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Tropical determinant of integer doubly-stochastic matrices

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ABSTRACT

Let $\mathcal{D}(m, n)$ be the set of all the integer points in the m-dilate of the Birkhoff polytope of doubly-stochastic $n \times n$ matrices. In this paper we find the sharp upper bound on the tropical determinant over the set $\mathcal{D}(m, n)$. We define a version of the tropical determinant where the maximum over all the transversals in a matrix is replaced with the minimum and then find the sharp lower bound on thus defined tropical determinant over $\mathcal{D}(m, n)$.

1. Introduction

We start with the following problem. Consider usual Rubik's cube with nine square stickers on each side colored in one of six colors. We want to solve Rubik's cube by peeling off the stickers and replacing them so that each of the faces has all stickers of one color. Doing this, we will ignore the structure of the Rubik's cube. For example, if initially the (solved) cube had the blue and the green squares on opposite faces, after removing and replacing the stickers we may end up having blue and green faces adjacent to each other. Here is our first problem.

Problem 1. How many stickers we would need to peel off and replace in the worst case scenario?

More generally, assume that we have n pails with m balls in each. Each ball is colored in one of n colors and we have m balls of each color. Same question:

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Problem 2. How many balls do we need to move from one pail to another in the worst case scenario so that the balls are sorted by color?

Consider an $n \times n$ matrix A where the rows represent the colors of the stickers (or balls) and the columns represent the faces of the cube (or pails). In the (i, j)th position in A we record the number of stickers (balls) of color i on face (in pail) j. All the entries of matrix A are nonnegative integers and the row and column sums of A are equal to m. We call such matrices *integer doubly-stochastic*.

We would like to assign each face a color so that the overall number of stickers that we need to move is the smallest possible. In other words, we would like to find a transversal of A with the largest possible sum of entries. That is, Problem 2 reformulates as:

Problem 3. Given positive integers m and n, find the sharp lower bound $L(m, n)$ on

$$\max_{\sigma \in S_n}\{a_{1\sigma(1)} + a_{2\sigma(2)} + \cdots + a_{n\sigma(n)}\}$$

over the set of all integer doubly-stochastic $n \times n$ matrices $A = (a_{ij})$ whose row and column sums are equal to m. The answer to Problem 2 would then be $mn - L(m, n)$.

The quantity above looks very similar to the determinant of a matrix, where multiplication is replaced with addition while the addition is replaced with taking maximum. Even more, it is almost identical with the definition of the *tropical determinant*

$$\operatorname{tropdet} A = \min_{\sigma \in S_n}\{a_{1\sigma(1)} + a_{2\sigma(2)} + \cdots + a_{n\sigma(n)}\},$$

which is related to the classical assignment problem. Consider n workers and n jobs. Let worker i charge a_{ij} dollars for job j. We would like to assign the jobs, one for each worker, so that the overall cost is as small as possible. Clearly, the tropical determinant solves this problem. A polynomial-time algorithm for solving the assignment problem was developed by Kuhn in 1955 [7]. See [2] for modern treatment and various versions of the problem as well as an overview of the literature.

In 1926 van der Waerden conjectured that the smallest value of the permanent of $n \times n$ doubly-stochastic (with row and column sums equal to one) matrices is attained on the matrix all of whose entries are equal to $1/n$, and this minimum is attained only once. This conjecture was proved independently by Egorychev [4] and Falikman [5] in 1979/80. In [3] Burkard and Butkovich proved a tropical version of the conjecture, where the permanent is replaced with the tropical determinant. Our results provide an integral tropical version of the van der Waerden conjecture.

It is natural to pose a question similar to Problem 3 where we compute the upper bound on the usual tropical determinant.

Problem 4. Given positive integers m and n, find the sharp upper bound $U(m, n)$ on

$$\operatorname{tropdet} A = \min_{\sigma \in S_n}\{a_{1\sigma(1)} + a_{2\sigma(2)} + \cdots + a_{n\sigma(n)}\}$$

over the set of all integer doubly-stochastic $n \times n$ matrices A whose row and column sums are equal to m.

The set of all doubly-stochastic $n \times n$ matrices forms a convex polytope in $\mathbb{R}^{n^2}$ [1], an m-dilate of the Birkhoff polytope. The tropical determinant defines a piece-wise linear function on that polytope. We would like to minimize that function over the integer points of the polytope, so the question we are interested in is an integer linear-programming problem.

In this paper we solve Problem 3 and Problem 4 completely. First three sections of the paper are devoted to Problem 3. In the last section we solve Problem 4, which turns out to be significantly easier than Problem 3. Our methods are elementary, combinatorial in nature; one of our tools is Hall's marriage theorem.

2. Definitions, examples, and easy cases

Definition 2.1. Let $A = (a_{ij})$ be an $n \times n$ matrix. Define

$$\text{tropdet}\, A = \min_{\sigma \in S_n} \{a_{1\sigma(1)} + a_{2\sigma(2)} + \cdots + a_{n\sigma(n)}\}$$

$$\text{tdet}\, A = \max_{\sigma \in S_n} \{a_{1\sigma(1)} + a_{2\sigma(2)} + \cdots + a_{n\sigma(n)}\}.$$

We will refer to both of these quantities as the tropical determinant of A, which should not cause confusion since throughout the paper we will mostly be dealing with tdet A except for the last section of the paper which is devoted to tropdet A.

Definition 2.2. Let A be an $n \times n$ matrix with non-negative integer entries. We say that A is integer doubly stochastic with sums m if the entries in each of the rows and columns sum up to a fixed $m \in \mathbb{N}$. We will denote the set of all such matrices by $\mathcal{D}(m, n)$.

Then Problem 3 reformulates as:

Problem 5. Fix $m, n \in \mathbb{N}$. Find the sharp lower bound $L(m, n)$ on the tropical determinant tdet A over the set $\mathcal{D}(m, n)$.

Example 1. Let $n = 5, m = 7$

$$\text{tdet} \begin{pmatrix} 1 & 0 & 2 & \boxed{2} & 2 \\ 0 & 1 & \boxed{2} & 2 & 2 \\ 2 & \boxed{2} & 1 & 1 & 1 \\ \boxed{2} & 2 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 & \boxed{1} \end{pmatrix} = 9.$$

We will later show that $L(7, 5) = 9$, that is, the minimum of the tropical determinant on the set of 5×5 doubly-stochastic integer matrices with sums 7 is attained on this matrix.

One of our tools is Hall's marriage theorem. The theorem in our formulation deals with a block of zeroes in a matrix with the largest sum of dimensions after all possible swaps of columns and rows. We will refer to such a block as the largest block of zeroes.

Theorem 2.3 (Philip Hall [6]). *Let A be an $n \times n$ 0–1 matrix. Then there is a transversal in A that consists of all 1's if and only if the largest block of zeroes in A has sum of dimensions less than or equal to n.*

Here the theorem is formulated in its weakest form and it can be easily proved by induction on n. For our future discussion we will need two of its corollaries.

Corollary 2.4. *Let A be an $m \times n$ 0–1 matrix. Then there is a transversal that consists of all 1's if and only if the largest block of zeroes in A has sum of dimensions less than or equal to* $\max(m, n)$.

Proof. Let us assume that $m \geqslant n$. Extend A to a square 0–1 matrix by appending to A $m - n$ columns consisting of all 1's and apply Hall's marriage theorem to the resulting matrix. □

Let A be an $n \times n$ 0–1 matrix and W be the block of zeroes in A with the largest sum of dimensions. Then after some row and column swaps A can be written in the form

$$A = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}.$$

Corollary 2.5. *Each of Y and Z has a transversal that consists of all 1's.*

Proof. Let W be of size d_1 by d_2 and let the largest block of zeroes in Y be of size s_1 by s_2. We can assume that the block of zeroes is in the lower right corner of Y, right on top of the zero block W. Then the lower right $s_1 + d_1$ by s_2 block of A consists of all zeroes and hence $s_1 + d_1 + s_2 \leqslant d_1 + d_2$, so $s_1 + s_2 \leqslant d_2$ and by Corollary 2.4 there exists a transversal in Y that consists of all 1's. Similarly, such a transversal exists in Z. $\square$

Here is the first instance where we are going to apply Hall's marriage theorem to integer doubly-stochastic matrices.

Lemma 2.6. *Let $A \in \mathcal{D}(m, n)$. Then A has a transversal all of whose entries are nonzero.*

Proof. Let's rearrange the rows and columns of A so that the block of zeroes with the largest sum of dimensions is in the lower right corner of A. That is, A is of the form

$$\begin{pmatrix} B & & C & \\ & 0 & \cdots & 0 \\ D & \vdots & \ddots & \vdots \\ & 0 & \cdots & 0 \end{pmatrix}.$$

Let us assume that the block of zeroes is of size $r \times s$. Let Σ_B and Σ_C denote the sums of all the entries in the matrices B and C correspondingly. Then we have

$$\Sigma_C = sm$$
$$\Sigma_B + \Sigma_C = (n - r)m.$$

Hence $\Sigma_B = (n - r - s)m$, which implies $r + s \leqslant n$. By Hall's marriage theorem A has a transversal all of whose entries are nonzero. $\square$

Proposition 2.7. $L(m, n) \leqslant L(m + 1, n)$.

Proof. It is enough to check that for each matrix $A \in \mathcal{D}(m+1, n)$ there is a matrix $A' \in \mathcal{D}(m, n)$ such that

$$\operatorname{tdet} A' \leqslant \operatorname{tdet} A.$$

By Lemma 2.6, A has a transversal all of whose entries are nonzero. Let A' be obtained from A by subtracting 1 from each element on such a transversal. Clearly, $A' \in \mathcal{D}(m, n)$ and $\operatorname{tdet} A' \leqslant \operatorname{tdet} A$. $\square$

Proposition 2.8. *If $1 \leqslant m \leqslant n$ then $L(m, n) = n$.*

Proof. Clearly, $L(1, n) = n$. Hence by Proposition 2.7 $L(m, n) \geqslant n$. To show that $L(m, n)$ equals n, we need to construct a matrix in $\mathcal{D}(m, n)$ with tropical determinant n. Let the first row of this matrix have m 1's in the first m slots and zeroes in the rest. The second row is the shift of the first row by one slot to the right, etc. For example, here is a matrix in $\mathcal{D}(4, 6)$ whose tropical determinant is 6.

$$\operatorname{tdet}\begin{pmatrix} 1&1&1&1&0&0\\ 0&1&1&1&1&0\\ 0&0&1&1&1&1\\ 1&0&0&1&1&1\\ 1&1&0&0&1&1\\ 1&1&1&0&0&1 \end{pmatrix} = 6. \quad \square$$

Lemma 2.9. *$L(m, n) \geqslant m$.*

Proof. If $A \in \mathcal{D}(m, n)$, the sum of its entries is mn. On the other hand, the sum of all the entries of A is the sum of its entries in n transversals, which is less than or equal to n times the tropical determinant of A. Hence, $\operatorname{tdet} A \geqslant m$ and $L(m, n) \geqslant m$. □

Proposition 2.10. *If m is a multiple of n, then $L(m, n) = m$.*

Proof. By the previous proposition, we only need to find a matrix in $\mathcal{D}(m, n)$ whose tropical determinant is m. Let $m = qn$ for some $q \in N$ and then the matrix is qI_n, where I_n is the $n \times n$ matrix that consists of all 1's. □

Proposition 2.10 also follows from the tropical van der Waerden conjecture proved in [3].

3. Not so easy cases

Let now $m = qn + r$, where $0 \leqslant r < n$. We will provide lower bounds on the tropical determinant of $A \in \mathcal{D}(m, n)$, first for the case when $r \geqslant n/2$, and next for the case when $r < n/2$. Our first bound will turn out to be sharp, while the second bound will be sharp only under the additional assumption that $qr \geqslant n - 2r$.

Theorem 3.1. *Let $m = qn + r$, where $n/2 \leqslant r < n$. Then for any matrix $A \in \mathcal{D}(m, n)$ we have*

$$\operatorname{tdet} A \geqslant m + (n - r) = n(q + 1).$$

That is, $L(m, n) \geqslant n(q + 1)$.

Proof. Let us assume that there exists a matrix $A \in \mathcal{D}(m, n)$ such that

$$\operatorname{tdet} A < m + (n - r) = n(q + 1).$$

We rearrange rows and columns of A so that the tropical determinant is equal to the sum of entries on the main diagonal of A and the entries are decreasing along the main diagonal. That is, we may assume that

$$A = \begin{pmatrix} a_1 & & & & b_1 \\ & a_2 & & & \vdots \\ & & \ddots & & \vdots \\ & & & a_{n-1} & b_{n-1} \\ c_1 & \dots & \dots & c_{n-1} & a_n \end{pmatrix} \in \mathcal{D}(m, n), \tag{3.1}$$

where $a_1 \geqslant \cdots \geqslant a_n$. By our assumption,

$$\operatorname{tdet} A = a_1 + \cdots + a_n < n(q+1),$$

and hence $a_n \leqslant q$ as it is the smallest of the a_i's. We next observe that

$$c_1 + b_1 \leqslant a_1 + a_n$$

since otherwise we could switch the first and the last rows of the matrix and get a bigger sum of entries on the main diagonal. Similarly, we get

$$c_2 + b_2 \leqslant a_2 + a_n$$

$$\ldots$$

$$c_{n-1} + b_{n-1} \leqslant a_{n-1} + a_n.$$

Summing up these inequalities we obtain

$$m - a_n + m - a_n \leqslant \operatorname{tdet} A + (n-2)a_n, \tag{3.2}$$

which implies

$$2m \leqslant \operatorname{tdet} A + na_n.$$

Using our assumption $\operatorname{tdet} A < qn + n$ and its consequence $a_n \leqslant q$ we get

$$2qn + 2r = 2m \leqslant \operatorname{tdet} A + na_n < qn + n + qn,$$

so $2r < n$, which contradicts the hypotheses of the theorem. □

Theorem 3.2. *Let $m = qn + r$, where $0 \leqslant r \leqslant n/2$. Then for any matrix $A \in \mathcal{D}(m, n)$ we have*

$$\operatorname{tdet} A \geqslant m + r = qn + 2r.$$

That is, $L(m, n) \geqslant m + r$.

Proof. As in the proof of the previous theorem, we assume that there exists a matrix $A \in \mathcal{D}(m, n)$ such that

$$\operatorname{tdet} A < m + r = qn + 2r$$

and that A is of the form 3.1 with non-increasing a_i's, and the sum of entries on the main diagonal equal to the tropical determinant.

Then, using the hypothesis that $2r \leqslant n$ we get

$$\operatorname{tdet} A = a_1 + \cdots + a_n < qn + 2r \leqslant n(q+1),$$

which implies $a_n \leqslant q$ as a_n is the smallest among the a_i's. As before, we have $2m \leqslant \operatorname{tdet} A + na_n$, which together with $\operatorname{tdet} A < qn + 2r$ and $a_n \leqslant q$ implies

$$2qn + 2r = 2m \leqslant \operatorname{tdet} A + na_n < qn + 2r + qn = 2qn + 2r,$$

that is, $2qn + 2r < 2qn + 2r$, a contradiction. □

We next show that the bound obtained in Theorem 3.1 is sharp.

Theorem 3.3. *Let $m = qn + r$, where $n/2 \leqslant r < n$. Then*

$$L(m, n) = m + n - r = n(q + 1).$$

Proof. We have already shown in Theorem 3.1 that $L(m, n) \geqslant n(q + 1)$. Hence we only need to come up with a matrix A in $\mathcal{D}(m, n)$ whose tropical determinant is $n(q + 1)$. Our matrix A will consist of four blocks

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix},$$

where $A_2 \in M(r, n - r)$ and $A_3 \in M(n - r, r)$ are matrices all of whose entries are equal to $q + 1$, and $A_4 \in M(n - r, n - r)$ has all of its entries equal to q . The upper left corner block A_1 is of size r by r and is constructed in the following way: let the first $2r - n$ entries in the first row be equal to $q + 1$ and all the other entries in that row be equal to q. Here we are using the fact that $0 \leqslant 2r - n < r$, which is equivalent to the assumption of the theorem that $n/2 \leqslant r < n$. The second row is then a shift by one slot of the first row, the third is the shift of the second, etc. For example, for $r = 5$ and $n = 7$ we get

$$A_1 = \begin{pmatrix} q+1 & q+1 & q+1 & q & q \\ q & q+1 & q+1 & q+1 & q \\ q & q & q+1 & q+1 & q+1 \\ q+1 & q & q & q+1 & q+1 \\ q+1 & q+1 & q & q & q+1 \end{pmatrix}.$$

Clearly, $A \in \mathcal{D}(m, n)$ and its tropical determinant is at most $n(q + 1)$ as all its entries are less than or equal to $q + 1$. By Theorem 3.1, the tropical determinant of A is at least $n(q + 1)$ and hence $\operatorname{tdet} A = n(q + 1)$. $\square$

We will show that under some additional assumptions the bound of Theorem 3.2 is sharp.

Theorem 3.4. *Let $m = qn + r$, where $0 \leqslant r \leqslant n/2$, and assume that $qr \geqslant n - 2r$. Then $L(m, n) = m + r = qn + 2r$.*

Proof. We need to construct a matrix A in $\mathcal{D}(m, n)$ whose tropical determinant is $qn + 2r$. We will start out in the same way as in Theorem 3.3. The matrix A will be of the form

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix},$$

where $A_2 \in M(r, n - r)$ and $A_3 \in M(n - r, r)$ are matrices all of whose entries are equal to $q + 1$, and $A_4 \in M(n - r, n - r)$ has all of its entries equal to q . Now we need the row and column sums of A_1 to be equal to $qr + 2r - n$. Notice that now $2r - n \leqslant 0$. We will first make every entry of A_1 to be equal to q and will then distribute $2r - n$ between r entries in each row. For this, let us divide $n - 2r$ by r with a remainder to get $n - 2r = rl + r'$, where $0 \leqslant r' < r$. Now subtract l from each entry in A_1 and an extra 1 from r' entries in each row and column. We get

$$A_1 = \begin{pmatrix} q-l-1 & \dots & q-l-1 & q-l & \dots & \dots & q-l \\ q-l & q-l-1 & \ddots & q-l-1 & q-l & \ddots & q-l \\ \vdots & q-l & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & q-l \\ q-l & \ddots & \ddots & \ddots & \ddots & \ddots & q-l-1 \\ q-l-1 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ q-l-1 & \dots & q-l-1 & q-l & \dots & \dots & q-l-1 \end{pmatrix},$$

where the band of $q-l-1$'s is of width r'. For this construction to work we need the entries of A to be nonnegative. That is, we need, $q-l-1 \geqslant 0$ for $r' \neq 0$, and $q-l \geqslant 0$ for $r' = 0$. This is where we use the additional assumption that $qr \geqslant n-2r$. If $r' = 0$, we get $qr \geqslant n-2r = rl$, which implies $qr \geqslant lr$ and $q \geqslant l$. Next, let $r' \neq 0$. Then $qr \geqslant n-2r = rl + r'$, which implies $q \geqslant l + r'/r$, or $q \geqslant l+1$. □

Corollary 3.5. *Let $m = qn + r$ where $n/3 \leqslant r < n/2$ and $q \neq 0$. Then $L(m, n) = m + r = qn + 2r$.*

Proof. We have

$$\frac{n-2r}{r} \leqslant \frac{3r-2r}{r} = 1$$

and hence the assumption of Theorem 3.4 that $qr \geqslant n-2r$ is satisfied for all $q \geqslant 1$. □

Example 2. Notice that we have solved the Rubik's cube Problem 1 stated in the introduction. For this problem, we have $n = 6$, $m = 9$, and $r = 3$, so we are under the assumptions of Theorem 3.3 and $L(9, 6) = m + n - r = 12$, so one needs to replace $mn - L(m, n) = 54 - 12 = 42$ stickers in the worst-case scenario represented by the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 \\ 1 & 1 & 1 & 2 & 2 & 2 \\ 1 & 1 & 1 & 2 & 2 & 2 \\ 2 & 2 & 2 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 & 1 \end{pmatrix}.$$

4. Hard cases

Let $m = nq + r$, where $0 \leqslant r < n$. It remains to take care of the situation when $n > 2r + rq$ and r and q are both nonzero. We will adjust the above construction of a matrix A with small tropical determinant to this case. The matrix A will still consist of four blocks

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \tag{4.1}$$

where A_4 is a square matrix with all its entries equal to q, but its size now could be smaller than $(n-r)$. Sub-matrices A_2 and A_3 will have entries that are equal to q and $q+1$ and the excess (smaller than in the previous construction) will then be distributed over the matrix A_1, whose entries will be less than or equal to q.

Example 3. Let us look at the example where $n = 6$, $m = 7$, and $q = 1$. If we use the construction of the previous section we would get the matrix

$$\begin{pmatrix} -3 & 2 & 2 & 2 & 2 & 2 \\ 2 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

with a negative entry. Let's make the block of 1's 4×4 instead of 5×5, then we won't need to put four 2's in the same row. We get

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 & 1 & 2 \\ 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 & 1 \end{pmatrix} \tag{4.2}$$

and the resulting matrix does not have any negative entries. By Theorem 3.2, $L(7, 6) \geqslant m + r = 8$. Notice that $\operatorname{tdet} A = 10$, which is bigger than the bound of 8. We will show in Example 5 that $L(7, 6) = 10$.

Example 4. The block of q's in our new construction does not have to be square. For example, if $n = 5$ and $m = 6$ (and hence $q = 1$), we will show that the following matrix with 3×4 block of 1's has the smallest possible tropical determinant in $\mathcal{D}(6, 5)$:

$$\begin{pmatrix} 0 & 1 & 2 & 1 & 2 \\ 0 & 2 & 1 & 2 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

For general m, n with $n > 2r + rq$ and nonzero q and r (which implies $n > 3$), we construct A in a similar way. Let $A_1 \in M(l_1, l_2)$, $A_2 \in M(l_1, n - l_2)$, $A_3 \in M(n - l_1, l_2)$, and $A_4 \in M(n - l_1, n - l_2)$ with $l_1, l_2 \geqslant r$. Set the entries of A_4 equal to q. Let the entries of A_2 be q's and $q + 1$'s with exactly r

$q + 1$'s in each column and with the $q + 1$'s evenly distributed among the rows so that the numbers of $q + 1$'s in different rows differ by no more than 1. This can be easily accomplished by shifting the $q + 1$'s along the columns. We then order the rows in A_2 so that the row sums are non-decreasing as we go from top of the matrix to the bottom. Matrix A_3 is constructed in a similar way.

Let us assume for now that we were able to construct A_1 and obtained a matrix A in $\mathcal{D}(m, n)$ with all the entries less than or equal to $q + 1$. The tropical determinant of such a matrix A is less than or equal to $nq + (l_1 + l_2)$. Since we are looking for a matrix in $\mathcal{D}(m, n)$ with the smallest possible tropical determinant, we want to make $l_1 + l_2$ as small as possible.

Let Σ_{A_i} be the sum of all the entries of the block A_i for $i = 1 \ldots 4$. Then $\Sigma_{A_4} = q(n - l_1)(n - l_2)$. Hence $\Sigma_{A_2} = (n - l_2)m - q(n - l_2)(n - l_1)$ and

$$\Sigma_{A_1} = l_1 m - \Sigma_{A_2} = (l_1 + l_2)r + l_1 l_2 q - nr.$$

In order for Σ_{A_1} to be non-negative, we need l_1 and l_2 to satisfy the inequality

$$(l_1 + l_2)r + l_1 l_2 q \geqslant nr.$$

To minimize $\operatorname{tdet} A$, we would like to find $l_1, l_2 \geqslant r$ that satisfy this inequality with $l_1 + l_2$ being as small as possible. If the sum of two integers $l_1 + l_2$ is fixed, their product $l_1 l_2$ is the largest when $l_1 = l_2$ or $l_2 = l_1 + 1$, so we can assume that either $l_1 = l_2 = l$ or $l_1 = l + 1, l_2 = l$. In the first case, we need l to satisfy $ql^2 + 2lr - rn \geqslant 0$ and in the second, $ql^2 + l(2r + q) + r - rn \geqslant 0$. For each of these cases let's finish the construction of the block A_1 of the matrix A.

Let us consider the case when l is the smallest positive integer that satisfies

$$ql^2 + 2lr - rn \geqslant 0$$

and A_1 is of size l by l. Notice that our assumption $n > 2r + rq$ implies that l is greater than or equal to r. Let each entry of A_1 be q, then the overall excess in the first l rows of A would be $r(n - l) - rl = rn - 2lr$. To check that this excess is positive we need to verify that $n \geqslant 2l$. If n is even it's enough to see that if the plug in $n/2$ for l into $ql^2 + 2lr - rn \geqslant 0$ the inequality is satisfied and by minimality of l we have $l \leqslant n/2$. If n is odd we plug in $(n - 1)/2$ and again (using $n > 2r + rq > 3r$ and $q \geqslant 1$) conclude that $n \geqslant 2l$.

We want to distribute this positive excess $rn - 2lr$ evenly (that is, the row sums differ by no more than 1) among the l rows of A_1. Let $ql^2 + 2lr - rn = a \geqslant 0$. We have

$$\frac{rn - 2lr}{l} = \frac{ql^2 - a}{l} = ql - \frac{a}{l},$$

where $a \geqslant 0$. Hence if $a = 0$ the excess in each row is equal to ql and this can be remedied by letting all the entries of A_1 be equal to 0. If $a = ql^2$ then there is no excess and A_1 consists of all q's.

In general, we need to construct A_1 whose row and column sums are equal to a, where $0 \leqslant a \leqslant ql^2$ and a is distributed as evenly as possible among the rows and columns of A_1 so that this distribution agrees with the distribution of the excess in matrices A_2 and A_3. For this, first distribute a among the rows making sure that the row sums are non-increasing as we go from the top of the matrix to the bottom. Then distribute each row sum evenly among the entries swapping entries in the same row to make sure that the distribution over the columns is even. Finally, we swap columns of A_1 so that the column sums are non-increasing as we go from left to right. We have proved.

Proposition 4.1. *Let $m = nq + r$, where $0 < r < n$. Additionally, assume that $n > 2r + rq$ and q is nonzero. Let l be the smallest positive integer that satisfies $ql^2 + 2lr - rn \geqslant 0$. Then there exists a matrix $A \in \mathcal{D}(m, n)$ such that* $\operatorname{tdet} A \leqslant nq + 2l$. *That is, $L(m, n) \leqslant nq + 2l$.*

We next consider the case when $l_1 = l + 1, l_2 = l$ and $ql^2 + l(2r + q) + r - rn = a \geqslant 0$. As before, we first let A_1 consist of all q's. The excess over first $(l + 1)$ rows is then equal to

$$(n - l)r - r(l + 1) = nr - 2lr - r = l^2 q + lq - a$$

and if we divide this by the number of the rows of A_2 we get

$$\frac{nr-2lr-r}{l+1} = lq - \frac{a}{l+1}.$$

Similarly, the overall excess over first l columns is $(n-l-1)r - rl = nr - 2lr - r$, and when we divide this by the number of columns we get

$$\frac{nr-2lr-r}{l} = \frac{l^2q+lq-a}{l} = (l+1)q - \frac{a}{l}.$$

It can be checked very similarly to the above that the excess $nr - 2lr - r$ is nonnegative, that is, that $n \geqslant 2l+1$ if $n > 2$. Hence $0 \leqslant a \leqslant l(l+1)q$ and we can construct an $l+1$ by l matrix A_1 with row and column sums equal to a by distributing a evenly among the entries as before. We have proved

Proposition 4.2. *Let $m = nq + r$, where $0 < r < n$. Additionally, assume that $n > 2r + rq$ and q is nonzero. Let l be the smallest positive integer that satisfies $ql^2 + l(2r+q) - rn \geqslant 0$. Then there exists a matrix $A \in \mathcal{D}(m,n)$ such that* tdet $A \leqslant nq + 2l + 1$. *That is, $L(m,n) \leqslant nq + 2l + 1$.*

Our last task is to show that the tropical determinant attains its minimum on the matrices that we have constructed.

Theorem 4.3. *Let $m = nq + r$, where $0 \leqslant r < n$. Additionally, assume that $n > 2r + rq$ and q and r are nonzero. Consider two inequalities*

$$l^2q + 2lr - rn \geqslant 0 \tag{4.3}$$

$$l^2q + l(2r+q) + r - rn \geqslant 0. \tag{4.4}$$

Let l be the smallest non-negative integer that satisfies at least one of these inequalities. Then

Case (1) If this l satisfies only (4.4) then $L(m,n) = qn + 2l + 1$.
Case (2) If this l satisfies (4.3) (and, consequently (4.4)) then $L(m,n) = qn + 2l$.

Example 5. Let us use this theorem to find $L(7,6)$. Here $m = 7$, $n = 6$, $q = 1$, $r = 1$. Clearly, $n > 2r + rq$. The two inequalities are $l^2 + 2l - 6 \geqslant 0$ and $l^2 + 3l - 5 \geqslant 0$. The smallest positive l that satisfies at least one of them is 2 and it happens to satisfy both inequalities. Hence $L(7,6) = qn + 2l = 6 + 4 = 10$. See (4.2) above for an example of a matrix in $\mathcal{D}(7,6)$ that has tropical determinant 10.

Proof. Notice that the hypotheses of the theorem imply that $n > 2r + rq \geqslant 2 + 1 = 3$, so in what follows we can use the fact that $n > 3$.

Let $A \in \mathcal{D}(m,n)$. We only need to show that tdet $A \geqslant qn+2l+1$ in the first case and tdet $A \geqslant qn+2l$ in the second case since we have constructed matrices in Propositions 4.1 and 4.2 whose tropical determinants are less than or equal to (and, hence, by this theorem are equal to) these bounds.

Swap the rows and columns of A so that the largest (in terms of sum of dimensions) block of elements that are less than or equal to q is in the lower right corner of A. Let us call this block W. Then A is of the form

$$A = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}.$$

Let X be of size k_1 by k_2.

Lemma 4.4. *Under the assumptions of the theorem, in both cases, we have $r(k_1+k_2) + k_1k_2q \geqslant rn$.*

Proof. Let Σ_W and Σ_Y be the sums of all entries in blocks W and Y. Then $\Sigma_W \leqslant q(n-k_1)(n-k_2)$. Hence

$$\Sigma_Y = (n-k_2)(qn+r) - \Sigma_W \geqslant (n-k_2)(qn+r) - q(n-k_1)(n-k_2).$$

On the other hand, $\Sigma_Y \leqslant k_1(qn+r)$. Putting these two inequalities together we get

$$(n-k_2)(qn+r) - q(n-k_1)(n-k_2) \leqslant k_1(qn+r),$$

which is equivalent to $r(k_1+k_2)+k_1k_2q \geqslant rn$. □

Let us assume that the conclusion of the theorem does not hold. That is, we assume in Case (1) that $\operatorname{tdet} A \leqslant qn+2l$ and in Case (2) $\operatorname{tdet} A \leqslant qn+2l-1$.

Lemma 4.5. *Under the assumptions of the proof of the Theorem that we have made so far, $k_1+k_2 \leqslant n$.*

Proof. If not, the sum of dimensions of W is less than n and by Hall's Marriage Theorem 2.3 A has a transversal with each of the entries being at least $q+1$. Hence $\operatorname{tdet} A \geqslant n(q+1)$. On the other hand, in Case (2) we are under the assumption that $\operatorname{tdet} A \leqslant qn+2l-1$, and hence we get $n(q+1) \leqslant qn+2l-1$ and then $n \leqslant 2l-1$, which rewrites $l \geqslant (n+1)/2$, but it is easy to see that $l \leqslant n/2$. For this, plug in $n/2$ into (4.3) if n is even and $(n-1)/2$ if n is odd and verify that both of them satisfy the inequality, using the assumption that $n \geqslant 4$. In Case(1) we get $n(q+1) \leqslant qn+2l$, that is, $l \geqslant n/2$. Again plugging in $(n-1)/2$ for l into (4.4) if n is odd and $(n-2)/2$ if n is even, we get a contradiction. □

Lemma 4.6. *Under the assumptions of the proof of the Theorem that we have made so far, we have $k_1+k_2 \geqslant 2l+1$ in Case (1) and $k_1+k_2 \geqslant 2l$ in Case (2).*

Proof. In Case (2) we have

$$(l-1)^2q + (l-1)(2r+q) + r - rn < 0,$$

which rewrites $r(2l-1)+l(l-1)q < rn$. Let us assume that $k_1+k_2 \leqslant 2l-1$. Then $k_1k_2 \leqslant l(l-1)$ and $r(k_1+k_2)+k_1k_2q \leqslant r(2l-1)+l(l-1)q < rn$ by the above, which contradicts Lemma 4.4.

Similarly, in Case (1) we have $l^2q+2lr-rn < 0$. Let us assume that $k_1+k_2 \leqslant 2l$. Then $k_1k_2 \leqslant l^2$ and $r(k_1+k_2)+k_1k_2q \leqslant 2lr+l^2q < rn$ by the above, which contradicts Lemma 4.4. □

By Corollary 2.5 there exist transversals in each Y and Z whose entries are at least $q+1$. Notice, that since $k_1+k_2 \leqslant n$, the transversals in Y and Z have respectively k_1 and k_2 entries.

Consider transversals in Y and Z that have largest possible sums of entries t_1 and t_2. The union of two such transversals extends (in many different ways) to a transversal of A. Consider all the transversals of A obtained by extending the union of a transversal in Y with sum of entries t_1 and a transversal in Z with sum of entries t_2. Among all these transversals, pick one transversal in Y and one in Z such that their union extends to the largest transversal in A. Let $a_1, \ldots, a_{k_1}$ and $b_1, \ldots, b_{k_2}$ be the entries of those transversals in Y and Z. We have

$$t_1 = a_1 + \cdots + a_{k_1} \geqslant k_1(q+1) \quad t_2 = b_1 + \cdots + b_{k_2} \geqslant k_2(q+1).$$

Cross out the rows and columns of A that intersect those largest transversals to get an $(n-k_1-k_2)$ by $(n-k_1-k_2)$ submatrix Q of W. Let us assume first that Q has a transversal of all q's.

Then, using Lemma 4.6, we get a contradiction in Case (2)

$$\begin{aligned} qn+2l-1 &\geqslant \operatorname{tdet} A \geqslant t_1+t_2+(n-k_1-k_2)q \\ &\geqslant k_1(q+1)+k_2(q+1)+qn-k_1q-k_2q \\ &= qn+k_1+k_2 \geqslant qn+2l, \end{aligned}$$

as well as in Case (1)

$$\begin{aligned} qn + 2l &\geqslant \operatorname{tdet} A \geqslant t_1 + t_2 + (n - k_1 - k_2)q \\ &\geqslant k_1(q+1) + k_2(q+1) + qn - k_1 q - k_2 q \\ &= qn + k_1 + k_2 \geqslant qn + 2l + 1. \end{aligned}$$

Next, assume that the largest transversal of Q has an entry that is less than or equal to $q - 1$. We can assume that this largest transversal is the main diagonal of Q and the smallest entry of the largest transversal is in the lower right corner. We have

$$A = \left(\begin{array}{cccc|cccc|cccc}
 & & & & & & & c_1 & a_1 & & & \\
 & & & & & & & c_2 & & a_2 & & \\
 & & & & & & & \vdots & & & \ddots & \\
 & & & & & & & c_{k_1} & & & & a_{k_1} \\ \hline
 & & & & e_1 & & & f_1 & & & & \\
 & & & & & e_2 & & f_2 & & & & \\
 & & & & & & \ddots & \vdots & & & & \\
d_1 & d_2 & \dots & d_{k_2} & g_1 & g_2 & \dots & e_{k_3} & i_1 & i_2 & \dots & i_{k_1} \\ \hline
b_1 & & & & & & & h_1 & & & & \\
 & b_2 & & & & & & h_2 & & & & \\
 & & \ddots & & & & & \vdots & & & & \\
 & & & b_{k_2} & & & & h_{k_2} & & & &
\end{array}\right),$$

where $e_{k_3} \leqslant q - 1$, $k_3 = n - k_1 - k_2$, and $\operatorname{tdet} Q = e_1 + e_2 + \cdots + e_{k_3}$. Let $f = f_1 + \cdots + f_{k_3 - 1} + e_{k_3}$, $g = g_1 + \cdots + g_{k_3-1} + e_{k_3}$, $c = c_1 + \cdots + c_{k_1}$, $d = d_1 + \cdots + d_{k_2}$, $h = h_1 + \cdots + h_{k_2}$, and $i = i_1 + \cdots + i_{k_1}$.

By an argument similar to that of Theorem 3.1 we get

$$f + g \leqslant k_3 e_{k_3} + \operatorname{tdet} Q. \tag{4.5}$$

Next notice that each $c_j \leqslant a_j$ since otherwise we could replace a_j with c_j and get a transversal of Y with sum of entries bigger than t_1. We also know that $i_j \leqslant q$ since it's in the block W where all the entries are less than or equal to q.

Assume that we simultaneously have $c_j = a_j$ and $i_j = q$. Then we can pick $\{a_1, a_2, \dots, a_{j-1}, c_j, a_{j+1} \dots, a_{k_1}\}$, a transversal of Y with sum of entries t_1, and $\{e_1, \dots, e_{k_3 - 1}, i_j\}$, a transversal of Q. Notice that since $e_{k_3} < q = i_j$, this contradicts the fact that we initially chose a transversal $\{a_1, \dots, a_{k_1}, e_1, \dots e_{k_3}, b_1, \dots, b_{k_2}\}$ of A so that the sum of the a's is t_1, the sum of b's is t_2, and the overall sum of all the entries in the transversal is the largest possible among such transversals.

This can be visualized by swapping columns $n - k_1$ and $n - k_1 + j$ in A and noticing that we do not change the value of the transversal in Y (as c_j replaces a_j) but make the transversal in Q bigger (since $i_2 = q$ replaces $e_{k_3} \leqslant q - 1$). For example, in the case $j = 1$ the swaps are: $a_1 \leftrightarrow c_1$ and $e_{k_3} \leftrightarrow i_1$.

Hence $c_j + i_j \leqslant a_j + q - 1$ and summing these up over j we get $c + i \leqslant t_1 + qk_1 - k_1$. Similarly, $d + h \leqslant t_2 + qk_2 - k_2$. Since the row and column sums of A are equal to $qn + r$ we get

$$qn + r = c + f + h, \quad qn + r = d + g + i.$$

Adding up these two equalities and using inequalities obtained above together with (4.5) we get

$$\begin{aligned}2qn + 2r &= (c + i) + (d + h) + (f + g)\\ &\leqslant t_1 + qk_1 - k_1 + t_2 + qk_2 - k_2 + k_3 e_{k_3} + \operatorname{tdet} Q\\ &\leqslant t_1 + qk_1 - k_1 + t_2 + qk_2 - k_2 + k_3(q-1) + \operatorname{tdet} Q\\ &= t_1 + t_2 + qn - n + \operatorname{tdet} Q.\end{aligned}$$

From here,

$$\operatorname{tdet} A \geqslant t_1 + t_2 + \operatorname{tdet} Q \geqslant qn + 2r + n > qn + k_1 + k_2.$$

Here the inequality is strict as we have assumed $r > 0$. Next, by Lemma 4.6, in Case (1) we have $k_1 + k_2 \geqslant 2l + 1$ and hence

$$\operatorname{tdet} A > qn + k_1 + k_2 \geqslant qn + 2l + 1.$$

In Case (2) we have $k_1 + k_2 \geqslant 2l$ and hence

$$\operatorname{tdet} A > qn + k_1 + k_2 \geqslant qn + 2l$$

as claimed.

Notice that the above inequalities are proved under the assumption that the largest transversal of Q has an entry that is less than or equal to $q - 1$. The fact that the inequalities are strict shows that the minimum of $\operatorname{tdet} A$ is achieved on a matrix A such that there is a transversal of all q's in Q. □

We sum up all the obtained results in:

Theorem 4.7. *Let $m = qn + r$ where $0 \leqslant r < n$. Then*

- *If $q = 0$ then $L(m, n) = n$ (Proposition 2.8).*
- *If $r = 0$ then $L(m, n) = m$ (Proposition 2.10).*
- *If $n/2 \leqslant r < n$ then $L(m, n) = n(q + 1)$ (Theorem 3.3).*
- *If $0 < r < n/2$ and $n \leqslant 2r + rq$ (in particular, if $n/3 \leqslant r < n/2$) then $L(m, n) = qn + 2r$ (Theorem 3.4 Corollary 3.5).*
- *If $n > 2r + rq$ and $r, q \neq 0$ then $L(m, n) = qn + 2l + 1$ or $qn + 2l$ (see the definition of l and details in Theorem 4.3).*

5. Upper bound on the tropical determinant

Let $A = (a_{ij})$ be an n by n matrix. Recall that

$$\operatorname{tropdet}(A) = \min_{\sigma \in S_n}\{a_{1\sigma(1)} + a_{2\sigma(2)} + \cdots + a_{n\sigma(n)}\}.$$

In this section we solve Problem 4 which reformulates as:

Problem 6. Fix $m, n \in \mathbb{N}$. Find the sharp upper bound $U(m, n)$ on the tropical determinant $\operatorname{tropdet} A$ over the set $\mathcal{D}(m, n)$.

This problem turns out to be much easier than our original problem. The solution is very similar to the “not so easy” cases of Problem 5.

Theorem 5.1. *Let $m = qn + r$ where $0 \leqslant r \leqslant n/2$. Then for any matrix $A \in \mathcal{D}(m, n)$ we have $\operatorname{tropdet} A \leqslant qn$. That is, $U(m, n) \leqslant qn$.*

Proof. Let's assume that there exists a matrix $A \in \mathcal{D}(m, n)$ such that

$$\operatorname{tropdet} A > qn.$$

We rearrange rows and columns of A so that the tropical determinant is equal to the sum of entries on the main diagonal of A and the entries are non-decreasing along the main diagonal. That is, we may assume that A is of the form

$$A = \begin{pmatrix} a_1 & & & & b_1 \\ & a_2 & & & \vdots \\ & & \ddots & & \vdots \\ & & & a_{n-1} & b_{n-1} \\ c_1 & \cdots & \cdots & c_{n-1} & a_n \end{pmatrix}, \tag{5.1}$$

where $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$ and $\operatorname{tropdet} A = a_1 + \cdots + a_n$. Notice that by our assumption we also have

$$\operatorname{tropdet} A = a_1 + \cdots + a_n > qn,$$

which implies $a_n \geqslant q + 1$ as a_n is the largest among the a_i's. Similarly to the proof of Theorem 3.1 we get $2m \geqslant \operatorname{tropdet} A + na_n$ and hence

$$2qn + 2r = 2m \geqslant \operatorname{tropdet} A + na_n > qn + n(q + 1),$$

which implies $2r > n$, and this contradicts the hypotheses of the theorem. □

Theorem 5.2. *Let $m = qn + r$ where $n/2 \leqslant r < n$. Then for any matrix $A \in \mathcal{D}(m, n)$ we have $\operatorname{tropdet} A \leqslant qn + (2r - n)$. That is, $U(m, n) \leqslant qn + (2r - n)$.*

Proof. As in the previous theorem, we assume that A is of the form 5.1 with non-decreasing a_i's and the sum of entries on the main diagonal equal to the tropical determinant.

Let us assume that $\operatorname{tropdet} A > qn + (2r - n)$. Then

$$\operatorname{tropdet} A = a_1 + \cdots + a_n > qn + (2r - n) \geqslant qn,$$

which implies that $a_n \geqslant q + 1$ as a_n is the largest among the a_i's. As before we get

$$2m \geqslant \operatorname{tropdet} A + na_n > qn + (2r - n) + n(q + 1) = 2qn + 2r = 2m,$$

that is, $2m < 2m$, a contradiction. □

Finally, we show that these bounds are sharp.

Theorem 5.3. *Let $m = qn + r$ where $0 \leqslant r < n/2$. Then $U(m, n) = qn$.*

Proof. We have already shown in Theorem 3.1 that $U(m, n) \leqslant nq$. Hence we only need to come up with a matrix A in $\mathcal{D}(m, n)$ whose tropical determinant is nq. Our matrix A will consist of four blocks

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix},$$

where $A_2 \in M(n - r, r)$ and $A_3 \in M(r, n - r)$ are matrices all of whose entries are equal to q, and $A_4 \in M(r, r)$ has all of its entries equal to $q + 1$. The upper left corner block A_1 is of size $n - r$ by $n - r$ and is constructed in the following way: Let the first r entries in the first row be equal to $q + 1$ and the remaining $n - 2r$ be equal to q. Here we are using the fact that $0 < n - 2r$, which is equivalent to the assumption of the theorem that $r < n/2$. The second row is then a shift by one slot of the first row, the third is the shift of the second, etc.

Clearly, the tropical determinant of A is at least nq as all its entries are greater than or equal to q. By the Theorem 5.1, the tropical determinant of A is at most nq and hence $\operatorname{tropdet} A = nq$. $\square$

Theorem 5.4. *Let $m = qn + r$ where $n/2 \leqslant r < n$. Then $U(m, n) = qn + 2r - n$.*

Proof. We need to construct a matrix A in $\mathcal{D}(m, n)$ whose tropical determinant is $qn + 2r$. As before, let matrix A be of the form

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix},$$

where $A_2 \in M(n-r, r)$ and $A_3 \in M(r, n-r)$ are matrices all of whose entries are equal to q, and $A_4 \in M(r, r)$ has all of its entries equal to $q + 1$. It remains to define A_1 so that the sums of first $n - r$ rows and columns of A are equal to $qn + r$. Let first all the entries of A_1 be equal to q and then distribute r between $n - r$ entries in each row and column of A_1. For this, divide r by $n - r$ with a remainder to get $r = (n-r)q' + r'$, where $0 \leqslant r' < r$. Notice that $q' \geqslant 1$ since $r \geqslant n - r$. Next, let the entries of A_1 be equal to $q + q'$ and then add a band of 1's of width $r' < r$ to A_1. Since $n - r \leqslant r$ we can pick a transversal of all q's in A_2 and A_3 each consisting of $n - r$ q's. If we extend this to a transversal of A we get a transversal with sum of entries equal to $qn + 2r - n$. This is the smallest transversal in A as the largest number of q's we could have in a transversal in A is $2(n - r)$ (at most $n - r$ from first $n - r$ rows and at most $n - r$ from first $n - r$ columns) and all the entries in a transversal that are not equal to q, are at least $q + 1$. We have shown

$$\operatorname{tropdet} A = qn + 2r - n. \quad \square$$

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