m-1)}. \]

**Corollary 4 ([14]).** The enumerating function of rooted planar 4-regular maps without loops is

\[
\frac{1}{y} \left. \frac{\partial^2 f(x, y, z, 0, 1, 1)}{\partial x^2} \right|_{x=0} = yz \cdot \left( \frac{1}{2!} \right) \frac{\partial^k f(x, y, z, t, 1, 1)}{\partial t^k} \bigg|_{t=0}. \]

From this and Lagrangian inversion [7] one may determine the number of rooted planar 4-regular maps without loops, i.e., the number of those having \( 2m \) edges is \( L_m = 2L_{m-1} \), where

\[
L_m = \sum_{l+n=m} \left( \frac{(-1)^l \cdot 2^l \cdot 3^n}{(n+1)(2n+1)} \right) \left( \frac{2n + 2}{n} \right) \left( \frac{n + m}{l} \right). \]

If we consider the case of \( y = q = t = 1 \) and \( w = 0 \), then we will have results for rooted 2-connected near-4-regular planar maps without nonroot-vertex loops (i.e., the root-vertex is the only possible cut-vertex).

**Theorem B.** The enumerating function of those in \( U \) having no nonroot-cut-vertices is

\[
f = 1 + \sum_{m \geq 2} \sum_{n \geq m} (-1)^l \cdot C(m, n, l) F^l \cdot 3^{n+l-m-1} \cdot x^{2(m-1)}. \]

**Remark.** The function in the above theorem heavily depends on the parameter \( F \) which will be handled further.

Recall that under the condition of \( y = q = t = 1 \) and \( w = 0 \) the discriminant of (12) becomes

\[ \Delta = (1 - x - 2xF)^2 - 4x^2z(x - 1 - xz + xF), \]

where \( x = x^2 \) as we have assumed at the beginning of this section. Suppose that \( x \) is a multi-root of \( \Delta \). Then from the double-root method [4] and \( \Delta = \frac{\partial \Delta}{\partial x} = 0 \) one may see that

\[
F = \frac{1 - x + 2x^3z(1 - z)}{2x(1 - x^2z)}; \quad x = 1 + \frac{z}{2} \phi(x, z), \quad \phi(x, z) = \frac{x^2(1 + x - 2xz)^2}{(1 + 2xz)(1 - x^2z)}. \]
This together with an application of Lagrangian inversion for the above relation provides a way to expand \( F \) (or \( F^1 \)) into a power series of \( z \), i.e.,

\[
F = z + \sum_{m \geq 1} \frac{1}{m!} \left( \frac{z}{2} \right)^m D_{x=1}^{(m-1)} \left\{ \frac{\partial F}{\partial x} \phi^m \right\},
\]

where the operator is defined as

\[
D_{x=1}^{(m-1)} f = \left. \frac{\partial^{m-1} f}{\partial x^{m-1}} \right|_{x=1}.
\]

Since

\[
\frac{\partial}{\partial x} \phi^m = \frac{x^{2m-2}((x^2 z - 1) + 2z x^2 z(1 + x - 2x z))(1 + x - 2x z)^{2m}}{2(1 - x^2 z)^{m+2}(1 + 2xz)^m},
\]

we obtain

\[
\frac{1}{m!} D_{x=1}^{(m-1)} \left\{ \frac{\partial F}{\partial x} \phi^m \right\} = \sum_{\substack{i,j \geq 0 \atop 2m \geq k \geq 0}} A(i, j, k, 2m) z^{i+j+1}(1-2z)^k
\]

\[
- \sum_{\substack{i,j \geq 0 \atop 2m \geq k \geq 0}} B(i, j, k, 2m) z^{i+j}(1-2z)^k
\]

\[
+ \sum_{\substack{i,j \geq 0 \atop 2m+1 \geq k \geq 0}} B(i, j, k, 2m) z^{i+j+1}(1-2z)^k,
\]

in which

\[
A(i, j, k, 2m) = \frac{L(i, j, k, l, 2m)}{m+2i+j+k+1} \binom{2m+2i+j+k}{m}:
\]

\[
B(i, j, k, 2m) = \frac{L(i, j, k, 2m)}{m+2i+j+k-1} \binom{2m+2i+j+k-2}{m}:
\]

\[
C(i, j, k, 2m) = \frac{2L(i, j, k, 2m)}{m+2i+j+k+1} \binom{2m+2i+j+k}{m}:
\]

\[
L(i, j, k, 2m) = (-1)^{i+j-1} \binom{m+i+1}{i} \binom{m+j-1}{j} \binom{2m}{k}:
\]

Based on this we may expand the function \( F \) into a power series of \( z \), i.e.,

\[
F = 2z^2 + 5z^3 + 20z^4 + 114z^5 + 758z^6 + 5461z^7 + 723795z^8 + 1827535z^9 + \cdots.
\]

One may see that the initial values of maps coincide with our calculations. For instance, there are 20 distinct rooted 2-connected 4-regular planar maps with four inner faces which are determined by the elements of \( \mathcal{L}(2) \) as depicted in Figure 7.

Although the function \( F \) presents a way to count a type of rooted 2-connected 4-regular planar maps, we may easily see that if loops are permitted, then there are another type of maps which must be considered. In fact, rooted 2-connected 4-regular planar maps consists of three types of maps according to whether the other end of the root-edge is a cut-vertex or not as depicted in Figure 8.
Figure 7. Twenty rooted 2-connected near-4-regular planar maps which correspond to 20 4-regular maps with four inner faces.

Figure 8. Three types of 2-valent maps which will result in three kinds of 4-regular planar maps.

Accordingly, we have the following:

Theorem C. The enumerating function of rooted 2-connected 4-regular planar maps is

\[ F(4) = (1 + 2z)F, \]

where \( F \) is defined in Theorem B.

By using an algebraic symbolic system such as MAPLE, the first few coefficients of the function \( F(4) \) may be determined:

\[ F(4) = 2z^2 + 9z^3 + 30z^4 + 154z^5 + 986z^6 + 6977z^7 + 52590z^8 + 415678z^9 + \cdots. \]

We may see that this formula coincides with the initial values of maps. For instance, there are a total of 30 rooted 2-connected 4-regular planar maps among which 20 are induced by those shown in Figure 7 while other ten are determined by the following group of 2-valent maps in Figure 9.

Remark. Liu raised an open problem for enumerating rooted 2-connected 4-regular maps on the plane [9, p. 167]. Late, Cai proposed the same question as cited in our acknowledgement. The above exact expression gives a solution to this question in a weak sense since we have assumed that a 2-connected graph (or corresponding map) may have loops. If we follow Tutte’s definition [18] of connectivity a graph (map) is \( k \)-connected if and only if the girth is at least \( k \) and it requires at least \( k \) vertices to separate the graph (map). Then it follows that a 2-connected graph (map) has no loops. One may see late that even in this case we may also give an affirmative answer to this problem.
Let \( t = w = 0 \) and \( y = q = 1 \). Then no loops or cut-vertices will be allowed to appear on the vertices other than the rooted one. It is clear that maps in \( \mathcal{I}(2) - L \) are all 2-connected without loops and have root-valencies 2. This time the discriminant of (12) is turned into
\[
\Delta = (1 - x - 2xz - 2xF)^2 - 4x^2z(x - 1 + xz + xF).
\]

Again by using the double-root method one may find that
\[
F = \frac{1 - x - 2xz + 2x^3z(1 + z)}{2x(1 - x^2z)};
\]
\[
x = 1 + z\phi(x, z), \quad \phi(x, z) = x^2 \left\{ \frac{(1 + x)^2}{2(1 - x^2z)} - 2 \right\}.
\]

**THEOREM D.** The enumerating function of rooted 2-connected 4-regular planar maps without loops is
\[
F = \sum_{m \geq 1} \frac{z^m}{2m+1 \cdot m!} \left\{ \frac{x^{2m-2}(x^2(3 + 2x)z - 1)((1 + x)^2 - 4(1 - x^2z))^m}{(1 - x^2z)^{m+1}} \right\}
\]
\[+ \frac{z^2}{1 - z}, \]
i.e.,
\[
F = \frac{z^2}{1 - z} + \sum_{m \geq 1} \sum_{i \geq 0} \frac{2^j \cdot m!}{m!} \binom{m+i}{i} \binom{m-1}{j} A(i, j, m) z^{m+i+j+1}
\]
\[= \sum_{i \geq 0} \sum_{m \geq 1} \sum_{j \geq 0} \frac{2^j \cdot m!}{m!} \binom{m+i}{i} \binom{m-1}{j} B(i, j, m) z^{m+i+j}
\]
with
\[
A(i, j, m) = D_{i+1}^{(m-1)} (x^{2m+2i+2j}(x - 3)^{m-j}(x - 1)^{m-j}(3 + 2x)),
\]
\[
B(i, j, m) = D_{i+1}^{(m-1)} (x^{2m+2i+2j-2}(x - 3)^{m-j}(x - 1)^{m-j}).
\]
According to this, we may expand the enumerating function of rooted 2-connected 4-regular planar maps without loops into a power series of \( z \), i.e.,
\[
F = z^3 + 2z^4 + 10z^5 + 42z^6 + 209z^7 + 1066z^8 + 5726z^9 + \cdots.
\]

Figure 10 shows a group of 3 distinct unrooted planar maps which will induce 10 rooted 2-connected loopless 4-regular planar maps with five inner faces.
4. Asymptotic Evaluations

In this section we shall evaluate the asymptotics of the numbers of the two types of 2-connected planar 4-regular maps obtained in the previous sections. We first state some basic facts which are useful in our evaluations.

Fact 4. Let \( \mathcal{M} \) be a set of infinite rooted planar maps with \( \mathcal{M}_1 \) as its subset. Suppose that their enumerating functions may be expanded into power series such as

\[
\begin{align*}
f_{\mathcal{M}}(z) &= \sum a_k z^k, \\
f_{\mathcal{M}_1}(z) &= \sum b_k z^k.
\end{align*}
\]

If \( R \) and \( R_1 \) are, respectively, the convergence radius of \( f_{\mathcal{M}}(z) \) and \( f_{\mathcal{M}_1}(z) \), then \( R \) and \( R_1 \) are, respectively, the singularities of \( f_{\mathcal{M}}(z) \) and \( f_{\mathcal{M}_1}(z) \). Furthermore, \( R \leq R_1 \).

Fact 5 [21]. The convergence radius of the power series expansion with the number of inner faces as the parameter of the enumerating function for the rooted planar 4-regular maps is \( \frac{1}{12} \). Furthermore, any kind of infinite rooted 4-regular planar maps must have its convergence radius, say \( r \), must satisfy

\[
\frac{1}{12} \leq r < 1.
\]

Let us begin with the asymptotic evaluation of the number of rooted 2-connected 4-regular planar maps (which may have loops). Let

\[
\begin{align*}
F &= \frac{1 - x + 2x^3z(1 - z)}{2x(1 - x^2z)}, \\
x &= 1 + \frac{z}{2} \frac{x^2(1 + x - 2xz)^2}{(1 + 2xz)(1 - x^2z)}. \tag{15}
\end{align*}
\]

Fact 6. All the possible singularities of the function \( F \) defined above are determined by eqn. (15).

In fact, by the definition of \( F \), its singularities are either induced by \( x = 0 \) or contained in those of (15). If a singularity, say \( z_0 \), is determined by \( x = 0 \), then it will also satisfy (15). A contradiction.
We now investigate the singularities of (15). The equation (15) may be rewritten as

\[ H(x, z) = 2(x - 1)(1 - x^2z) - zx^2(1 + x - 2xz)^2 = 0. \]  

(16)

By the implicit function theorem, (16) will determine a function \( x = x(z) \) with its singularities also satisfying \( \frac{\partial H(x, z)}{\partial x} = 0 \). From this we may obtain a group of two equations as

\[
\begin{align*}
2(x - 1)(1 - x^2z) - zx^2(1 + x - 2xz)^2 &= 0, \\
2z(1 - 2z)x^3 - 5zx^2 - 3(1 - 2z)x + 4 &= 0.
\end{align*}
\]

(17)

After simplification we may extract the variable \( z \) as

\[ z = \frac{6x^3 - 39x^2 + 62x - 24}{x(8x^4 - 20x^3 + 57x^2 - 76x + 36)}. \]  

(18)

Substitute this into (17) we find a high order equation for \( x \), i.e.,

\[ 96x^6 - 112x^5 - 324x^4 + 1296x^3 - 1012x^2 - 2664x^1 + 4224x^1 - 928x^2 - 576x = 0. \]  

(19)

In order to solve this equation and simplify our further calculations we have to factorize the left-hand side of (19) as

\[ 4x(x^2 - 2x + 4)(24x^2 - 28x - 9)(x + 2)^2(x - 1)^2. \]

Hence, all the wanted zeros are

\[ \begin{align*}
x_1 &= 0, & x_2 = x_3 &= 1, & x_4 = x_5 &= -2, & x_6 &= 1 - \sqrt{3}i, \\
x_7 &= 1 + \sqrt{3}i, & x_8 &= \frac{7 + \sqrt{103}}{12}, & x_9 &= \frac{7 - \sqrt{103}}{12}.
\end{align*} \]

By (18), the corresponding singularities are

\[ \begin{align*}
z_1 &= \infty, & z_2 = z_3 &= 1, & z_4 = z_5 &= \frac{i}{3}, & z_6 &= 0.2500000000 - 0.4330127016i, \\
z_7 &= 0.2500000000 + 0.4330127016i, & z_8 &= 0.0918103549, & z_9 &= 2.723004459.
\end{align*} \]

By a result as we will show latter that the convergence radius of the enumerating function for the rooted 2-connected loopless 4-regular planar maps is \( \frac{27}{196} \), together with the Facts 4 and 5, one may see that the only possible singularity is \( z_8 \) and the corresponding value of \( x \) is \( x_8 = \frac{7 + \sqrt{103}}{12} \).

Let \( m = z_8 \) be the convergence radius of \( F \). In order to apply Darboux’s method [1, Theorem 4], we have to investigate the behaviour of \( x = x(z) \) near \( m \) which will lead to the asymptotics of the number of maps considered. Since both \( \frac{\partial H(x, z)}{\partial x} \) and \( \frac{\partial H(x, z)}{\partial z} \) are not equal to zero at \( m \), the implicit function \( x = x(z) \) may be expanded into a power series of \( (z - m)^{\frac{1}{2}} \) near \( m \). Let

\[ \begin{align*}
x &= a + b\sqrt{1 - \frac{z}{m}} + c\left(1 - \frac{z}{m}\right) + d\left(1 - \frac{z}{m}\right)^{\frac{1}{2}}, \\
z &= m - m\left(1 - \frac{z}{m}\right).
\end{align*} \]

(20)
Substitute those into the equation system (15) and let the coefficients of \((1 - \frac{z}{m})^3\) be zero. Furthermore, we use 24\(a^2 - 28a - 9\) to reduce the orders of the corresponding polynomials appearing in the identity we obtained. Then we have a group of relations such as
\[
a = x_8, \quad b^2 = \frac{\alpha}{\beta}, \quad d = \frac{p}{q},
\]
\[
c = m(8a^3m - 12mb^2a - 2a^3 + 5a^2 + 6b^2a - 6a - 5b^2) = -6m + 12m^2a^2 - 6ma^2 + 10ma + 3.
\]
where
\[
\alpha = -\frac{1}{6718464}(-41713449335m^3 + 250833038930m^4
- 782701056 + 5892516720m - 570212178016m^5 - 14112941904m^2
+ 850504242688m^6 + 108370079216m^8 - 639600563984m^7)(1 + 2m)a
- \frac{1}{2985984}(-2073097368m^2 + 912571920m - 4611953453m^3
- 127650816 + 29236351190m^4 - 66494831920m^5 + 99188769952m^6
- 7459357232m^7 + 12638804000m^8)(1 + 2m),
\]
\[
\beta = \left( \frac{26564888}{729}m^8 + 36 + \frac{265175224}{324}m^5 - \frac{4074965}{729}m^2 - \frac{123273008}{729} \right) a
+ 2388m + \frac{3840665}{54}m^3 - \frac{302624902}{729}m^6 + \frac{397907300}{729}m^7 + \frac{253344784}{729}a
+ \frac{12977}{16}m + \frac{18561181}{1296}m^7 + \frac{501300547}{144}m^3 - \frac{742181}{258501473}m^6
+ \frac{24}{324}m^8 + \frac{29546737}{324}m^9 + \frac{18284803}{768}m^3 - \frac{258501473}{5184}m^4,
\]
\[
p = 2mb(-36m^2a^2 + 61ma^2 + 100m^2b^2a - 9 + 48m^3a^4 + 18m + 120m^3a^2b^2
- 36m^2a^4 - 50mb^2a + 15a - 120m^2a^2b^2 - 20ma^3 + 6ma^4 + 30ma^2b^2
+ 80m^2a^3 - 9a^2 + 3b^2 + 13mb^2 - 30ma + 12m^2b^2),
\]
\[
q = (-6m + 12ma^2 - 6ma^2 + 10ma + 3)^2.
\]

Note that we only need to expand \(x(z)\) into the form of \(x = a + b\sqrt{1 - \frac{z}{m}} + c(1 - \frac{z}{m}) + d\left(1 - \frac{z}{m}\right)^3\) here. If needed, higher terms have to be introduced and hence more parameters such as \(a, b, c, d\) will appear.

We now substitute (20) into \(F\) and find that
\[
F = H_0 + H_1(1 - \frac{z}{m})^\frac{1}{2} + H_2(1 - \frac{z}{m}) + H_3(1 - \frac{z}{m})^3 + \cdots,
\]
where \(H_1 = 0, H_3 = \frac{A_3}{m}\) with
\[
A_3 = \left( -\frac{242}{27} mbc + 2bc + \frac{2311}{1296} b^3 m^2 + \frac{50191}{2592} dm - \frac{24655}{2592} mb + \frac{35939521}{13436928} m^3 b
+ \frac{1583114113}{13436928} m^3 d + \frac{14}{3} b^3 m + \frac{3888}{729} m^2 cb - \frac{394962625}{6718464} m^4 d - \frac{7}{6} d
- \frac{2686067}{46656} m^4 b^3 + \frac{4753793}{93312} m^3 b^3 - \frac{455669}{5832} dm^2 + \frac{281879}{15552} m^2 b + \frac{1439459}{11664} m^4 bc \right)
\]
By Darboux’s method we may obtain an asymptotic expression for the coefficients of the power series in $z$ of $F$, i.e., the following theorem.

**Theorem E.** The number of rooted 2-connected 4-regular planar maps (which may have loops) with $n+1$ faces is asymptotic to

$$H_3(1 + 2m) \frac{1}{n^2 m^n \Gamma(-\frac{1}{2})} \left(1 + O\left(\frac{1}{n}\right)\right),$$

where $m = 0.0918103549$, $H_3 = 0.2221398938$.

Next, we study the asymptotics of the number of rooted 2-connected 4-regular planar maps without loops.

We are now faced with another group of relations as follows:

$$G = \frac{1 - x - 2xz + 2x^3z(1 + z)}{2x(1 - x^2z)},$$

$$x = 1 + z(x^2 \left[\frac{(1 + x)^2}{2(1 - x^2z)} - 2\right])$$

which were defined in Theorem C.

As we have reasoned in the proof of Fact 6, all the singularities of $G$ appear in (21). By derivating (21) for $x$ one may find another group of two equations in the form of

$$zx^2[(1 + x)^2 - 4(1 - x^2z)] - 2(x - 1)(1 - x^2z) = 0,$$

$$z(1 + 4z)x^3 + 3zx^2 - \frac{5}{2}zx - \frac{1}{2} = 0.$$  

(22)

Solving the above equations for $z$ we obtain that

$$z = \frac{3x - 4}{x^2(2x - 5)}.$$  

Substitute this into (22) we find an equation for $x$ in the form of

$$(3x - 4)(x^2(2x - 5) + 12x - 16) + 3x(3x - 4)(2x - 5) - \frac{5}{2}(3x - 4)(2x - 5) - \frac{1}{2}x(2x - 5)^2 = 0.$$  

(23)

Factorizing the left-hand side of the above equation we have that

$$(6x - 7)(x - 2)(x + 1)^2 = 0.$$  

Thus, the zeros and the corresponding singularities are listed in the following table:

<table>
<thead>
<tr>
<th>x</th>
<th>7/6</th>
<th>2</th>
<th>-1</th>
<th>-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>z</td>
<td>27/196</td>
<td>-2</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

One may see from Fact 5 that \( z = 1 \) will not be the convergence radius as expected. If \( z = -\frac{1}{2} \) is a wanted singularity which also corresponds to the convergence radius, then by Fact 4 \( z = \frac{1}{2} \) must be a singularity of \( G \), a contradiction with the fact that all the singularities are listed in the above table. Thus, the only possible candidate for the wanted singularity corresponding to the convergence radius is \( R_1 = \frac{27}{196} \). This also fills a gap left in our proof that \( x_8 = 7 + \sqrt{103} \) corresponds to the convergence radius for the rooted 2-connected 4-regular planar maps before.

After a very similar procedure in our proof of Theorem D, we may expand \( G \) into a power series of \( \sqrt{1 - \frac{z}{R_1}} \) such that

\[
G = G_0 + G_1 \left( 1 - \frac{z}{R_1} \right)^{\frac{1}{2}} + G_2 \left( 1 - \frac{z}{R_1} \right) + G_3 \left( 1 - \frac{z}{R_1} \right)^{\frac{3}{2}} + \cdots ,
\]

where \( G_1 = 0 \), \( G_3 = \frac{C_3}{D_3} \) with

\[
C_3 = \frac{b}{544195584} (-668675284992 b^2 + 7907772096 + 1522802804544 m b^2 + 3402216385344 m^2 b^2 - 79668752688 m + 6231629731392 m^5 b^2 - 7529660237344 m^5 - 11535327241236 m^3 b^2 + 1106778379189 m^3 + 16933019154912 m^4 b^2 - 1698232726260 m^2 + 7345486374996 m^4 ) ,
\]

\[
D_3 = \frac{618376825392666025}{705277476864} ,
\]

in which

\[
a = \frac{7}{6} , \quad b^2 = \frac{b_1}{b_2} , \quad d = \frac{d_1}{d_2} , \quad m = \frac{27}{196} ,
\]

\[
b_1 = \frac{16357}{1957561711} m^4 + \frac{5832}{1957561711} m^3 + \frac{2420791051}{1957561711} m^2 + \frac{2635089576}{1957561711} m + 629856 ,
\]

\[
b_2 = -\frac{1822}{3} + \frac{2010049}{162} m + \frac{34992}{81} m^2 + \frac{1014604976}{2187} m^3 ,
\]

\[
c = -5 + 12 a + 6 a^2 + 24 m a^2 - 6 b^2 - 6 a b^2 - 24 a m b^2 + 16 m a^3 + 2 a^3 ,
\]

\[
d_1 = b (12 a^4 + 144 m a^4 + 384 m^2 a^4 + 48 a^3 + 384 m a^3 + 60 a^2 b^2 + 480 m a^2 b^2 - 120 m a^2 + 72 a^2 + 960 m^2 a^2 b^2 + 480 m a b^2 - 60 a + 120 a b^2 + 25 + 40 m b^2 + 82 b^2 ) ,
\]

\[
d_2 = (-5 + 12 a + 6 a^2 + 24 m a^2 )^2 .
\]

Again Darboux’s method applies and we arrive at
THEOREM F. The number of rooted 2-connected 4-regular planar maps without loops with $n+1$ faces is asymptotic to

$$\frac{G_3}{n^5 m^9 \Gamma\left(-\frac{3}{2}\right)} \left(1 + O\left(\frac{1}{n}\right)\right),$$

where $m = \frac{27}{196}$, $G_3 = 0.001787827325$.

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REFERENCES

6. Z. C. Gao, I. M. Wanless and N. C. Wormald, Counting 5-connected planar triangulations, private communication.