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Differential Geometry and its Applications 24 (2006) 235–259

DIFFERENTIAL GEOMETRY AND ITS APPLICATIONS

www.elsevier.com/locate/difgeo

On variational approach to differential invariants of rank two distributions

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Received 8 February 2005
Available online 20 October 2005
Communicated by R.L. Bryant

Abstract

We construct differential invariants for generic rank 2 vector distributions on n-dimensional manifolds, where $n \geqslant 5$. Our method for the construction of invariants is completely different from the Cartan reduction-prolongation procedure. It is based on the dynamics of the field of so-called abnormal extremals (singular curves) of rank 2 distribution and on the theory of unparameterized curves in the Lagrange Grassmannian, developed in [A. Agrachev, I. Zelenko, Geometry of Jacobi curves I, J. Dynam. Control Syst. 8 (1) (2002) 93–140; II, 8 (2) (2002) 167–215]. In this way we construct the *fundamental form* and the *projective Ricci curvature* of rank 2 vector distributions for arbitrary $n \geqslant 5$. In the next paper [I. Zelenko, Fundamental form and Cartan's tensor of (2,5)-distributions coincide, J. Dynam. Control. Syst., in press, SISSA preprint, Ref. 13/2004/M, February 2004, math.DG/0402195] we show that in the case $n = 5$ our fundamental form coincides with the Cartan covariant biquadratic binary form, constructed in 1910 in [E. Cartan, Les systemes de Pfaff a cinque variables et les equations aux derivees partielles du second ordre, Ann. Sci. Ecole Normale 27 (3) (1910) 109–192; reprinted in: Oeuvres completes, Partie II, vol. 2, Gautier-Villars, Paris, 1953, pp. 927–1010]. Therefore first our approach gives a new geometric explanation for the existence of the Cartan form in terms of an invariant degree four differential on an unparameterized curve in Lagrange Grassmannians. Secondly, our fundamental form provides a natural generalization of the Cartan form to the cases $n > 5$. Somewhat surprisingly, this generalization yields a rational function on the fibers of the appropriate vector bundle, as opposed to the polynomial function occurring when $n = 5$. For $n = 5$ we give an explicit method for computing our invariants and demonstrate the method on several examples.

MSC: 58A30; 53A55

Keywords: Nonholonomic distributions; Differential invariants; Abnormal extremals; Jacobi curves; Lagrange Grassmannian

1. Introduction

A rank l vector distribution D on an n-dimensional manifold M or an (l, n)-distribution (where $l < n$) is a subbundle of the tangent bundle TM with l-dimensional fibers. Two germs of vector distributions D_1 and D_2 at the point $q_0 \in M$ are called equivalent, if there exist neighborhoods U and $\tilde{U}$ of q_0 and a diffeomorphism $F : U \mapsto \tilde{U}$ such that $F_* D_1(q) = D_2(F(q))$ for all $q \in U$ and $F(q_0) = q_0$. The natural problem is to construct invariants of distributions

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0926-2245/$ – see front matter
doi:10.1016/j.difgeo.2005.09.004

w.r.t. this equivalence relation. Distributions are associated with Pfaffian systems and with control systems linear in the control. So invariants of distributions are also invariants of the corresponding Pfaffian systems and state-feedback invariants of the corresponding control systems.

The obvious (but very rough in the most cases) invariants of a distribution D at q are so-called *the small and the big growth vectors* at q: The small growth vector is the tuple $(\dim D(q), \dim D^2(q), \dim D^3(q), \dots)$, where D^j is the jth power of the distribution D, i.e., $D^j = D^{j-1} + [D, D^{j-1}]$, $D^1 = D$, $j \geqslant 2$; the big growth vector is the tuple $(\dim D(q), \dim D_2(q), \dim D_3(q), \dots)$, where D_j is defined by the following recursive formula $D_j = D_{j-1} + [D_{j-1}, D_{j-1}]$, $D_1 = D$. A simple estimation shows that at least $l(n-l) - n$ functions of n variables are required to describe generic germs of (l, n)-distribution, up to the equivalence (see [17] and [25] for precise statements). There are only three cases, where $l(n-l) - n$ is not positive: $l = 1$ (line distributions), $l = n-1$, and $(l, n) = (2, 4)$. Moreover, it is well known that in these cases generic germs of distributions are equivalent: for $l = 1$ it is just the classical theorem about the rectification of vector fields without stationary points, for $l = n-1$ all generic germs are equivalent to Darboux's model, while for $(l, n) = (2, 4)$ they are equivalent to Engel's model (see [9] or [24]). In all other cases functional invariants should appear for generic distributions.

The case $l = 2$, $n = 5$ (the smallest dimensions, when functional parameters appear) was treated by E. Cartan in [11] with his method of equivalence. For any $(2, 5)$-distribution D with the small growth vector $(2, 3, 5)$ he constructed the canonical coframe in some 14-dimensional manifold and used some structural functions of this coframe in order to obtain an invariant homogeneous polynomial of degree 4 on each plane $D(q)$. He called it the covariant biquadratic binary form. We will call it for shortness *the Cartan form*. If the roots of the projectivization of the Cartan form are different, then their cross-ratio is the functional invariant of the distribution. E. Cartan proved also that the largest possible Lie algebra of infinitesimal symmetries for $(2, 5)$-distributions with the small growth vector $(2, 3, 5)$ is the split real form $\widetilde{G}_2$ of the exceptional Lie algebra G_2, there is only one, up to the equivalence, germ of $(2, 5)$-distribution with algebra of infinitesimal symmetries equal to $\widetilde{G}_2$, and this germ is also the unique one with the identically vanishing Cartan form.

In the modern terminology, to any generic $(2, 5)$-distribution the canonical Cartan $\widetilde{G}_2$-valued connection can be assigned. The Cartan reduction-prolongation procedure was systematized by N. Tanaka in [15,16] for a special class of distributions (see also survey [18]). This class contains $(2, 5)$-distributions, but it does not contain, for example, generic $(2, n)$-distributions with $n > 5$. To be more precise let us briefly describe the class of distributions for which the Tanaka theory works. Suppose that a distribution D satisfies $D^m = TM$ for some $m \in \mathbb{N}$. Set $D^0(q) = 0$ and $\mathfrak{g}_{-l}(q) = D^l(q)/D^{l-1}(q)$, $1 \leqslant 1 \leqslant m$. Given a point $q \in M$ the space $\mathfrak{g}(q) = \bigoplus_{l=-m}^{-1} \mathfrak{g}_l(q)$ can be endowed naturally with the structure of the graded nilpotent Lie algebra, called also *symbol algebra of D at q*. Suppose that all symbol algebras $\mathfrak{g}(q)$ are isomorphic to one graded nilpotent Lie algebra $\mathfrak{g} = \bigoplus_{l=-m}^{-1} \mathfrak{g}_l$. In [15] it was shown that there exists the maximal graded Lie algebra $\mathfrak{S}_\mathfrak{g} = \bigoplus_{l=-m}^{\infty} \bar{\mathfrak{g}}_l$ such that its "negative part" $\bigoplus_{l=-m}^{-1} \bar{\mathfrak{g}}_l$ is isomorphic to $\mathfrak{g}$. The Lie algebra $\mathfrak{S}_\mathfrak{g}$ is called the *universal prolongation* of the symbol algebra $\mathfrak{g}$. In [16] N. Tanaka proved that if the Lie algebra $\mathfrak{S}_\mathfrak{g}$ is semisimple, then to any distribution with the same symbol algebra $\mathfrak{g}$ at any point the canonical Cartan $\mathfrak{S}_\mathfrak{g}$-valued connection can be assigned.

But the class of $(2, n)$-distributions, where $n \geqslant 5$, such that the universal prolongations of their symbol algebras are semisimple, consists of distributions of only two types: $(2, 5)$-distributions with the small growth vector $(2, 3, 5)$, considered by Cartan, and $(2, 6)$-distributions with the small growth vector $(2, 3, 4, 5, 6)$ and the big growth vector $(2, 3, 4, 6)$ at any point (in both cases the universal prolongation is the split real form of G_2, the last case is not generic). The proof of this fact follows without difficulties from the results of [18] about the classification of all graded simple Lie algebras $\mathfrak{S} = \bigoplus_{l=-m}^{\infty} \mathfrak{g}_l$, which are universal prolongations of their negative part $\bigoplus_{l=-m}^{-1} \mathfrak{g}_l$: just consider the case $\dim \mathfrak{g}_{-1} = 2$ and make an appropriate analysis of Dynkin diagrams. So, the Tanaka theory cannot be directly applied for generic $(2, n)$-distributions with $n > 5$. Other important points in the Tanaka approach are that the symbol algebras should be isomorphic at different points and all constructions are heavily based on the type of symbol. Note that already in the case of $(2, 6)$-distributions with maximal possible small growth vector $(2, 3, 5, 6)$ three different symbol algebras are possible.

In the present paper we develop a completely different method for the construction of functional (differential) invariants of generic germs of bracket-generating $(2, n)$-distributions for arbitrary $n \geqslant 5$, which is based on a new variational approach to differential invariants proposed recently by A. Agrachev (see [1,2], and also the Introduction to [3]). Our constructions are independent of the symbol algebra and actually also of the small growth vector of a distribution D. The algebras of infinitesimal symmetries do not play any role in the constructions. Besides, to construct our invariants we do not look for some canonical frame (on the contrary, the canonical frame may be found with the help of our invariants, see, for example, [20, Section 10.5] there).

The main objects in the method are abnormal extremals of distributions, which are some special unparameterized curves in the cotangent bundle (see Section 2.1 below). To a given extremal one can assign a special curve, the Jacobi curve, in some Lagrange Grassmannian, i.e., in the set of all Lagrangian subspaces of some symplectic space (see Section 2.2 below). The Jacobi curves can be seen as generalizations of the spaces of Jacobi fields along Riemannian geodesics. Any symplectic invariant of the Jacobi curve (i.e., an invariant w.r.t. the natural action of the linear Symplectic group on the Lagrange Grassmannian, where this Jacobi curve lives) produces an invariant of the distribution itself. In [3] for any curve of so-called constant *weight* in a Lagrange Grassmannian we constructed *the canonical projective structure* and the following two invariants w.r.t. the action of the linear Symplectic Group and reparameterizations: a special degree 4 differential, the *fundamental form*, and a special function, the *projective Ricci curvature*. These constructions are based on the notion of the cross-ratio of four "points" in a Lagrange Grassmannian and some universal asymptotic of the cross-ratio of points on a curve in a Lagrange Grassmannian, when one glues them together (see formula (2.5) below). Roughly speaking, the fundamental form of the curve is the first nontrivial term of this asymptotic. We briefly describe all these constructions in Section 2.3.

The next steps of the method are to interpret the condition for the Jacobi curve of an abnormal extremal of a distribution to be of the constant weight in terms of the distribution (Section 3.1) and to pass from the mentioned invariants defined on a single Jacobi curve of each regular abnormal extremal of the distribution to the corresponding invariants of the distribution itself (Sections 3.2 and 3.3). In this way for generic germ of $(2,n)$-distribution we construct the fundamental form and the projective Ricci curvature. To describe what kind of object is the fundamental form of the distribution D, let us denote by $(D^l)^\perp \subset T^*M$ the annihilator of the lth power D^l, namely

$$(D^l)^\perp = \{(q,p) \in T^*M\colon\ p \cdot v = 0\ \forall v \in D^l(q)\}. \tag{1.1}$$

The fundamental form at the point $q \in M$ is a special degree 4 homogeneous rational function defined, up to multiplication on positive constant, on the linear space

$$(D^2)^\perp(q) = (D^2)^\perp \cap T_q^*M. \tag{1.2}$$

We show that for $(2,5)$-distribution with the small growth vector $(2,3,5)$ the fundamental form at any point is a polynomial (Theorem 3), while for $n > 5$ for generic $(2,n)$-distributions the fundamental form is a rational function, which is not a polynomial (Theorem 5).

In the case of $(2,5)$-distribution with small growth vector $(2,3,5)$ the fundamental form can be also realized as a degree 4 polynomial on the plane $D(q)$ for all $q \in M$ (we call it the *tangential fundamental form*), i.e., it is an object of the same nature as the Cartan form. In the next paper [22] we prove that for $n = 5$ our tangential fundamental form coincides (up to constant factor -35) with the Cartan form. Therefore first the existence of the special degree four differential on a curve in Lagrange Grassmannians gives the new geometric explanation for the existence of the Cartan covariant biquadratic binary form. Secondly, our fundamental form can be seen as a natural generalization of the Cartan form to the cases $n > 5$.

The projective Ricci curvature of the distribution is a function, defined on the subset of $(D^2)^\perp \backslash (D^3)^\perp$, where the fundamental form does not vanish. Note that the notion of the projective Ricci curvature is new even for $n = 5$. Using this notion, we construct, in addition to the fundamental form, a special degree 10 homogeneous rational function, defined, up to multiplication on a positive constant, on $(D^2)^\perp(q)$ for any $q \in M$ (for $n = 5$ this function is again a polynomial).

In the case $n = 5$ we also give an explicit method for calculation of the fundamental form and the projective Ricci curvature. The method is given by three very compact formulas in Theorem 2. The main advantage of it is that one can work all the time with one arbitrary chosen local basis of the distribution, while in the Cartan method one had to repeat the whole Cartan reduction-prolongation procedure for this distribution from the very beginning. We apply the obtained formulas for several examples. In particular, we calculate our invariants for the distribution generated by the rolling of two spheres of radiuses r and $\hat r$ ($r \leqslant \hat r$) without slipping and twisting. We show that the fundamental form of such distribution is equal to zero iff $\frac{\hat r}{r} = 3$ and that the distributions with different ratios $\frac{\hat r}{r}$ are not equivalent. Also we give some sufficient conditions for the rigidity of an abnormal trajectory in terms of the canonical projective structure and the fundamental form on it (see Proposition 4.5 below). In the forthcoming paper [23] we classify distributions with the constant projective Ricci curvature and a big group of symmetries, giving models and proving uniqueness results (we announce some of these results at the end of the Section 3, see Theorem 4). In the forthcoming paper [12] we construct the canonical frame for generic $(2,n)$-distributions with $n>5$ and find the most symmetric case for any such n. The main tool in this construction is the canonical projective structure on each abnormal extremal.

2. Preliminaries

2.1. Abnormal extremals

For $(2,n)$-distributions (where $n\geqslant 4$) with small growth vector of the type $(2,3,4,\text{ or }5,\dots)$ one can distinguish special (unparameterized) curves in the cotangent bundle T^*M of M. For this let $\pi: T^*M\mapsto M$ be the canonical projection. For any $\lambda\in T^*M$, $\lambda=(p,q)$, $q\in M$, $p\in T_q^*M$, let $\varsigma(\lambda)(\cdot)=p(\pi_*\cdot)$ be the canonical Liouville form and $\sigma=-d\varsigma$ be the standard symplectic structure on T^*M (here we prefer the sign "−" in the right-hand side, although usually one defines the standard symplectic form on T^*M without this sign). Let $(D^l)^\perp\subset T^*M$ be as in (1.1). The set $D^\perp$ is a codimension 2 submanifold of T^*M. Consider the restriction $\sigma|_{D^\perp}$ of the form σ on $D^\perp$. It is not difficult to check that (see, for example [19, Section 2]): The set of points, where the form $\sigma|_{D^\perp}$ degenerates, coincides with $(D^2)^\perp$. The set $(D^2)^\perp\backslash(D^3)^\perp$ is a codimension 1 submanifold of $D^\perp$. For each $\lambda\in(D^2)^\perp\backslash(D^3)^\perp$ the kernel of $\sigma|_{D^\perp}(\lambda)$ is a two-dimensional subspace of $T_\lambda D^\perp$, which is transversal to $T_\lambda(D^2)^\perp$. Hence $\forall\lambda\in(D^2)^\perp\backslash(D^3)^\perp$ we have

$$\ker\sigma|_{(D^2)^\perp}(\lambda)=\ker\sigma|_{D^\perp}(\lambda)\cap T_\lambda(D^2)^\perp.$$

This equality implies that these kernels form a line distribution in $(D^2)^\perp\backslash(D^3)^\perp$ and this line distribution defines a *characteristic 1-foliation* Ab_D of $(D^2)^\perp\backslash(D^3)^\perp$. The leaves of this foliation will be called the *characteristic curves* of the distribution D. These characteristic curves are also called regular abnormal extremals of D (as in [14,19]).

Remark 2.1. The term abnormal extremal comes from Pontryagin Maximum Principle in Optimal Control. Defining on the set of all curves tangent to D some functional (for example, length w.r.t. some Riemannian metric on M), one can consider the corresponding optimal control problem with fixed endpoints. Abnormal extremals are the extremals of this problem with vanishing Lagrange multiplier near the functional, so they do not depend on the functional but on the distribution D itself. Projections of abnormal extremals to the base manifold M will be called *abnormal trajectories*. Conversely, an abnormal extremal projected to the given abnormal trajectory will be called its *lift*. If some lift of the abnormal trajectory is a regular abnormal extremal, then this abnormal trajectory will be called regular. Again from Pontryagin Maximum Principle it follows that the set of all lifts of given abnormal trajectory can be provided with the structure of linear space. The dimension of this space is called *corank of the abnormal trajectory*.

2.2. Jacobi curves

Given a segment γ of characteristic curve we construct a curve of Lagrangian subspaces, called the Jacobi curve, in the appropriate symplectic space. For this for any $\lambda\in(D^2)^\perp$ denote by $\mathcal{J}(\lambda)$ the following subspace of $T_\lambda(D^2)^\perp$

$$\mathcal{J}(\lambda)=\bigl(T_\lambda(T^*_{\pi(\lambda)}M)+\ker\sigma|_{D^\perp}(\lambda)\bigr)\cap T_\lambda(D^2)^\perp=\bigl\{v\in T_\lambda(D^2)^\perp:\ \pi_*v\in D\bigl(\pi(\lambda)\bigr)\bigr\}.\tag{2.1}$$

Here $T_\lambda(T^*_{\pi(\lambda)}M)$ is tangent to the fiber $T^*_{\pi(\lambda)}M$ at the point λ (or vertical subspace of $T_\lambda(T^*M)$). Actually $\mathcal{J}$ is rank $(n-1)$ distribution on the manifold $(D^2)^\perp\backslash(D^3)^\perp$.

Let O_γ be a neighborhood of γ in $(D^2)^\perp$ such that $N=O_\gamma/(Ab_D|_{O_\gamma})$ is a well-defined smooth manifold. The quotient manifold N is a symplectic manifold endowed with a symplectic structure $\bar\sigma$ induced by $\sigma|_{(D^2)^\perp}$. Let $\phi: O_\gamma\to N$ be the canonical projection on the factor. It is easy to check that $\phi_*(\mathcal{J}(\lambda))$ is a Lagrangian subspace of the symplectic space $T_\gamma N$ for all $\lambda\in\gamma$. Let $L(T_\gamma N)$ be the Lagrangian Grassmannian of the symplectic space $T_\gamma N$, i.e., $L(T_\gamma N)=\{\Lambda\subset T_\gamma N:\ \Lambda^\angle=\Lambda\}$, where $\Lambda^\angle$ is the skew-symmetric complement of the subspace Λ, $\Lambda^\angle=\{v\in T_\gamma N:\ \bar\sigma(v,\Lambda)=0\}$. The *Jacobi curve of the characteristic curve* (*regular abnormal extremal*) γ is the mapping $J_\gamma:\gamma\mapsto L(T_\gamma N)$ satisfying

$$J_\gamma(\lambda)=\phi_*\bigl(\mathcal{J}(\lambda)\bigr),\quad\forall\lambda\in\gamma.\tag{2.2}$$

Remark 2.2. In [1] and [2] Jacobi curves of extremals were constructed in a purely variational way using the notion of Lagrangian derivative ($\mathcal{L}$-derivative) of the endpoint map associated with geometric structure (control system). The reason to call these curves Jacobi curves is that they can be considered as the generalization of spaces of "Jacobi fields" along Riemannian geodesics: in terms of these curves one can describe some optimality properties of corresponding extremals. Namely, if the Jacobi curve of the abnormal extremal is a simple curve in the Lagrange Grassmannian, then the corresponding abnormal trajectory is W^1_∞-isolated (rigid) curve in the space of all curves tangent to the distribution D with fixed endpoints (the curve in the Lagrange Grassmannian is called simple if one can choose a Lagrangian subspace transversal to each Lagrange subspace belonging to the image of the curve). This result can be found in [6] or [19]. Moreover, if some Riemannian metric is given on M, then under the same conditions on the Jacobi curve the corresponding abnormal trajectory is the shortest among all curves tangent to the distribution D, connecting its endpoints and sufficiently closed to this abnormal trajectory in W^1_1-topology (see [7]) and even in C^0-topology (see [8]).

Jacobi curves are invariants of the distribution. They are unparameterized curves in the Lagrange Grassmannians. In this way the problem of finding invariants of rank 2 distributions is reduced to the much more treatable problem of finding symplectic invariants of unparameterized curves in Lagrange Grassmannians.

2.3. Principal invariants of curves in Grassmannian of half-dimensional subspaces

Let W be a $2m$-dimensional linear space and $G_m(W)$ be the set of all m-dimensional subspaces of W (i.e., the Grassmannian of half-dimensional subspaces). Below we give definitions of the weight and the rank of the curve in $G_m(W)$ and describe briefly the construction of the fundamental form and the projective Ricci curvature for a curve of constant weight in $G_m(W)$ (for the details see [3]), which are invariants w.r.t. the action of General Linear Group $GL(W)$. Since any curve of Lagrange subspaces w.r.t. some symplectic form on W is obviously the curve in $G_m(W)$, all constructions below are valid for the curves in the Lagrange Grassmannian.

For given $\Lambda \in G_m(W)$ denote by $\Lambda^{\pitchfork}$ the set of all m-dimensional subspaces of W transversal to Λ, $\Lambda^{\pitchfork} = \{\Gamma \in G_m(W)\colon \Gamma \cap \Lambda = 0\}$. Fix some $\Delta \in \Lambda^{\pitchfork}$. Then for any subspace $\Gamma \in \Lambda^{\pitchfork}$ there exists a unique linear mapping from Δ to Λ with graph Γ. We denote this mapping by $\langle \Delta, \Gamma, \Lambda\rangle$. So, $\Gamma = \{v + \langle \Delta, \Gamma, \Lambda\rangle v \mid v \in \Delta\}$. Choosing the bases in Δ and Λ one can assign to any $\Gamma \in \Lambda^{\pitchfork}$ the matrix of the mapping $\langle \Delta, \Gamma, \Lambda\rangle$ w.r.t. these bases. In this way we define the coordinates on the set $\Lambda^{\pitchfork}$.

Remark 2.3. Assume that W is endowed with some symplectic form $\bar\sigma$ and Δ, Λ are Lagrange subspaces w.r.t. $\bar\sigma$. Then the map $v \mapsto \bar\sigma(v, \cdot)$, $v \in \Delta$, defines the canonical isomorphism between Δ and Λ^*. It is easy to see that Γ is a Lagrange subspace iff the mapping $\langle \Delta, \Gamma, \Lambda\rangle$, considered as the mapping from Λ^* to Λ, is self-adjoint.

Let $\Lambda(t)$ be a smooth curve in $G_m(W)$ defined on some interval $I \subset \mathbb{R}$. We are looking for invariants of $\Lambda(t)$ w.r.t. the action of $GL(W)$. Suppose that in some coordinates $W \cong \mathbb{R}^m \times \mathbb{R}^m$ and $\Lambda(t) = \{(x, S_t x)\colon x \in \mathbb{R}^m\}$ for some $m \times m$-matrix S_t. The curve $\Lambda(\cdot)$ is called *ample at the point* τ if the function $t \mapsto \det(S_t - S_\tau)$ has a zero of *finite order* at τ. It is easy to see that this definition does not depend on the choice of the coordinates. The curve $\Lambda(\cdot)$ is called *ample* if it is ample at any point.

Definition 1. *The order of zero of the function $t \mapsto \det(S_t - S_\tau)$ at τ, where S_t is a coordinate representation of the curve $\Lambda(\cdot)$, is called a weight of the curve $\Lambda(\cdot)$ at τ.*

It is clear that the weight of $\Lambda(\tau)$ is an integral valued upper semicontinuous function of τ. Therefore it is locally constant on the open dense subset of the interval of definition I.

Now suppose that the curve has the constant weight k on some subinterval $I_1 \subset I$. It implies that for any two distinct and sufficiently closed parameters $t_0, t_1 \in I_1$ one has $\Lambda(t_0) \cap \Lambda(t_1) = 0$. Hence for such t_0, t_1 the following linear mappings

$$\frac{d}{ds}\bigl\langle \Lambda(t_0), \Lambda(s), \Lambda(t_1)\bigr\rangle\Big|_{s=t_0} : \Lambda(t_0) \mapsto \Lambda(t_1), \tag{2.3}$$

$$\frac{d}{ds}\langle \Lambda(t_1), \Lambda(s), \Lambda(t_0)\rangle\Big|_{s=t_1} : \Lambda(t_1) \mapsto \Lambda(t_0) \tag{2.4}$$

are well defined. Taking the composition of mapping (2.4) with mapping (2.3) we obtain an operator from the subspace $\Lambda(t_0)$ to itself, which is actually the infinitesimal cross-ratio of two points $\Lambda(t_i)$, $i = 0, 1$, together with two tangent vectors $\dot{\Lambda}(t_i)$ at these points in $G_m(W)$ (see [3] for the details).

Theorem 1. (See [3, Lemma 4.2]) *If the curve has the constant weight k on some subinterval $I_1 \subset I$, then the following asymptotic holds*

$$\operatorname{tr}\left(\frac{d}{ds}\langle \Lambda(t_1), \Lambda(s), \Lambda(t_0)\rangle\Big|_{s=t_1} \circ \frac{d}{ds}\langle \Lambda(t_0), \Lambda(s), \Lambda(t_1)\rangle\Big|_{s=t_0}\right) = -\frac{k}{(t_0 - t_1)^2} - g_\Lambda(t_0, t_1), \tag{2.5}$$

where $g_\Lambda(t_0, t_1)$ is a smooth function in a neighborhood of diagonal $\{(t, t) \mid t \in I_1\}$.

The function $g_\Lambda(t_0, t_1)$ is a "generating" function for invariants of the parametrized curve by the action of $GL(2m)$. The first coming invariant of the parametrized curve, *the generalized Ricci curvature*, is just $g_\Lambda(t, t)$, the value of g_Λ at the diagonal.

In order to obtain invariants for unparameterized curves (i.e., for one-dimensional submanifold of $G_m(W)$) we use a simple reparameterization rule for the function g_Λ. Indeed, let $t = \varphi(\tau)$ be a reparameterization of the curve Λ (with original parameter t). It follows directly from (2.5) that

$$g_{\Lambda\circ\varphi}(\tau_0, \tau_1) = \dot{\varphi}(\tau_0)\dot{\varphi}(\tau_1) g_\Lambda\big(\varphi(\tau_0), \varphi(\tau_1)\big) + k\left(\frac{\dot{\varphi}(\tau_0)\dot{\varphi}(\tau_1)}{(\varphi(\tau_0) - \varphi(\tau_1))^2} - \frac{1}{(\tau_0 - \tau_1)^2}\right). \tag{2.6}$$

In particular, putting $\tau_0 = \tau_1 = \tau$, one obtains the following reparameterization rule for the generalized Ricci curvature

$$g_{\Lambda\circ\varphi}(\tau, \tau) = \dot{\varphi}(\tau)^2 g_\Lambda\big(\varphi(\tau), \varphi(\tau)\big) + \frac{k}{3}\mathbb{S}(\varphi), \tag{2.7}$$

where $\mathbb{S}(\varphi)$ is a Schwarzian derivative of φ,

$$\mathbb{S}(\varphi) = \frac{1}{2}\frac{\varphi^{(3)}}{\varphi'} - \frac{3}{4}\left(\frac{\varphi''}{\varphi'}\right)^2 = \frac{d}{dt}\left(\frac{\varphi''}{2\varphi'}\right) - \left(\frac{\varphi''}{2\varphi'}\right)^2. \tag{2.8}$$

From (2.7) it follows that the class of local parameterizations, in which the generalized Ricci curvature is identically equal to zero, defines a *canonical projective structure* on the curve (i.e., any two parameterizations from this class are transformed one to another by Möbius transformation). This parameterizations are called projective. From (2.6) it follows that if t and τ are two projective parameterizations on the curve $\Lambda(\cdot)$, $t = \varphi(\tau) = \frac{a\tau+b}{c\tau+d}$, and g_Λ is the generating function of $\Lambda(\cdot)$ w.r.t. the parameter t, then

$$\frac{\partial^2 g_{\Lambda\circ\varphi}}{\partial \tau_1^2}(\tau, \tau_1)\Big|_{\tau_1=\tau} = \frac{\partial^2 g_\Lambda}{\partial t_1^2}(t, t_1)\Big|_{t_1=t=\varphi(\tau)} \big(\varphi'(\tau)\big)^4, \tag{2.9}$$

which implies that the following degree four differential $\mathcal{A} = \frac{1}{2}\frac{\partial^2 g_\Lambda}{\partial t_1^2}(t, t_1)|_{t_1=t}(dt)^4$ on the curve $\Lambda(\cdot)$ does not depend on the choice of the projective parametrization (by degree four differential on the curve we mean the following: for any point of the curve a degree 4 homogeneous function is given on the tangent line to the curve at this point). This degree four differential is called the *fundamental form* of the curve.

If t is an arbitrary (not necessarily a projective) parametrization on the curve $\Lambda(\cdot)$, then the fundamental form in this parametrization is of the form $A(t)(dt)^4$, where $A(t)$ is a smooth function, called the *density* of the fundamental form w.r.t. the parametrization t. The density $A(t)$ can be expressed by the generating function g_Λ in the following way (see [3, Lemma 5.1]):

$$A(t) = \frac{1}{2}\frac{\partial^2}{\partial t_1^2} g_\Lambda(t, t_1)\Big|_{t_1=t} - \frac{3}{5k} g_\Lambda(t, t)^2 - \frac{3}{20}\frac{d^2}{dt^2} g_\Lambda(t, t). \tag{2.10}$$

Remark 2.4. Fix a parametrization t of the curve $\Lambda(\cdot)$. Suppose that in some coordinates $W \cong \mathbb{R}^m \times \mathbb{R}^m$ and $\Lambda(t) = \{(x, S_t x)\colon x \in \mathbb{R}^m\}$ for some $m \times m$-matrix S_t. Then

$$g_\Lambda(t_0, t_1) = \frac{\partial^2}{\partial t_0 \partial t_1} \ln \frac{\det(S_{t_0} - S_{t_1})}{(t_0 - t_1)^k} \tag{2.11}$$

(see relations (4.9), (4.11), and Lemma 4.2 in [3]). From this and (2.10) it follows that for any t_0 the density $A(t_0)$ w.r.t. the parametrization t of the fundamental form of $\Lambda(\cdot)$ at t_0 is a rational expression w.r.t. some entries of the matrices $\dot S(t_0), \ddot S(t_0), \dots, S^{(j)}(t_0)$ for some $j > 0$.

If the fundamental form $\mathcal{A}$ of the curve $\Lambda(\cdot)$ is not zero at any point of $\Lambda(\cdot)$, then the canonical length element $\sqrt[4]{|\mathcal{A}|}$ is defined on $\Lambda(\cdot)$. The length w.r.t. this length element gives the *canonical* parametrization of the unparameterized curve, well defined up to translation. The Ricci curvature w.r.t. this parametrization is a functional invariant of the unparameterized curve, which is called its *projective Ricci curvature*. If $t = \varphi(\tau)$ is the transition function between the canonical parametrization τ and some projective parametrization t, then by (2.7) it follows that the projective Ricci curvature is equal to $\frac{k}{3}\mathbb{S}(\varphi(\tau))$.

Remark 2.5. By construction, if τ is the canonical parametrization then $\mathcal{A} = \pm(d\tau)^4$.

Remark 2.6. Note that in the case $m = 1$ the fundamental form is always identically zero (see [3], Lemma 5.2 there): in this case the only invariant of an unparameterized curve in the corresponding Lagrange Grassmannian is the canonical projective structure on it.

Another important characteristic of the curve $\Lambda(\cdot)$ in $G_m(W)$ is the rank of its velocities. Take a smooth moving frame $(E_1(\tau), \dots, E_m(\tau))$ which spans the subspace $\Lambda(\tau)$ for any τ. Set

$$\mathcal{D}^{(i)}\Lambda(\tau) = \operatorname{span}\left\{\frac{d^k E_j(\tau)}{d\tau^k}\colon 0 \leqslant k \leqslant i,\ 1 \leqslant j \leqslant m\right\}. \tag{2.12}$$

It is clear that the subspaces $\mathcal{D}^{(i)}\Lambda(\tau)$ do not depend on the choice of frames $(E_1(\tau), \dots, E_m(\tau))$. The *rank* $r(\tau)$ *of* $\Lambda(\cdot)$ at τ is equal by definition to the difference $\dim \mathcal{D}^{(1)}\Lambda(\tau) - \dim \Lambda(\tau)$.

Remark 2.7. Note that the tangent space $T_\Lambda G_m(W)$ to any subspace $\Lambda \in G_m(W)$ can be identified with the space $\mathrm{Hom}(\Lambda, W/\Lambda)$ of linear mappings from Λ to W/Λ. Namely, take a curve $\Lambda(t) \in G_m(W)$ with $\Lambda(0) = \Lambda$. Given some vector $l \in \Lambda$, take a curve $l(\cdot)$ in W such that $l(t) \in \Lambda(t)$ for all sufficiently small t and $l(0) = l$. Denote by $\mathrm{pr}\colon W \mapsto W/\Lambda$ the canonical projection on the factor. It is easy to see that the mapping $l \mapsto \mathrm{pr}\, l'(0)$ from Λ to W/Λ is a linear mapping depending only on $\frac{d}{dt}\Lambda(0)$. In this way we identify $\frac{d}{dt}\Lambda(0) \in T_\Lambda G_m(W)$ with some element of $\mathrm{Hom}(\Lambda, W/\Lambda)$ (a simple counting of the dimensions shows that these correspondence between $T_\Lambda G_m(W)$ and $\mathrm{Hom}(\Lambda, W/\Lambda)$ is a bijection). By construction, the rank of the curve $\Lambda(t)$ at the point τ in $G_m(W)$ is actually equal to the rank of the linear mapping corresponding to its velocity $\frac{d}{dt}\Lambda(t)$ at τ. If W is endowed with some symplectic form $\bar\sigma$ and $L(W)$ is the corresponding Lagrange Grassmannian, then the tangent space $T_\Lambda L(W)$ to any $\Lambda \in L(W)$ can be identified with the space of quadratic forms $Q(\Lambda)$ on the linear space Λ. Namely, let $\Lambda(t)$ and $l(t)$ be as above (where $G_m(W)$ is substituted by $L(W)$). It is easy to see that the quadratic form $l \mapsto \bar\sigma(l'(0), l)$ depends only on $\frac{d}{dt}\Lambda(0)$. In this way we identify $\frac{d}{dt}\Lambda(0) \in T_\Lambda L(W)$ with some element of $Q(\Lambda)$ (a simple counting of the dimensions shows that these correspondence between $T_\Lambda L(W)$ and $Q(\Lambda)$ is a bijection).

Using the identification in the previous remark one can define the notion of monotone curves in the Lagrange Grassmannian: the curve $\Lambda(t)$ in $L(W)$ is called *nondecreasing* (*nonincreasing*) if its velocities $\frac{d}{dt}\Lambda(t)$ at any point are nonnegative (nonpositive) definite quadratic forms.

As we will see in the next section the rank of Jacobi curves of characteristic curves of a rank 2 distribution is identically equal to 1. There is a simple criterion for rank 1 curves in the Lagrange Grassmannian to be of constant weight in terms of the subspaces $\mathcal{D}^{(i)}\Lambda(\tau)$, defined above:

Proposition 2.1. *The curve $\Lambda(\cdot)$ of constant rank* 1 *in the Lagrange Grassmannian $L(W)$ of a symplectic space W,* dim $W = 2m$*, has a constant finite weight in a neighborhood of the point τ iff*

$$\dim \mathcal{D}^{(m)}\Lambda(\tau) = 2m. \tag{2.13}$$

In this case the weight is equal to m^2.

The proof of the proposition can be easily obtained by application of some formulas and statements of Sections 6 and 7 of [3] (for example, formulas (6.15), (6.16), (6.18), (6.19), Proposition 4, and Corollary 2 there).

Note also that from the fact that the rank of the curve is equal to 1 it follows easily that

$$\dim \mathcal{D}^{(i)}\Lambda(\tau) - \dim \mathcal{D}^{(i-1)}\Lambda(\tau) \leqslant 1. \tag{2.14}$$

Therefore the condition (2.13) is equivalent to the relation $\dim \mathcal{D}^{(i)}\Lambda(\tau) = m + i$ for all $i = 1, \dots, m$.

3. Fundamental form and projective Ricci curvature of rank 2 distribution

3.1. Properties of Jacobi curves of regular abnormal extremals of rank 2 distributions

In this subsection we find under what assumption on germ of a $(2, n)$-distribution ($n \geqslant 4$) with small growth vector of the type $(2, 3, 4$ or $5, \dots)$ one can apply the theory of Section 2.3. First note that the Jacobi curve J_γ of a characteristic curve γ of the distribution D defined by (2.2) is not ample, because all subspaces $J_\gamma(\lambda)$ have a common line. Indeed, let $\delta_a : T^*M \mapsto T^*M$ be the homothety by $a \neq 0$ in the fibers, namely,

$$\delta_a(p, q) = (ap, q), \quad q \in M,\ p \in T_q^*M. \tag{3.1}$$

Denote by $\vec{e}(\lambda)$ the following vector field called Euler field: $\vec{e}(\lambda) = \frac{\partial}{\partial a}\delta_a(\lambda)|_{a=1}$.

Remark 3.1. Obviously, if γ is a characteristic curve of D, then also $\delta_a(\gamma)$ is.

It implies that the vectors $\phi_*(\vec{e}(\lambda))$ coincide for all $\lambda \in \gamma$, so the line $E_\gamma \stackrel{def}{=} \{\mathbb{R}\phi_*(\vec{e}(\lambda))\}$ is common for all subspaces $J_\gamma(\lambda)$, $\lambda \in \gamma$ (here, as in Introduction, $\phi : O_\gamma \to N$ is the canonical projection on the factor $N = O_\gamma/(Ab_D|_{O_\gamma})$, where O_γ is a sufficiently small tubular neighborhood of γ in $(D^2)^\perp$). Therefore it is natural to make an appropriate factorization by this common line E_γ. Namely, by above all subspaces $J_\gamma(\lambda)$ belong to the skew-symmetric complement $E_\gamma^\angle$ of E_γ in $T_\gamma N$. Denote by $p : T_\gamma N \mapsto T_\gamma N/E_\gamma$ the canonical projection on the factor-space.

The mapping $\widetilde{J}_\gamma(\lambda) : \gamma \mapsto L(E_\gamma^\angle/E_\gamma)$, defined by $\widetilde{J}_\gamma(\lambda) = p(J_\gamma(\lambda))$, is called the *reduced Jacobi curve* of the characteristic curve γ. Note that

$$\dim \widetilde{J}_\gamma(\lambda) = n - 3. \tag{3.2}$$

Now the question is at which points $\lambda \in \gamma$ the germ of the reduced Jacobi curve has constant weight? The answer on this question can be easily done in terms of rank $(n-1)$ distribution $\mathcal{J}$ defined by (2.1) on $(D^2)^\perp \backslash (D^3)^\perp$.

First note that for any $\lambda \in \gamma$ one can make the following identification

$$T_\gamma N \sim T_\lambda (D^2)^\perp / \ker \sigma|_{(D^2)^\perp}(\lambda). \tag{3.3}$$

Take on O_γ any vector field H tangent to the characteristic 1-foliation Ab_D and without stationary points, i.e., $H(\lambda) \in \ker \sigma|_{(D^2)^\perp}(\lambda)$, $H(\lambda) \neq 0$ for all $\lambda \in O_\gamma$. Then it is not hard to see that under identification (3.3) one has

$$\widetilde{J}_\gamma(e^{tH}\lambda) = (e^{-tH})_*\bigl(\mathcal{J}(e^{tH}\lambda)\bigr)/\operatorname{span}\bigl(\ker \sigma|_{(D^2)^\perp}(\lambda), \vec{e}(\lambda)\bigr) \tag{3.4}$$

where e^{tH} is the flow generated by the vector field H. Recall that for any vector field ℓ in $(D^2)^\perp$ one has

$$\frac{d}{dt}\bigl((e^{-tH})_*\ell\bigr) = (e^{-tH})_*[H, \ell]. \tag{3.5}$$

Set $\mathcal{J}^{(0)} = \mathcal{J}$ and define inductively

$$\mathcal{J}^{(i)}(\lambda) = \mathcal{J}^{(i-1)}(\lambda) + \big\{[H, V](\lambda)\colon\ H \in \ker\sigma|_{(D^2)^\perp},\ V \in \mathcal{J}^{(i-1)} \text{ are vector fields}\big\} \tag{3.6}$$

or shortly $\mathcal{J}^{(i)} = \mathcal{J}^{(i-1)} + [\ker\sigma|_{(D^2)^\perp}, \mathcal{J}^{(i-1)}]$. Then by definition of the operation $\mathcal{D}^{(i)}$ (see (2.12)) and formulas (3.4), (3.5) it follows that

$$\mathcal{D}^{(i)}\widetilde{J}_\gamma(e^{tH}\lambda)|_{t=0} = \mathcal{J}^{(i)}(\lambda)/\operatorname{span}\big(\ker\sigma|_{(D^2)^\perp}(\lambda), \vec{e}(\lambda)\big), \tag{3.7}$$

which in turn implies that

$$\dim\mathcal{D}^{(i)}\widetilde{J}_\gamma(e^{tH}\lambda)|_{t=0} - \dim\mathcal{D}^{(i-1)}\widetilde{J}_\gamma(e^{tH}\lambda)|_{t=0} = \dim\mathcal{J}^{(i)}(\lambda) - \dim\mathcal{J}^{(i-1)}(\lambda). \tag{3.8}$$

Proposition 3.1. *The (reduced) Jacobi curve of a characteristic curve of a $(2,n)$-distribution ($n \geqslant 4$) with small growth vector of the type $(2, 3, 4$ or $5, \dots)$ is of rank 1 at any point.*

Proof. First show that the (reduced) Jacobi curve has rank 1 at any point. For this, according to (3.8), it is sufficient to prove that

$$\dim\mathcal{J}^{(1)}(\lambda) - \dim\mathcal{J}(\lambda) = 1. \tag{3.9}$$

Let X_1, X_2 be two vector fields, constituting a basis of the distribution D, i.e., for any $q \in M$ $D(q) = \operatorname{span}(X_1(q), X_2(q))$. Since our study is local, we can always suppose that such basis exists, restricting ourselves, if necessary, on some coordinate neighborhood instead of whole M. Given the basis X_1, X_2 one can construct a special vector field tangent to the characteristic 1-foliation Ab_D. For this suppose that

$$X_3 = [X_1, X_2] \bmod D, \qquad X_4 = [X_1, X_3] \bmod D^2, \qquad X_5 = [X_2, X_3] \bmod D^2. \tag{3.10}$$

Let us introduce the "quasi-impulses" $u_i : T^*M \mapsto \mathbb{R}$, $1 \leqslant i \leqslant 5$,

$$u_i(\lambda) = p \cdot X_i(q), \quad \lambda = (p, q),\ q \in M,\ p \in T_q^*M. \tag{3.11}$$

Then by definitions $(D^2)^\perp = \{\lambda \in T^*M\colon\ u_1(\lambda) = u_2(\lambda) = u_3(\lambda) = 0\}$. For given function $G : T^*M \mapsto \mathbb{R}$ denote by $\vec{G}$ the corresponding Hamiltonian vector field defined by the relation $\sigma(\vec{G}, \cdot) = dG(\cdot)$. Then it is easy to show (see, for example [19]) that

$$\ker\sigma|_{D^\perp}(\lambda) = \operatorname{span}\big(\vec{u}_1(\lambda), \vec{u}_2(\lambda)\big), \quad \forall\lambda \in D^\perp, \tag{3.12}$$

$$\ker\sigma|_{(D^2)^\perp}(\lambda) = \mathbb{R}\big((u_4\vec{u}_2 - u_5\vec{u}_1)(\lambda)\big), \quad \forall\lambda \in (D^2)^\perp\backslash(D^3)^\perp. \tag{3.13}$$

The last relation implies that the following vector field

$$\vec{h}_{X_1,X_2} = u_4\vec{u}_2 - u_5\vec{u}_1 \tag{3.14}$$

is tangent to the characteristic 1-foliation (this field is actually the restriction on $(D^2)^\perp$ of the Hamiltonian vector field of the function $h_{X_1,X_2} = u_4u_2 - u_5u_1$).

Suppose that $\dim D^3(q) = 5$ for any q (the case when $\dim D^3(q) = 4$ for some q can be treated similarly and it is left to the reader). Let us complete the tuple $(X_1, X_2, X_3, X_4, X_5)$ to a local frame $X_1, \dots, X_n$ on M. Similarly to (3.11) define the "quasi-impulses" $u_i : T^*M \mapsto \mathbb{R}$, $5 < i \leqslant n$. The tuple of functions $\{u_i\}_{i=1}^n$ defines coordinates on any fiber T_q^*M. Let us denote

$$\partial_\theta = u_4\partial_{u_5} - u_5\partial_{u_4}, \qquad \mathcal{X} = u_5\vec{u}_2 + u_4\vec{u}_1 - \big(u_4^2 + u_5^2\big)\partial_{u_3}, \qquad F = \vec{u}_3 + u_4\partial_{u_1} + u_5\partial_{u_2}. \tag{3.15}$$

On $(D^2)^\perp\backslash(D^3)^\perp$, using (2.1) and (3.12), one has

$$\mathcal{J} = \operatorname{span}(\vec{h}_{X_1,X_2}, \vec{e}, \mathcal{X}, \partial_\theta, \partial_{u_6}, \dots, \partial_{u_n}). \tag{3.16}$$

By direct computations, one can obtain that

$$[\vec{h}_{X_1,X_2}, \partial_{u_i}] \in \operatorname{span}(\vec{e}, \partial_\theta, \partial_{u_6}, \dots, \partial_{u_n}), \quad 6 \leqslant i \leqslant n, \tag{3.17}$$

$$[\vec{h}_{X_1,X_2}, \partial_\theta] \equiv \mathcal{X} \ \bigl(\mathrm{mod}\bigl(\mathrm{span}(\vec{h}_{X_1,X_2}, \vec{e}, \partial_\theta, \partial_{u_6}, \ldots, \partial_{u_n})\bigr)\bigr), \tag{3.18}$$

$$[\vec{h}_{X_1,X_2}, \mathcal{X}] \equiv -\bigl(u_4^2 + u_5^2\bigr) F \ (\mathrm{mod}\, \mathcal{J}). \tag{3.19}$$

From this and the definition of $\mathcal{J}^{(1)}$ it follows that

$$\mathcal{J}^{(1)} = \mathbb{R}F \oplus \mathcal{J}, \tag{3.20}$$

which implies (3.9). Finally, from (3.15) and (3.19), it follows easily that $\bar{\sigma}([h, \mathcal{X}], \mathcal{X}) = (u_4^2 + u_5^2)^2 > 0$, which implies that the curve $t \mapsto \widetilde{J}_\gamma(e^{t\vec{h}_{x_1,x_2}}\lambda)$ is nondecreasing (see Remark 2.7 and the sentence after it). □

Proposition 2.1, relation (2.14), Proposition 3.1 and relation (3.8) imply immediately the following characterization of the points of $(D^2)^\perp \backslash (D^3)^\perp$ in which the germ of the corresponding reduced Jacobi curve has a constant weight:

Proposition 3.2. *For any $\lambda \in (D^2)^\perp \backslash (D^3)^\perp$ the following relation holds*

$$\dim \mathcal{J}^{(i)}(\lambda) - \dim \mathcal{J}^{(i-1)}(\lambda) \leqslant 1, \quad \forall i = 1, \ldots, n-3. \tag{3.21}$$

The germ of the reduced Jacobi curve $\widetilde{J}_\gamma$ at $\lambda \in \gamma$ has a constant weight iff

$$\dim \mathcal{J}^{(n-3)}(\lambda) = 2n - 4. \tag{3.22}$$

In this case the weight is equal to $(n-3)^2$.

From (3.21) it follows that (3.22) is equivalent to relations $\mathcal{J}^{(i)}(\lambda) = n - 1 + i$ for all $i = 1, \ldots, n-3$.

Denote by $\mathcal{R}_D$ the set of all $\lambda \in (D^2)^\perp \backslash (D^3)^\perp$ such that the germ of the reduced Jacobi curve $\widetilde{J}_\gamma$ at $\lambda \in \gamma$ has a constant weight. By the previous proposition,

$$\mathcal{R}_D = \bigl\{\lambda \in (D^2)^\perp \backslash (D^3)^\perp \colon \dim \mathcal{J}^{(n-3)}(\lambda) = 2n - 4\bigr\}. \tag{3.23}$$

Also $\forall q \in M$ let

$$\mathcal{R}_D(q) = \mathcal{R}_D \cap T_q^* M \tag{3.24}$$

and $(D^2)^\perp(q)$ be as in (1.2). The question is whether for generic germ of a rank 2 distribution at q the set $\mathcal{R}_D(q)$ is not empty so that we can apply the theory, presented in Section 2.3.

For this first we will investigate the following question: suppose that the reduced Jacobi curve of the regular abnormal extremal γ has constant weight; what can be said about the corresponding abnormal trajectory $\xi = \pi(\gamma)$? Take some basis (X_1, X_2) for the distribution D in a neighborhood of the curve ξ such that ξ is tangent to the line distribution spanned by X_1 (since our considerations are local we always can do it, restricting ourselves, if necessary, to some subinterval of ξ). For any $q \in \xi$ denote by $\mathcal{T}_\xi^{(i)}(q)$ the following subspace of $T_q M$:

$$\mathcal{T}_\xi^{(i)}(q) = \mathrm{span}\bigl(X_1(q), X_2(q), \mathrm{ad}\, X_1(X_2)(q), \ldots, (\mathrm{ad}\, X_1)^i(X_2)(q)\bigr). \tag{3.25}$$

It is easy to see that the subspaces $\mathcal{T}_\xi^{(i)}(q)$ do not depend on the choice of the local basis (X_1, X_2) with the above property, but only on the germs of the distribution D and the curve ξ at q. The property of the curve ξ to be an abnormal trajectory can be described in terms of $\mathcal{T}_\xi^{(i)}(q)$:

Proposition 3.3. *If γ is an abnormal extremal in $(D^2)^\perp \backslash (D^3)^\perp$ and ξ is the corresponding abnormal trajectory, $\xi = \pi(\gamma)$, then $\forall \lambda \in \gamma$ the following relations hold*

$$\mathcal{T}_\xi^{(i)}\bigl(\pi(\lambda)\bigr) = \pi_* \mathcal{J}^{(i)}(\lambda), \tag{3.26}$$

$$\dim \mathcal{T}_\xi^{(i)}\bigl(\pi(\lambda)\bigr) = \dim \mathcal{J}^{(i)}(\lambda) - (n-3). \tag{3.27}$$

Proof. Let, as before, H be some vector field without stationary points tangent to the characteristic 1-foliation Ab_D in a neighborhood of γ. Also, let $\widetilde{\mathcal{X}}$ be some vector field in a neighborhood of γ such that $\pi_*(\mathrm{span}(H(\lambda), \widetilde{\mathcal{X}}(\lambda))) = D(\pi(\lambda))$. Then from construction of $\mathcal{J}^{(i)}$ and relations (3.17)–(3.19) it follows easily that

$$\mathcal{J}^{(i)}(\lambda) = \mathrm{span}\big(T_\lambda\big((D^2)^\perp\big(\pi(\lambda)\big)\big), H(\lambda), \widetilde{\mathcal{X}}(\lambda), \mathrm{ad}\, H(\widetilde{\mathcal{X}})(\lambda), \ldots, (\mathrm{ad}\, H)^i(\widetilde{\mathcal{X}})(\lambda)\big). \tag{3.28}$$

Take some n-dimensional submanifold Σ of $(D^2)^\perp$, passing through γ transversal to the fibers $(D^2)^\perp(\pi(\lambda))$ for any $\lambda \in \gamma$. By construction, π projects some neighborhood $\widetilde{\Sigma}$ of γ in Σ bijectively to some neighborhood V of ξ in M. Taking

$$X_1\big(\pi(\lambda)\big) = \pi_* H(\lambda), \quad X_2\big(\pi(\lambda)\big) = \pi_*\widetilde{\mathcal{X}}(\lambda), \qquad \forall \lambda \in \widetilde{\Sigma}$$

and using Eqs. (3.25), (3.28), one obtains (3.26). Relation (3.27) follows from (3.26) and the fact that the fiber $(D^2)^\perp(q)$ is $(n-3)$-dimensional. This concludes the proof. □

Corollary 1. *The reduced Jacobi curve of the regular abnormal extremal γ has constant weight iff*

$$\dim \mathcal{T}_\xi^{(n-3)}(q) = n-1, \quad \forall q \in \xi, \tag{3.29}$$

where $\xi = \pi(\gamma)$ is the abnormal trajectory corresponding to γ.

Remark 3.2. Note that a smooth curve ξ in M, satisfying (3.29) together with the following relation

$$\dim \mathcal{T}_\xi^{(n-2)}(q) = n-1, \quad \forall q \in \xi, \tag{3.30}$$

is a corank 1 abnormal trajectory (see Remark 2.1 for definition of corank). If in addition to (3.29) and (3.30) the following relation holds

$$\mathcal{T}_\xi^{(n-3)}(q) + D^3(q) = T_q M, \quad \forall q \in \xi, \tag{3.31}$$

then the curve ξ is a regular abnormal extremal. In terms of a local basis (X_1, X_2) such that ξ is tangent to the line distribution spanned by X_1 the condition (3.31) is equivalent to the fact that for all $q \in \xi$ the vectors $X_1(q), X_2(q), \mathrm{ad}\, X_1(X_2)(q), \ldots, (\mathrm{ad}\, X_1)^{n-3}(X_2)(q)$, and $[X_2, [X_1, X_2]](q)$ span the whole tangent space $T_q M$. The assertions of this remark can be deduced without difficulties from the fact that abnormal trajectories are critical points of certain endpoint mapping (or time × input/state mapping) and from the expression for the first differential for this mapping (one can use, for example, [6, Section 4]).

Remark 3.3. If the germ of a regular abnormal trajectory ξ at some point q_0 has corank 1, then the set of $q \in \xi$, satisfying (3.29), is open and dense set in some neighborhood of q_0 in ξ.

Now we are ready to prove the following genericity result:

Proposition 3.4. *For a generic germ of $(2, n)$-distribution D at $q_0 \in M$ $(n \geqslant 4)$ the set $\mathcal{R}_D(q_0)$, defined in (3.24), is a nonempty open set in Zariski topology on the linear space $(D^2)^\perp(q_0)$, i.e., $\mathcal{R}_D(q_0)$ is a complement to some proper algebraic variety of $(D^2)^\perp(q_0)$.*

Proof. First note that the set $(D^2)^\perp(q_0)\backslash\mathcal{R}_D(q_0)$ is an algebraic variety in the linear space $(D^2)^\perp(q_0)$. Indeed, choose again a local frame $\{X_i\}_{i=1}^n$ on M such that X_1, X_2 constitute a local basis of D and X_3, X_4, X_5 satisfy (3.10). Then from (3.16), the definitions of the subspaces $\mathcal{J}^{(i)}(\lambda)$ and the vector field $\vec{h}_{X_1,X_2}$ it follows that as a basis of spaces $\mathcal{J}^{(i)}(\lambda)$ one can take some vector fields, which are linear combinations of the fields $\vec{u}_k$, ∂_{u_l} with polynomial in u_j coefficients (here $k, l = 1, \ldots, n$, $j = 4, \ldots, n$). Therefore the set

$$(D^2)^\perp(q_0)\backslash\mathcal{R}_D(q_0) = \big\{\lambda \in (D^2)^\perp(q_0)\colon\ \mathcal{J}^{(n-3)}(\lambda) < 2n-4\big\} \tag{3.32}$$

can be represented as a zero level set of some polynomial in u_j, $j = 4, \ldots, n$.

Further the coefficients of this polynomial are some polynomials in the space of l_n-jets of $(2, n)$-distributions for some natural l_n. We will denote this space by $\mathrm{Jet}_{2,n}(l_n)$. It implies that there exists an open set $\mathcal{U}_n$ in Zariski topology of $\mathrm{Jet}_{2,n}(l_n)$ such that the set $\mathcal{R}_D(q_0)$ is not empty iff the l_n-jet of D at q_0 belongs to $\mathcal{U}_n$. Note that if the set $\mathcal{U}_n$ is not empty, then it is dense in $\mathrm{Jet}_{2,n}(l_n)$. Therefore in order to prove our proposition it is sufficient to give an example of germ of $(2,n)$-distribution such that $\mathcal{R}_D$ is nonempty. As such example one can take the distribution D_0 spanned by the following vector fields

$$X_1 = \frac{\partial}{\partial x_1}, \qquad X_2 = \frac{\partial}{\partial x_2} + \sum_{i=1}^{n-3} \frac{x_1^i}{i!}\frac{\partial}{\partial x_{i+2}} + x_1 x_2 \frac{\partial}{\partial x_n}, \tag{3.33}$$

where $(x_1,\dots,x_n)$ are some local coordinates on M, $q_0 = (0,\dots,0)$. Using Remark 3.2 and Corollary 1, it is easy to see that the curve $(x_1,0,\dots,0)$ is a regular abnormal trajectory and its lift has the reduced Jacobi curve of constant weight. This implies that $\mathcal{R}_{D_0}(q_0) \neq \emptyset$. $\Box$

Below we give an explicit description of the set $\mathcal{R}_D$ for $n = 4, 5$ and 6. In the case $n = 4$, small growth vector $(2,3,4)$, from (3.19) it follows immediately that $\mathcal{R}_D = (D^2)^\perp \backslash (D^3)^\perp$. A similar result holds in the case $n = 5$:

Proposition 3.5. *For any* $(2,5)$*-distribution with small growth vector* $(2,3,5)$ *the following relation holds*

$$\mathcal{R}_D = (D^2)^\perp \backslash (D^3)^\perp. \tag{3.34}$$

Proof. Let the vector fields $\vec{h}_{X_1,X_2}$ and F be as in (3.14) and (3.15) respectively. Then, using (3.20), one can obtain by direct computations that

$$[\vec{h}_{X_1,X_2}, F] = u_4\vec{u}_5 - u_5\vec{u}_4 \ (\mathrm{mod}\, J^{(1)}) \tag{3.35}$$

(actually this formula holds for all $n \geqslant 5$). Hence $\dim J^{(2)}(\lambda) = \dim J^{(1)}(\lambda) + 1 = 6$ for all $\lambda \in (D^2)^\perp \backslash (D^3)^\perp$, which implies (3.34). $\Box$

Remark 3.4. Let D be a $(2,n)$-distribution ($n \geqslant 5$) such that $\dim D^3(q) = 4$ for any q in some neighborhood U. Then from (3.35) it follows easily that $J^{(2)}(\lambda) = J^{(1)}(\lambda)$ for any $q \in \pi(U)$. It implies that $\mathcal{R}_D(q) = \emptyset$ for any such q and the theory of Section 2.3 cannot be directly applied for the reduced Jacobi curves. However it is easy to see that in the considered case D is either the Goursat distribution or by the factorization of the ambient manifold by the corresponding characteristics of D^2 (or series of such factorizations) one can get from this distribution the distribution $\widetilde{D}$, satisfying $\dim \widetilde{D}^3 = 5$.

In the case of a $(2,6)$-distribution D with growth vector of the type $(2,3,5,\dots)$ the set $\mathcal{R}_D$ can be described as follows: Take some $\bar{\lambda} = (\bar{p}, q) \in (D^3)^\perp(q)$ and some vector $v \in D(q)$. Let ν be some vector field tangent to D such that $\nu(q) = v$. Also, let (X_1, X_2) be a local basis of the distribution. Then it is easy to see that the number $\bar{p}\cdot[\nu,[\nu,[X_1,X_2]]](q)$ does not depend on the choice of the vector field ν. Consider the following quadratic form

$$v \mapsto Q_{\bar{\lambda},X_1,X_2}(v) \stackrel{def}{=} \bar{p}\cdot\big[\nu,\big[\nu,[X_1,X_2]\big]\big](q) \tag{3.36}$$

on $D(q)$. A change of the local basis of the distribution causes to the multiplication of this quadratic form on a nonzero constant (which is equal to the determinant of the transition matrix between the bases). For the $(2,6)$-distribution D the linear space $(D^3)^\perp(q)$ is one-dimensional. Therefore the zero level set $\mathcal{K}(q) = \{v \in D(q)\colon Q_{\bar{\lambda},X_1,X_2}(v) = 0\}$ of $Q_{\bar{\lambda},X_1,X_2}$ is the same for all $\bar{\lambda} \in (D^3)^\perp(q)\backslash(0,q)$ and any local basis X_1, X_2 of the distribution.

Proposition 3.6. *For a* $(2,6)$*-distribution* D *with small growth vector of the type* $(2,3,5,\dots)$ *the following relation holds*

$$\mathcal{R}_D(q) = \big\{\lambda \in (D^2)^\perp(q)\colon \pi_*\big(\ker\sigma|_{(D^2)^\perp}(\lambda)\big) \notin \mathcal{K}(q)\big\}. \tag{3.37}$$

The set $\mathcal{R}_D(q) \neq \emptyset$ *iff the small growth vector of* D *at* q *is equal to* $(2,3,5,6)$.

Proof. As before, complete some basis X_1, X_2 of D to the frame $\{X_i\}_{i=1}^6$ on M such that X_3, X_4, X_5 satisfy (3.10). Let c_{ji}^k be the structural functions of this frame, i.e., the functions, satisfying $[X_i, X_j] = \sum_{k=1}^6 c_{ji}^k X_k$. Then from

(3.35) by straightforward calculation it follows that

$$[\vec{h}_{X_1,X_2}, u_4\vec{u}_5 - u_5\vec{u}_4] = [u_4\vec{u}_2 - u_5\vec{u}_1, u_4\vec{u}_5 - u_5\vec{u}_4] = \alpha_6\vec{u}_6 \ \big(\operatorname{mod}\operatorname{span}(\vec{u}_4, \vec{u}_5, J^{(1)})\big), \tag{3.38}$$

where

$$\alpha_6 = c_{52}^6 u_4^2 - (c_{42}^6 + c_{51}^6)u_4u_5 + c_{41}^6 u_5^2. \tag{3.39}$$

From (3.35) and (3.38) it follows that if $J^{(3)}(\lambda) = J^{(2)}(\lambda)$, then $\alpha_6 = 0$. Conversely, if $\alpha_6 = 0$ then by (3.38) $J^{(3)}(\lambda) \subset \operatorname{span}(\vec{u}_4, \vec{u}_5, J^{(1)})$. But by construction $J^{(3)}(\lambda) \subset \vec{e}(\lambda)^{\angle}$ (where $\vec{e}(\lambda)^{\angle}$ is the skew-symmetric complement of $\vec{e}(\lambda)$ in $T_\lambda T^*M$). This together with (3.35) implies that $J^{(3)}(\lambda) = J^{(2)}(\lambda)$. So, $\lambda \in \mathcal{R}_D(q)$ iff $\alpha_6(\lambda) \neq 0$. To prove (3.37) it remains to note that

$$Q_{\bar{\lambda},X_1,X_2}\big(\pi_*(\vec{h}_{X_1,X_2}(\lambda)\big) = C\alpha_6(\lambda),$$

where C is a nonzero constant. The last assertion of the proposition follows from the fact that $\alpha_6 \equiv 0$ iff $c_{ji}^6 = 0$, where $i = 1, 2,\ j = 4, 5$, which is equivalent to the condition $\dim D^4(q) = 5$. □

3.2. Fundamental form of distribution and its properties

For any $\lambda \in \mathcal{R}_D$ take the characteristic curve γ, passing through λ. Let $\mathcal{A}_\lambda$ be the fundamental form of the reduced Jacobi curve $\widetilde{J}_\gamma$ of γ at λ. By construction $\mathcal{A}_\lambda$ is a degree 4 homogeneous function on the tangent line to γ at λ. In the previous subsection to any (local) basis (X_1, X_2) of the distribution D we assigned the vector field $\vec{h}_{X_1,X_2}$ tangent to the characteristic 1-foliation Ab_D (see (3.14)). Let

$$A_{X_1,X_2}(\lambda) = \mathcal{A}_\lambda\big(\vec{h}_{X_1,X_2}(\lambda)\big). \tag{3.40}$$

In this way to any (local) basis (X_1, X_2) of the distribution D we assign the function A_{X_1,X_2} on $\mathcal{R}_D$. If we consider the parametrization $t \mapsto \widetilde{J}_\gamma(e^{t\vec{h}_{X_1,X_2}}\lambda)$ of the reduced Jacobi curve of γ, then $A_{X_1,X_2}(\lambda)$ is the density of the fundamental form of this curve w.r.t. the parametrization t at $t = 0$.

Let $\tilde{X}_1, \tilde{X}_2$ be another basis of the distribution D. By direct computation one has

$$\vec{h}_{\tilde{X}_1,\tilde{X}_2}(\lambda) = \Delta^2\big(\pi(\lambda)\big)\vec{h}_{X_1,X_2}(\lambda), \tag{3.41}$$

where Δ is equal to the determinant of the transition matrix from the basis (X_1, X_2) to the basis $(\tilde{X}_1, \tilde{X}_2)$. From this and the homogeneity of $\mathcal{A}$ it follows that

$$A_{\tilde{X}_1,\tilde{X}_2}(\lambda) = \Delta\big(\pi(\lambda)\big)^8 A_{X_1,X_2}(\lambda). \tag{3.42}$$

Therefore for any $q \in M$ such that $\mathcal{R}_D(q) \neq \emptyset$ the restriction of A_{X_1,X_2} to $\mathcal{R}_D(q)$ is the well defined function, up to the multiplication on a positive constant, or the well defined element of the "positive projectivization" of the space of functions on $\mathcal{R}_D(q)$. We will call it the *fundamental form of the rank* 2 *distribution D at the point q*. From now on we will write $\vec{h}$ instead of $\vec{h}_{X_1,X_2}$ and A instead of A_{X_1,X_2} without special mentioning.

Remark 3.5. According to Section 2.3 (see the sentence after formula (2.8)) any abnormal extremals of $(2, n)$-distribution D lying in $\mathcal{R}_D$ carries the canonical projective structure. It can be shown that in the case $n = 4$, small growth vector $(2, 3, 4)$, our canonical projective structure defined on abnormal extremals (and therefore also on abnormal trajectories) coincides with the projective structure on abnormal trajectories, introduced in [10] (see Proposition 5 there). Note also that by Remark 2.6 and relation (3.2) in the case $n = 4$ the fundamental form is identically equal to zero.

Remark 3.6. Using Remark 2.4 it is easy to see that the fundamental form $A(\lambda)$ is a smooth function for all $\lambda \in \mathcal{R}_D$: one can choose the coordinate representation of the curves $t \mapsto \widetilde{J}_\gamma(e^{t\vec{h}}\lambda)$ smoothly depending on λ and use the fact that the operation of differentiation by t in coordinates corresponds to the operation $\operatorname{ad}\vec{h}$ due to the relation (3.5).

In fact one can say much more about the algebraic structure of the fundamental form of distribution.

Proposition 3.7. *For any $q \in M$ such that $\mathcal{R}_D(q) \neq \emptyset$ the fundamental form of the rank 2 distribution D at the point q is a degree 4 homogeneous rational function on $(D^2)^\perp(q)$, defined up to the multiplication on a positive constant.*

Proof. First let us prove that the fundamental form at q is a rational function on $(D^2)^\perp(q)$. From Remark 2.4 it follows that in order to do this it is sufficient to show that the parametrized reduced Jacobi curves $t \mapsto \widetilde{J}_\gamma(e^{t\vec{h}}\lambda)$ have coordinate representations $t \mapsto S_\lambda(t)$ such that for any natural l all entries of $S_\lambda^{(l)}(0)$, as functions of λ, are rational functions on the fibers $(D^2)^\perp(q)$. For this choose the following $(2n-3)$ vector fields on $(D^2)^\perp$:

$$\partial_\theta,\ X,\ \partial_{u_6},\ \dots,\ \partial_{u_n},\ F,\ Y_4,\ \dots,\ Y_{n-1},\ Z,\ \vec{e},\ \vec{h}, \tag{3.43}$$

where

$$Y_k = u_{k+1}\vec{u}_k - u_k\vec{u}_{k+1} + \sum_{i=1}^{3}\bigl(u_{k+1}\{u_i,u_k\} - u_k\{u_i,u_{k+1}\}\bigr)\partial_{u_i},$$

$$Z = u_4\vec{u}_5 + u_5\vec{u}_4 + \sum_{i=1}^{3}\bigl(u_4\{u_i,u_5\} + u_4\{u_i,u_4\}\bigr)\partial_{u_i}$$

(here $\{u_i,u_j\}$ are Poisson brackets of the Hamiltonians u_i and u_j, i.e., $\{u_i,u_j\} = du_j(\vec{u}_i)$). Let

$$W_\lambda = \bigl(\vec{e}(\lambda)^\angle \cap T_\lambda(D^2)^\perp\bigr)/\operatorname{span}\bigl(\ker\sigma|_{(D^2)^\perp}(\lambda), \vec{e}(\lambda)\bigr) \tag{3.44}$$

(here by $\vec{e}(\lambda)^\angle$ we mean the skew-symmetric complement of $\vec{e}(\lambda)$ in $T_\lambda T^*M$). Then under identification (3.3) the reduced Jacobi curve $\widetilde{J}_\gamma$ lives in the Lagrange Grassmannian $L(W_\lambda)$ of the symplectic space W_λ. Denote by $\mathcal{P}$ the set of all $\lambda \in (D^2)^\perp$ such that the vector fields (3.43) at λ constitute a basis of $T_\lambda(D^2)^\perp$. Evidently, for any $q \in M$ the set $\mathcal{P} \cap (D^2)^\perp(q)$ is a nonempty open set in Zariski topology on the linear space $(D^2)^\perp(q)$. For any $\lambda \in (D^2)^\perp$ the first $2(n-3)$ vectors in (3.43) belong to $\vec{e}(\lambda)^\angle$. Therefore, for any $\lambda \in \mathcal{P}$ the images of the first $2(n-3)$ vectors in (3.43) under the canonical projection from $\vec{e}(\lambda)^\angle \cap T_\lambda(D^2)^\perp$ to W_λ constitute the basis of the space W_λ. Introduce in W_λ the coordinates w.r.t. this basis and suppose that $t \mapsto S_\lambda(t)$ is the corresponding coordinate representation of the curve $t \mapsto \widetilde{J}_\gamma(e^{t\vec{h}}\lambda)$, $\widetilde{J}_\gamma(e^{t\vec{h}}\lambda) = \{x, S_\lambda(t)\}\colon x \in \mathbb{R}^{n-3}\}$. Then from (3.4) and (3.5) it follows that for any natural l all entries of the matrix $S_\lambda^{(l)}(0)$ are some rational combinations of some coordinates of the vectors of the type $(\operatorname{ad}\vec{h})^j(\partial_\theta)(\lambda)$, $(\operatorname{ad}\vec{h})^j(X)(\lambda)$, or $(\operatorname{ad}\vec{h})^j(\partial_{u_i})(\lambda)$, w.r.t. the basis (3.43) (here $6 \leqslant i \leqslant n$, $1 \leqslant j \leqslant l$). But from the form of the vector fields Y_i and Z it is clear that these coordinates are also rational functions on the fibers $(D^2)^\perp(q)$. So, for any q the fundamental form at q is a rational function on the fiber $(D^2)^\perp(q)$.

Now let us show that the fundamental form is homogeneous of degree 4. Indeed, it is clear that $\delta_{a*}\mathcal{J}(\lambda) = J(\delta_a(\lambda))$, where δ_a is the homothety defined by (3.1). This together with Remark 3.1 implies that δ_{a*} induces the symplectic transformation from W_λ to $W_{\delta_a(\lambda)}$, which transforms the curve $\widetilde{J}_\gamma$ to the curve $\widetilde{J}_{\delta_a(\gamma)}$. Therefore the following identity holds

$$\mathcal{A}_{\delta_a(\lambda)}\bigl(\delta_{a*}\vec{h}(\lambda)\bigr) = \mathcal{A}_\lambda\bigl(\vec{h}(\lambda)\bigr). \tag{3.45}$$

On the other hand, one has $\vec{h}(\delta_a(\lambda)) = a\delta_{a*}\vec{h}(\lambda)$. Hence

$$A(\delta_a\lambda) = \mathcal{A}_{\delta_a(\lambda)}\bigl(\vec{h}\bigl(\delta_a(\lambda)\bigr)\bigr) = a^4\mathcal{A}_{\delta_a(\lambda)}\bigl(\delta_{a*}\vec{h}(\lambda)\bigr) = a^4\mathcal{A}_\lambda\bigl(\vec{h}(\lambda)\bigr) = a^4A(\lambda).$$

So A is homogeneous of degree 4. □

In the case $n = 5$ and small growth vector $(2,3,5)$ one can look at the fundamental form of the distribution D from the different point of view. In this case (in contrast to generic $(2,n)$-distributions with $n > 5$) there is only one abnormal trajectory starting at the given point $q \in M$ in the given direction (tangent to $D(q)$). All lifts of this abnormal trajectory can be obtained one from another by the homothety. So they have the same, up to a symplectic transformation, Jacobi curve. It means that one can consider the Jacobi curve and the fundamental form of this curve on the abnormal trajectory instead of the abnormal extremal. Therefore, to any $q \in M$ one can assign a homogeneous degree 4 rational function $\mathring{A}_q$ on the plane $D(q)$ in the following way: $\mathring{A}_q(v) = \mathcal{A}_\lambda(H)$, where $v \in D(q)$ and λ, H

satisfy

$$\pi(\lambda)=q,\qquad \pi_*H=v,\qquad H\in\ker\sigma|_{(D^2)^\perp}(\lambda). \tag{3.46}$$

By above $\mathcal{A}_\lambda(H)$ is the same for any choice of λ and H, satisfying (3.46). $\mathring{A}_q$ will be called the *tangential fundamental form* of the distribution D at the point q. We stress that the tangential fundamental form is the well defined function on $D(q)$ and not the class of functions under the positive projectivization.

The analysis of the algebraic structure presented in the proof of Proposition 3.7 is rather rough. In the sequel we will show that for $n=5$ the fundamental form is always a polynomial on $(D^2)^\perp(q)$ (defined up to the multiplication on a positive constant), while for $n>5$ it is a nonpolynomial rational function for generic distributions.

3.3. Projective curvature of rank 2 distribution with nonzero fundamental form

Denote by $\aleph_D=\{\lambda\in\mathcal{R}_D\colon \mathcal{A}_\lambda\neq 0\}$. Suppose that the set $\aleph_D$ is not empty. Note that for $n=5$ the set $\aleph_D$ is empty iff the distribution is locally equivalent to so-called free nilpotent $(2,5)$-distribution (see Example 1 and Remark 4.4 in Section 3).

For any $\lambda\in\aleph_D$ take the characteristic curve γ, passing through λ. Since by assumptions the fundamental form at λ is not zero, the projective Ricci curvature $\rho D(\lambda)$ of the reduced Jacobi curve $\widetilde{J}_\gamma$ is well defined at λ. So, to the given rank 2 distribution D we assign canonically the function $\rho_D:\aleph_D\mapsto\mathbb{R}$. This function is called *the projective Ricci curvature* of the distribution D.

Note also that on the germ of γ at λ the canonical parameter is defined, up to the shift (see the paragraph before Remark 2.5). Therefore one can define the vector field $\overrightarrow{h_A}$ on $\aleph_D$ by taking the velocities of the characteristic curves parameterized by their canonical parameters. The vector field $\overrightarrow{h_A}$ is invariant of the distribution D and it will be used in [23] for the construction of the canonical frame for rank 2 distributions with nonzero fundamental form.

Now we give a method for computation of the projective curvature ρ_D. Take some local basis X_1,X_2 of D. Let again $\vec{h}=\vec{h}_{X_1X_2}$ and $A=A_{X_1,X_2}$ be as in (3.14), and (3.40) respectively. Also denote by $\rho(\lambda)$ the Ricci curvature of the parameterized curve $t\mapsto\widetilde{J}_\gamma(e^{t\vec{h}}\lambda)$ at the point $t=0$. Note that in contrast to $\rho_D(\lambda)$, the function $\rho(\lambda)$ certainly depends on the local basis of distribution. Using the reparameterization rule (2.7) for the Ricci curvature, one can easily express the projective curvature $\rho_D(\lambda)$ by $\rho(\lambda)$ and $A(\lambda)$. Indeed, let τ be the canonical parameter on γ and t be the parameter defined by the field $\vec{h}$ such that $\tau=0$ and $t=0$ correspond to the point λ. Then by Remark 2.5

$$d\tau=\left|A(e^{t\vec{h}}\lambda)\right|^{1/4}dt. \tag{3.47}$$

Suppose that $t=\varphi(\tau)$. Then by (3.47)

$$\varphi'(\tau)=\left|A(e^{t\vec{h}}\lambda)\right|^{-1/4}. \tag{3.48}$$

Recall that the Jacobi curves under consideration have the weight equal to $(n-3)^2$. So, by (2.7)

$$\rho_D(e^{\tau\vec{h}_A}\lambda)=\rho(e^{\varphi(\tau)\vec{h}}\lambda)\big(\varphi'(\tau)\big)^2+\frac{(n-3)^2}{3}\mathbb{S}\big(\varphi(\tau)\big), \tag{3.49}$$

where $\mathbb{S}(\varphi)$ is Schwarzian of the function φ, defined by (2.8). One can check that Schwarzian satisfies the following relation

$$\mathbb{S}\big(\varphi(\tau)\big)=-y''(\tau)/y(\tau), \tag{3.50}$$

where $y(\tau)=(\varphi'(\tau))^{-1/2}$. By (3.48), $y(\tau)=|A(e^{\tau\overrightarrow{h_A}}\lambda)|^{1/8}$. Substituting this in (3.50) and using (3.47) one can easily obtain

$$\mathbb{S}\big(\varphi(\tau)\big)=\vec{h}\circ\vec{h}\big(\left|A(e^{t\vec{h}}\lambda)\right|^{-1/8}\big)\left|(A(e^{t\vec{h}}\lambda)\right|^{-3/8}. \tag{3.51}$$

Finally, substituting (3.51) with $\tau=0$ in (3.49) we get

$$\rho_D=\frac{\rho}{\sqrt{|A|}}+\frac{(n-3)^2}{3}\frac{\vec{h}\circ\vec{h}(|A|^{-1/8})}{\sqrt[8]{|A|^3}}=\frac{\rho A^2-\frac{(n-3)^2}{24}\vec{h}\circ\vec{h}(A)A+\frac{3(n-3)^2}{64}(\vec{h}(A))^2}{|A|^{5/2}}. \tag{3.52}$$

Since ρ_D is the well defined function on $\aleph_D$ and A is the degree 4 homogeneous rational function on $(D^2)^{\perp}(q)$, defined up to the multiplication on a positive constant, the numerator

$$\mathcal{C} \stackrel{def}{=} \rho A^2 - \frac{(n-3)^2}{24}\vec{h} \circ \vec{h}(A)A + \frac{3(n-3)^2}{64}\big(\vec{h}(A)\big)^2 \tag{3.53}$$

of the right-hand side of (3.52) is a degree 10 homogeneous function on $(D^2)^{\perp}(q)$, defined up to the multiplication on a positive constant. This function will be called the *second fundamental form* of the distribution D. The second fundamental form is a rational function on $(D^2)^{\perp}(q)$, because both A and ρ are rational (the rationality of ρ follows from the same arguments as in Proposition 3.7). In the case $n = 5$ the second fundamental form is a polynomial, which will follow from Theorem 3 below.

4. Calculation of invariants of (2, 5)-distributions

In the present section we give explicit formulas for the computation of the fundamental form and the projective Ricci curvature in the case of rank 2 distributions on a 5-dimensional manifold (as before we assume that the small growth vector is $(2, 3, 5)$). We demonstrate these formulas on several examples, showing simultaneously the efficiency of our invariants in proving that rank 2 distributions are not equivalent.

4.1. Preliminaries

In order to obtain these formulas we need more facts from the theory of curves in the Grassmannian $G_m(W)$ of half-dimensional subspaces (here $\dim W = 2m$) and in the Lagrange Grassmannian $L(W)$ w.r.t. to some symplectic form on W, developed in [3,4]. Below we present all necessary facts from the mentioned papers together with several new useful arguments.

Fix some $\Lambda \in G_m(W)$. As before, let $\Lambda^{\pitchfork}$ be the set of all m-dimensional subspaces of W transversal to Λ. Note that any $\Delta \in \Lambda^{\pitchfork}$ can be canonically identified with W/Λ. Keeping in mind this identification and taking another subspace $\Gamma \in \Lambda^{\pitchfork}$ one can define the operation of subtraction $\Gamma - \Delta$ as follows: $\Gamma - \Delta \stackrel{def}{=} \langle \Delta, \Gamma, \Lambda \rangle \in \mathrm{Hom}(W/\Lambda, \Lambda)$. It is clear that the set $\Lambda^{\pitchfork}$ provided with this operation can be considered as an affine space over the linear space $\mathrm{Hom}(W/\Lambda, \Lambda)$.

Consider now some ample curve $\Lambda(\cdot)$ in $G_m(W)$. Fix some parameter τ. By assumptions $\Lambda(t) \in \Lambda(\tau)^{\pitchfork}$ for all t from a punctured neighborhood of τ. We obtain the curve $t \mapsto \Lambda(t) \in \Lambda(\tau)^{\pitchfork}$ in the affine space $\Lambda(\tau)^{\pitchfork}$ with the pole at τ. We denote by $\Lambda_\tau(t)$ the identical embedding of $\Lambda(t)$ in the affine space $\Lambda(\tau)^{\pitchfork}$. First note that the velocity $\frac{\partial}{\partial t}\Lambda_\tau(t)$ is well defined element of $\mathrm{Hom}(W/\Lambda, \Lambda)$. Fixing an "origin" in $\Lambda(\tau)^{\pitchfork}$ we make $\Lambda_\tau(t)$ a vector function with values in $\mathrm{Hom}(W/\Lambda, \Lambda)$ and with the pole at $t = \tau$. Obviously, only free term in the expansion of this function to the Laurent series at τ depends on the choice of the "origin" we did to identify the affine space with the linear one. More precisely, the addition of a vector to the "origin" results in the addition of the same vector to the free term in the Laurent expansion. In other words, for the Laurent expansion of a curve in an affine space, the free term of the expansion is an element of this affine space. Denote this element by $\Lambda^0(\tau)$. The curve $\tau \mapsto \Lambda^0(\tau)$ is called the *derivative curve* of $\Lambda(\cdot)$. If we restrict ourselves to the Lagrange Grassmannian $L(W)$, i.e., if all subspaces under consideration are Lagrangian w.r.t. some symplectic form $\bar{\sigma}$ on W, then from Remark 2.3 it follows easily that the set $\Lambda_L^{\pitchfork}$ of all Lagrange subspaces transversal to Λ can be considered as an affine space over the linear space of all self-adjoint mappings from Λ^* to Λ, the velocity $\frac{\partial}{\partial t}\Lambda_\tau(t)$ is the well defined self-adjoint mappings from Λ^* to Λ, and the derivative curve $\Lambda^0(\cdot)$ consists of Lagrange subspaces.

Now suppose that the curve $\Lambda(\cdot)$ is nondecreasing rank 1 curve in $L(W)$. Then $\frac{\partial}{\partial t}\Lambda_\tau(t)$ is a nonpositive definite rank 1 self-adjoint map from Λ^* to Λ and for $t \neq \tau$ there exists a unique, up to the sign, vector $w(t, \tau) \in \Lambda(\tau)$ such that for any $v \in \Lambda(\tau)^*$ one has $\langle v, \frac{\partial}{\partial t}\Lambda_\tau(t)v\rangle = -\langle v, w(t, \tau)\rangle^2$. The properties of the vector function $t \mapsto w(t, \tau)$ for a rank 1 curve of constant weight in $L(W)$ can be summarized as follows (see [3, Section 7, Proposition 4 and Corollary 2]):

Proposition 4.1. *If $\Lambda(\cdot)$ is a rank* 1 *curve of constant weight in L(W), then for any $\tau\in I$ the function $t\mapsto w(t,\tau)$ has a pole of order m at $t=\tau$. Moreover, if we write down the expansion of $t\mapsto w(t,\tau)$ in Laurent series at $t=\tau$,*

$$w(t,\tau)=\sum_{i=1}^{m}e_i(\tau)(t-\tau)^{i-1-l}+\mathrm{O}(1),$$

then the vector coefficients $e_1(\tau),\dots,e_m(\tau)$ constitute a basis of the subspace $\Lambda(t)$.

The basis of the vectors $e_1(\tau),\dots,e_m(\tau)$ from the previous proposition is called the *canonical basis* of $\Lambda(\tau)$. Further for given τ take the derivative subspace $\Lambda^0(\tau)$ and let $f_1(\tau),\dots,f_m(\tau)$ be a basis of $\Lambda^0(\tau)$ dual to the canonical basis of $\Lambda(\tau)$, i.e. $\bar\sigma(f_i(\tau),e_j(\tau))=\delta_{i,j}$. The basis $(\{e_i(\tau)\}_{i=1}^m,\{f_i(\tau)\}_{i=1}^m)$ of whole symplectic space W is called *the canonical moving frame* of the curve $\Lambda(\cdot)$. Calculation of the structural equation for the canonical moving frame is another way to obtain symplectic invariants of the curve $\Lambda(\cdot)$.

For the reduced Jacobi curves of abnormal extremals of a $(2,5)$-distribution $m=2$. In this case the structural equation for the canonical moving frame has the following form:

$$\begin{cases} e_1'=3e_2,\\ e_2'=\frac14\rho e_1+4f_2,\\ f_1'=-\big(\frac{35}{36}A-\frac18\rho^2+\frac1{16}\rho''\big)e_1-\frac7{16}\rho' e_2-\frac14\rho f_2,\\ f_2'=-\frac7{16}\rho' e_1-\frac94\rho e_2-3f_1,\end{cases}\tag{4.1}$$

where ρ and A are the Ricci curvature and the density of the fundamental form of the parametrized curve $\Lambda(\cdot)$ respectively (for the proof see [4, Section 2, Proposition 7]). One can express $e_2(\tau)$ by $e_1'(\tau)$ using the first equation of (4.1), then $f_2(\tau)$ by $e_1(\tau)$ and $e_1''(t)$ using the second equation of (4.1), after that $f_1(\tau)$ by $e_1(\tau)$, $e_1'(\tau)$ and $e_1^{(3)}$ using the forth equation of (4.1). Finally substituting all this to the third equation of (4.1) one obtains the following useful

Proposition 4.2. *Suppose that $\Lambda(t)$ is a rank* 1 *curve of constant weight in $L(W)$ and $e_1(t)$ is the first vector in the canonical basis of $\Lambda(t)$. Then $e_1(t)$ satisfies the following relation*:

$$e_1^{(4)}=\left(35A-\frac{81}{16}\rho^2-\frac94\rho''\right)e_1-\frac{15}{2}\rho' e_1'-\frac{15}{2}\rho e_1''.\tag{4.2}$$

The previous proposition says that in order to find ρ and A it is sufficient to know the first vector $e_1(\tau)$ in the canonical basis of $\Lambda(\cdot)$. The following proposition gives a simple way to find the vector $e_1(\tau)$.

Proposition 4.3. *Let $\Lambda(\tau)$ be a rank* 1 *nondecreasing curve of constant weight in the Lagrange Grassmannian $L(W)$, where* $\dim W=4$. *Then the first vector $e_1(\tau)$ of the canonical basis of $\Lambda(\tau)$ can be uniquely* (*up to the sign*) *determined by the following two relations*

$$\mathbb{R}e_1(\tau)=\mathcal{D}^{(1)}\Lambda(\tau)^{\angle},\qquad \bar\sigma\big(e_1''(\tau),e_1'(\tau)\big)=36,\tag{4.3}$$

where the subspace $\mathcal{D}^{(1)}\Lambda(\tau)$ is as in (2.12) and $\mathcal{D}^{(1)}\Lambda(\tau)^{\angle}$ is its skew-symmetric complement.

Proof. The second relation of (4.3) follows directly from the first two equations of (4.1). Further from (4.1) it is clear that $\mathcal{D}^{(1)}\Lambda(\tau)=\operatorname{span}(e_1(\tau),e_2(\tau),f_2(\tau))$, which implies the first relation of (4.3) (see definition of the canonical moving frame). Finally, the vector $e_1(t)$ is determined by (4.3) uniquely, up to the sign: the first relation gives the direction of $e_1(t)$ and the second one "normalizes" this direction. □

4.2. Application to $(2,5)$*-distributions*

Choose some local basis (X_1,X_2) of a $(2,5)$-distribution D and complete it by the fields X_3, X_4, and X_5, satisfying (3.10), to the local frame on M. Such frame (X_1,X_2,X_3,X_4,X_5) will be called *adapted* to the distribution D. If instead of (3.10) one has

$$X_3=[X_1,X_2],\qquad X_4=\bigl[X_1,[X_1,X_2]\bigr]=[X_1,X_3],\qquad X_5=\bigl[X_2,[X_2,X_1]\bigr]=[X_3,X_2],\tag{4.4}$$

the frame (X_1,X_2,X_3,X_4,X_5) will be called *strongly adapted* to D.

We are going to show how to calculate our invariants starting from some adapted frame to the distribution. Let again $\vec{h}=\vec{h}_{X_1,X_2}$ as in (3.14). For any $\lambda\in(D^2)^\perp\backslash(D^3)^\perp$ consider the characteristic curve γ of D passing through λ. Under identification (3.3) the reduced Jacobi curve $\widetilde{J}_\gamma$ lives in the Lagrange Grassmannian $L(W_\lambda)$ of the symplectic space W_λ, defined by (3.44). Let $\epsilon_1(\lambda)$ be the first vector in the canonical basis of the curve $t\mapsto\widetilde{J}_\gamma(e^{t\vec{h}}\lambda)$ at the point $t=0$. Note that it is more convenient to work directly with vector fields of $(D^2)^\perp$, keeping in mind that the symplectic space W_λ belongs to the factor space $T_\lambda((D^2)^\perp)/\operatorname{span}(\vec{h}(\lambda),\vec{e}(\lambda))$. So, in the sequel by $\epsilon_1(\lambda)$ we will mean both the element of W_λ and some representative of this element in $T_\lambda((D^2)^\perp)$, depending smoothly on λ. In the last case all equalities, containing $\epsilon_1(\lambda)$, will be assumed modulo $\operatorname{span}(\vec{h}(\lambda),\vec{e}(\lambda))$. Now we are ready to prove the following

Proposition 4.4. *The vector $\epsilon_1(\lambda)$ can be chosen in the form*

$$\epsilon_1(\lambda)=6\bigl(\gamma_4(\lambda)\partial_{u_4}+\gamma_5(\lambda)\partial_{u_5}\bigr),\tag{4.5}$$

where

$$\gamma_4(\lambda)u_5-\gamma_5(\lambda)u_4\equiv 1.\tag{4.6}$$

Proof. First note that by (3.20) one has that $\operatorname{span}(\partial_{u_4},\partial_{u_5})\subset(\mathcal{J}^1)^\angle$. Hence from the first relation of (4.3) it follows that $\epsilon_1=6(\gamma_4\partial_{u_4}+\gamma_5\partial_{u_5})\ (\operatorname{mod}\operatorname{span}(\vec{h},\vec{e}))$, where $\gamma_4u_5-\gamma_5u_4\neq 0$. Further, denote by $e_1(t)$ the first vector in the canonical basis of the curve $t\mapsto\widetilde{J}_\gamma(e^{t\vec{h}}\lambda)$. Then

$$e_1(t)=(e^{-t\vec{h}})_*\epsilon\bigl(e^{t\vec{h}}(\lambda)\bigr).\tag{4.7}$$

Hence by (3.5)

$$\bar{\sigma}\bigl(e_1'(t),e_1''(t)\bigr)\big|_{t=0}=\sigma\bigl(\bigl[\vec{h},[\vec{h},\epsilon_1]\bigr](\lambda),[\vec{h},\epsilon_1](\lambda)\bigr).\tag{4.8}$$

By direct computation one can show that

$$\begin{aligned}&[\vec{h},\epsilon_1]=6\bigl(\gamma_5\vec{u}_1-\gamma_4\vec{u}_2+(\gamma_4u_4-\gamma_5u_4)\partial_{u_3}\bigr)\ \bigl(\operatorname{mod}\operatorname{span}(\vec{h},\vec{e},\epsilon_1)\bigr),\\&\bigl[\vec{h}[\vec{h},\epsilon_1]\bigr]=6(\gamma_4u_4-\gamma_5u_4)(\vec{u}_3+u_4\partial_{u_1}+u_5\partial_{u_2})\ \bigl(\operatorname{mod}\operatorname{span}(\vec{h},\vec{e},\epsilon_1,[\vec{h},\epsilon_1])\bigr).\end{aligned}\tag{4.9}$$

From (4.9) it is easy to show that the right-hand side of (4.8) is equal to $36(\gamma_4u_5-\gamma_5u_4)^2$, which together with the second relation of (4.3) implies (4.6). □

As a direct consequence of the previous proposition, Proposition 4.2, and relations (3.5), (4.7), (3.52) we obtain

Theorem 2. *Let $\epsilon_1(\lambda)$ be as in (4.5) and (4.6). Then there exist functions A_0, A_1 on $(D^2)^\perp$ such that*

$$(\operatorname{ad}\vec{h})^4(\epsilon_1)=A_0\epsilon_1+\vec{h}(A_1)\operatorname{ad}\vec{h}(\epsilon_1)+A_1(\operatorname{ad}\vec{h})^2(\epsilon_1)\ \operatorname{mod}\bigl(\operatorname{span}(\vec{h},\vec{e})\bigr).\tag{4.10}$$

The fundamental form $A(\lambda)$ and the projective Ricci curvature $\rho_D(\lambda)$ of the distribution D satisfy:

$$35A=A_0+\frac{9}{100}A_1^2-\frac{3}{10}(\vec{h})^2(A_1),\tag{4.11}$$

$$\rho_D=\left(-\frac{2}{15}A_1A^2-\frac{1}{6}\vec{h}\circ\vec{h}(A)A+\frac{3}{16}\bigl(\vec{h}(A)\bigr)^2\right)|A|^{-5/2}.\tag{4.12}$$

Remark 4.1. It is clear that in the previous theorem we can take ϵ_1 satisfying (4.5) and the relation $\gamma_4(\lambda)u_5-\gamma_5(\lambda)u_4\equiv\text{const}$ along any characteristic curve of D (instead of (4.6)). In particular one can take as ϵ_1 one of the following vector fields: $\frac{1}{u_5}\partial_{u_4}$, $\frac{1}{u_4}\partial_{u_5}$, $\frac{(u_5\partial_{u_4}-u_4\partial_{u_5})}{u_4^2+u_5^2}$ or $\frac{(u_5\partial_{u_4}+u_4\partial_{u_5})}{u_5^2-u_4^2}$

Formulas (4.10), (4.11) and (4.12) give an explicit way to calculate the fundamental form and the projective Ricci curvature of the distribution D, starting from some adapted frame to D. The previous theorem allows also to prove the following theorem about the algebraic structure of $(2,5)$-distributions

Theorem 3. *For a $(2,5)$-distribution with small growth vector $(2,3,5)$ the fundamental form at any point q is a degree 4 homogeneous polynomial on the fiber $(D^2)^\perp(q)$, up to the multiplication on a positive constant.*

Proof. Let $\epsilon_1 = \frac{1}{u_5}\partial_{u_4}$. Also denote $\tilde{\mathcal{X}} = \vec{u}_2 - u_5\partial_{u_3}$ and $\tilde{Y}_j = \vec{u}_j + \sum_{i=1}^3 \{u_i, u_j\}\partial_{u_i}$, $j = 4, 5$. Let F be as in (3.15). Then the tuple of the vector fields $(\epsilon_1, \tilde{\mathcal{X}}, F, Y_4, Y_5, \vec{h}, \vec{e})$ constitute a frame on $(D^2)^\perp$. By direct calculations

$$[\vec{h}, \epsilon_1] = -\frac{1}{u_5}\tilde{\mathcal{X}} + p_1\epsilon_1 \mod \mathbb{R}\vec{e}, \tag{4.13}$$

$$(\operatorname{ad}\vec{h})^2(\epsilon_1) = F + p_2\tilde{\mathcal{X}} + p_3\epsilon_1 \mod\bigl(\operatorname{span}(\vec{h}, \vec{e})\bigr), \tag{4.14}$$

where p_i, $i = 1, 2, 3$, are some rational functions in u_4, u_5 with denominator of the form u_5^l. From the form of the vector fields $\vec{h}$ and ϵ_1 it follows that the coordinates of the vector field $(\operatorname{ad}\vec{h})^4(\epsilon_1)$ w.r.t. the frame $(\epsilon_1, \tilde{\mathcal{X}}, F, Y_4, Y_5, \vec{h}, \vec{e})$ are also rational functions in u_4, u_5 with denominator of the form u_5^l. But from (4.10), (4.13) and (4.14) it follows that $(\operatorname{ad}\vec{h})^4\epsilon_1 \subset \operatorname{span}(\epsilon_1, \tilde{\mathcal{X}}, F, \vec{h}, \vec{e})$. Expressing $\tilde{\mathcal{X}}$ and F by ϵ_1, $[\vec{h}, \epsilon_1]$, and $(\operatorname{ad}\vec{h})^2(\epsilon_1)$ from (4.13) and (4.14) $\operatorname{mod}(\operatorname{span}(\vec{h}, \vec{e}))$, one obtains that coefficients A_0, A_1 from (4.10) and hence also the fundamental form A are rational functions in u_4, u_5 with denominator of the form u_5^l. But by Proposition 3.5 and Remark 3.6 A is smooth at the points with $u_5 = 0$, $u_4 \neq 0$. It implies that A has to be a polynomial. □

Corollary 2. *For any $q \in M$ the tangential fundamental form $\mathring{A}_q$ is a degree 4 homogeneous polynomial on $D(q)$.*

Remark 4.2. From the previous corollary it follows that the tangential fundamental form has the same algebraic nature, as the *covariant binary biquadratic form*, constructed by E. Cartan in [11, Chapter VI, Paragraph 33]. In the next paper [22] we prove that our tangential fundamental form coincides (up to constant factor -35) with the Cartan form.

In terms of the canonical projective structure on an abnormal extremal (see Remark 3.5) and the fundamental form one can obtain sufficient conditions for rigidity of the corresponding abnormal trajectory of a $(2,5)$-distribution D: A smooth curve ξ tangent to D and connecting two fixed points q_0 and q_1 is called rigid, if in some C^1-neighborhood of ξ the only curves tangent to D and connecting q_0 with q_1 are reparameterizations of ξ. Rigid curves are automatically abnormal trajectories of D. In [10] for $(2,4)$-distribution it was proved that an abnormal extremal trajectory ξ is rigid if and only if a global projective parameter exists on ξ (see Proposition 3.4 there). The following extension of the if part of this result to $(2,5)$-distributions can be obtained immediately from the result formulated in Remark 2.2 (see also [19, Theorem 4.2]) and the comparison theorems from [4] (Theorem 5, item 1 there):

Proposition 4.5. *For abnormal trajectory ξ of $(2,5)$-distribution to be rigid it is sufficient the existence of a global projective parameter on ξ together with the nonpositivity of the fundamental form along ξ (equivalently nonnegativity of the Cartan form along ξ).*

Moreover, if some Riemannian metric is given on M, then under the same conditions the corresponding abnormal trajectory is the shortest among all curves tangent to the distribution D, connecting its endpoints and sufficiently closed to this abnormal trajectory in C^0-topology. It follows again from the mentioned comparison theorem and from the fact that the simplicity of the Jacobi curve of the abnormal extremal implies the minimality of the length of the corresponding abnormal trajectory in C^0-topology (see [7,8]).

4.3. Examples

Now we will give the results of computations of the fundamental form and the projective Ricci curvature for five examples of concrete distributions or families of distributions. We will omit the calculations but they are straightforward using Theorem 2 (in fact Examples 2 and 3 are included in Example 4; Examples 1–3 and another example similar to Examples 4 and 5 with the detailed computations can be found in [20]).

Example 1 *(Free nilpotent* $(2,5)$*-distribution).* Let L_1 be a 5-dimensional nilpotent Lie algebra with the following commutative relations in some basis $X_1,\dots,X_5$:

$$[X_1,X_2]=X_3,\qquad [X_1,X_3]=X_4,\qquad [X_2,X_3]=X_5,\qquad \operatorname{ad}X_4=0,\qquad \operatorname{ad}X_5=0. \tag{4.15}$$

Actually L_1 is the free nilpotent 3-step Lie algebra with two generators. Let M_1 be the Lie group with the Lie algebra L_1. We consider $X_1,\dots,X_5$ as the left-invariant vector fields on M_1. Let $D_1=\operatorname{span}(X_1,X_2)$. Such distribution is called the *free nilpotent* $(2,5)$*-distribution*.

By (4.15) the tuple of the left-invariants fields (X_1,X_2,X_3,X_4,X_5) constitutes a strong adapted frame to distribution D_1. Applying Theorem 2 to this frame, it is easy to show that the fundamental form A_{D_1} of distribution of D_1 vanishes identically.

Example 2 *(Left-invariant rank 2 distribution on* $SO(3)\times\mathbb{R}^2$*).* One can take the basis a_1,a_2,a_3 on $so(3)$, satisfying the following commutative relations: $[a_1,a_2]=a_3$, $[a_2,a_3]=a_1$, and $[a_3,a_1]=a_2$. Also let b_1,b_2 be a basis of $\mathbb{R}^2$. Denote $D_2=\operatorname{span}((a_1,b_1),(a_2,b_2))$. Consider D_2 as a left-invariant distribution on $SO(3)\times\mathbb{R}^2$.

Remark 4.3. It can be shown easily (see [20]) that the distribution D_2 is a unique, up to a group automorphism of $SO(3)\times\mathbb{R}^2$, left-invariant completely nonholonomic rank 2 distribution on $SO(3)\times\mathbb{R}^2$ and its small growth vector is $(2,3,5)$. Distribution D_2 appears, when one studies the problem of rolling ball on the plane without slipping and twisting (see Example 4 below and also [13] for the details).

Completing the chosen basis (a_1,b_1), (a_2,b_2) of D_2 to the strong adapted frame and applying Theorem 2 to this frame, one has easily that the fundamental form A_{D_2} and the projective Ricci curvature ρ_{D_2} of D_2 satisfy

$$A_{D_2}\sim\left(u_4^2+u_5^2\right)^2,\qquad \rho_{D_2}=4\sqrt{35}/9 \tag{4.16}$$

(here as in the sequel we use the sign $\sim$ to emphasize that the fundamental form at a point is defined up to the multiplication on a positive constant).

Conclusion 1. *Since by (4.16)* A_{D_2} *is not zero the germs of distributions* D_1 *and* D_2 *are not equivalent.*

Remark 4.4. Actually, a $(2,5)$-distribution has the identically zero fundamental form iff it is locally equivalent to the distribution D_1. It follows from the fact that our fundamental form coincides with the Cartan form (see [22]) and the fact that the Cartan form of a distribution is identically zero iff it is locally equivalent to the distribution D_1 (see Chapter VIII of [11]).

Example 3 *(Left-invariant rank 2 distributions on* $SL(2,\mathbb{R})\times\mathbb{R}^2$*).* One can take the basis a_1,a_2,a_3 in $sl(2,\mathbb{R})$, satisfying the following commutative relations: $[a_1,a_2]=a_3$, $[a_2,a_3]=a_1$, $[a_3,a_1]=-a_2$. Let b_1,b_2 be a basis of $\mathbb{R}^2$. Suppose that

$$D_{3,h}=\operatorname{span}\big((a_1,b_1),(a_2,b_2)\big),\qquad D_{3,e}=\operatorname{span}\big((a_1,b_1),(a_3,b_2)\big). \tag{4.17}$$

We consider $D_{3,h}$ and $D_{3,e}$, as left-invariant distributions on the Lie group $SL(2,\mathbb{R})\times\mathbb{R}^2$.

Remark 4.5. It can be shown easily (see [20]) that distributions $D_{3,h}$ and $D_{3,e}$ are the only two different left-invariant rank 2 completely nonholonomic distributions on $SL(2,\mathbb{R})\times\mathbb{R}^2$, up to Lie group automorphisms of $SL(2,\mathbb{R})\times\mathbb{R}^2$, and their small growth vector is $(2,3,5)$. Note that the distribution $D_{3,e}$ appears, when one studies the problem of rolling hyperbolic plane on the Euclidean plane without slipping and twisting, (see geometric model in Example 4 below).

Completing the bases, chosen in (4.17), to the strong adapted frames of $D_{3,h}$ and $D_{3,e}$ and applying Theorem 2 to these frames, one has easily that the fundamental form $A_{D_{3,h}}$ and the projective curvature $\rho_{D_{3,h}}$ of $D_{3,h}$ satisfy

$$A_{D_{3,h}} \sim \left(u_4^2 - u_5^2\right)^2, \qquad \rho_{D_{3,h}} = \begin{cases} -4\sqrt{35}/9 & u_4 > u_5, \\ 4\sqrt{35}/9 & u_4 < u_5, \end{cases} \tag{4.18}$$

while the fundamental form $A_{D_{3,e}}$ and the projective curvature $\rho_{D_{3,e}}$ of $D_{3,e}$ satisfy

$$A_{D_{3,e}} \sim \left(u_4^2 + u_5^2\right)^2, \qquad \rho_{D_{3,e}} = -4\sqrt{35}/9. \tag{4.19}$$

Conclusion 2. *From the first relations of (4.16) and (4.18) it follows that germs of the distributions $D_{3,h}$ and D_2 are not equivalent; from the first relations of (4.18) and (4.19) it follows that germs of the distributions $D_{3,h}$ and $D_{3,e}$ are not equivalent; finally, from the second relations of (4.16) and (4.19) it follows that germs of the distributions D_2 and $D_{3,e}$ are not equivalent* (*in the last case the distributions are distinct by their projective Ricci curvatures but not by the fundamental forms*).

Example 4 *(Rolling of two surfaces of constant curvatures without slipping and twisting).* $(2,5)$-distributions appear naturally when one studies the possible motions of two surfaces S and $\widehat{S}$ in $\mathbb{R}^3$, which roll one on another without slipping and twisting. Here we follow the geometric model of this problem given in [5] (this model ignores the state constraints that correspond to the admissibility of contact of the bodies embedded in $\mathbb{R}^3$). The state space of the problem is the 5-dimensional manifold $M_4 = \{B\colon T_x S \mapsto T_{\hat{x}}\widehat{S} \mid B \text{ is an isometry}\}$.

Let $B(t) \subset M_4$ be an admissible curve, corresponding to the motion of the rolling surfaces. Let $x(t)$ and $\hat{x}(t)$ be trajectories of the contact points in S and $\widehat{S}$ respectively (so, $B(t)$ can be considered as an isometry from $T_{x(t)}S$ to $T_{\hat{x}(t)}\widehat{S}$).The condition of absence of slipping means that

$$B(t)\dot{x}(t) = \dot{\hat{x}}(t), \tag{4.20}$$

while the condition of absence of twisting can be written as follows

$$B(t)\bigl(\text{vector field parallel along } x(t)\bigr) = \text{vector field parallel along } \hat{x}(t). \tag{4.21}$$

From conditions (4.20) and (4.21) it follows that a curve $x(t) \in S$ determines completely the whole motion $B(t) \in M_4$ and the velocities of admissible motions define a $(2,5)$-distribution $D_{4,S,\widehat{S}}$ on M_4. If (v_1, v_2) and $(\hat{v}_1, \hat{v}_2)$ are some local orthonormal frames on S and $\widehat{S}$ respectively and β is the angle of rotation from the frame $(Bv_1(x), Bv_2(x))$ to the frame $(\hat{v}_1(\hat{x}), \hat{v}_2(\hat{x}))$, then the points of M_4 are parametrized by $(x, \hat{x}, \beta)$ and one can choose a local basis of distribution $D_{4,S,\widehat{S}}$ as follows

$$\begin{aligned} X_1 &= v_1 + \cos\beta\,\hat{v}_1 + \sin\beta\,\hat{v}_2 - (-\sigma_1 + \hat{\sigma}_1\cos\beta + \hat{\sigma}_2\sin\beta)\partial_\beta, \\ X_2 &= v_2 - \sin\beta\,\hat{v}_1 + \cos\beta\,\hat{v}_2 + (-\sigma_2 - \hat{\sigma}_1\sin\beta + \hat{\sigma}_2\cos\beta\partial_\beta, \end{aligned} \tag{4.22}$$

where σ_i, $\hat{\sigma}_i$ are structural functions of the frames: $[v_1, v_2] = \sigma_1 v_1 + \sigma_2 v_2$, $[\hat{v}_1, \hat{v}_2] = \hat{\sigma}_1\hat{v}_1 + \hat{\sigma}\hat{v}_2$.

Let us restrict ourselves to the case, when S and $\widehat{S}$ are surfaces of the constant curvatures k and $\hat{k}$ respectively. We will denote the corresponding $(2,5)$-distribution by $D_{4,k,\hat{k}}$. Take spherical, euclidean, or hyperbolic coordinates on S and $\widehat{S}$, according to the sign of the corresponding curvature, and the orthonormal frames (v_1, v_2) and $(\hat{v}_1, \hat{v}_2)$ tangent to the coordinate net. Complete the basis X_1, X_2 from (4.17) with these (v_1, v_2), $(\hat{v}_1, \hat{v}_2)$ to the frame strongly adapted to $D_{4,k,\hat{k}}$. Applying Theorem 2 to this frame, one can obtain by straightforward computations that the fundamental form $A_{D_{4,k,\hat{k}}}$ and the projective curvature $\rho_{D_{4,k,\hat{k}}}$ of $D_{4,k,\hat{k}}$ satisfy

$$A_{D_{4,k,\hat{k}}} \sim \operatorname{sgn}\bigl((9\hat{k} - k)(\hat{k} - 9k)\bigr)(u_4^2 + u_5^2)^2, \tag{4.23}$$

$$\rho_{D_{4,k,\hat{k}}} = \frac{4\sqrt{35}}{3}(k + \hat{k})\bigl|(9\hat{k} - k)(\hat{k} - 9k)\bigr|^{-1/2}. \tag{4.24}$$

Note that by geometry of the problem for the given k and $\hat{k}$ any distributions from the set $\mathrm{Fam}_{k,\hat{k}} = \{D_{4,\alpha k,\alpha\hat{k}}, D_{4,\alpha\hat{k},\alpha k}, \alpha > 0\}$ is equivalent to $D_{4,k,\hat{k}}$. An elementary analysis of the functions involved in the formulas (4.23) and (4.24) implies:

Proposition 4.6. *The germ of distribution $D_{4,k_1,\hat{k}_1}$ is equivalent to the germ of $D_{4,k,\hat{k}}$ if and only if $D_{4,k_1,\hat{k}_1}$ belongs to the set* $\mathrm{Fam}_{k,\hat{k}}$.

In particular, the set of the distributions $\{D_{4,k,1}, k \in \mathbb{R}\}$ gives an example of the one-parametric family of distributions with nonequivalent germs for the different values of the parameter k.

Remark 4.6. From (4.23) and Remark 4.4 it follows that if $\frac{\hat{k}}{k} = 9$ or $\frac{1}{9}$, then a germ of $D_{4,k,\hat{k}}$ is equivalent to a germ of the free nilpotent $(2,5)$-distribution D_1 from Example 1.

Example 5 (*Distributions generated by curves of constant torsion on* 3*-dimensional manifold of constant curvature*). These distributions were mentioned already in [11, Chapter XI, Paragraphs 52, 53]. Let Q be an oriented 3-dimensional Riemannian manifold. Then for given τ the curves of constant torsion τ together with their binormals are admissible curves of a rank 2 distribution on 5-dimensional manifold $M_5 = Q \times S^2$. Indeed, let $\gamma(t)$ be a curve in Q without inflection points, and let $n(t) \in S^2$ be the corresponding binormal. Then γ has a constant torsion τ iff

$$\dot{\gamma}(t) = \frac{1}{\tau} n(t) \times \nabla_{\dot{\gamma}(t)} n(t), \tag{4.25}$$

where by $\times$ we mean the vector product induced on each (oriented) tangent space $T_{\gamma(t)}Q$ by the Riemannian metric and ∇ denotes the covariant derivative, corresponding to this metric. Obviously, relation (4.25) defines the rank 2 distribution on M_5. We restrict ourselves to the case when Q has constant curvature K and denote by $D_{5,\tau,K}$ the corresponding $(2,5)$-distribution. It can be shown that the corresponding fundamental form $A_{D_{5,\tau,K}}$ and the projective Ricci curvature $\rho_{D_{5,\tau,K}}$ satisfy

$$A_{D_{5,\tau,K}} \sim \operatorname{sgn}\big((\tau^2 K^{-1} - 4)(1 - 4\tau^2 K^{-1})\big)(\text{sign definite quadratic form})^2, \tag{4.26}$$

$$\rho_{D_{5,\tau,K}} = \frac{2\sqrt{35}}{3}(\tau^2 K^{-1} + 1)\big|(\tau^2 K^{-1} - 4)(1 - 4\tau^2 K^{-1})\big|^{-1/2}. \tag{4.27}$$

Remark 4.7. Suppose that S is three-dimensional sphere of radius R. If $\tau R = 2$ or $\frac{1}{2}$, then from (4.26) the fundamental form is equal to zero. Hence by Remark 4.4 any germ of the corresponding distribution is equivalent to any germ of the free nilpotent $(2,5)$-distribution D_1 and by Remark 4.6 it is equivalent to any germ of the distribution $D_{4,k,\hat{k}}$ with $\frac{\hat{k}}{k} = 9$ or $\frac{1}{9}$.

Till now we used our invariants in order to prove the nonequivalence of distributions. But what to do, if both the fundamental form and the projective Ricci curvature do not distinct distributions? Comparing (4.23) with (4.26) and (4.24) with (4.27) it is not difficult to show that in the case of positive k and $\hat{k}$ for any ratio $\frac{\hat{k}}{k} \neq 1$ there exists distribution $D_{5,\tau,1/R^2}$, which has the fundamental form of the same type and the same projective Ricci curvature as $D_{4,k,\hat{k}}$. Does it imply that these distributions are equivalent? We will treat the questions of this kind in the forthcoming paper [23]. Below we formulate a theorem, which will be proved in the mentioned paper:

Theorem 4. *For given $s \in \{1, -1\}$ and $\rho \in R$ there exists a unique, up to a diffeomorphism, germ of* $(2,5)$*-distribution satisfying the following three conditions*:

1. *Its fundamental form is s multiplied by the square of a nondegenerated quadratic form Q*;
2. *Its symmetry group is* 6*-dimensional*;
3. *If Q is sign definite, then its projective Ricci curvature is identically equal to ρ, if Q is sign indefinite, then the absolute value of its projective Ricci curvature is identically equal to $|\rho|$.*

Remark 4.8. It can be shown that if a distribution D satisfies condition 1 of Theorem 4, then the dimension of the group of symmetries of D is not greater than 6. It can be shown also that conditions 1 and 2 of Theorem 4 imply that the projective Ricci curvature or its absolute value is identically equal to some constant.

It is easy to see that for positive k and $\hat{k}$ the group of symmetries of distribution $D_{4,k,\hat{k}}$ contains a subgroup isomorphic to $SO(3)\times SO(3)$ and therefore by Remark 4.8 it is 6-dimensional for $\frac{\hat{k}}{k}\neq 9$ or $\frac{1}{9}$, while the group of symmetries of distribution $D_{5,\tau,1/R^2}$ contains a subgroup isomorphic to $SO(4)$ and therefore by Remark 4.8 it is also 6-dimensional for $\tau R\neq 2$ or $\frac{1}{2}$. Therefore Theorem 4 implies

Corollary 3. *If the distributions $D_{4,k,\hat{k}}$ ($\hat{k}>k>0$) and $D_{5,\tau,1/R^2}$ have the fundamental forms of the same sign and their projective Ricci curvatures are equal, then germs of these distributions are equivalent.*

5. Algebraic structure of fundamental form in the case $n>5$

In the present section we show that in the case $n>5$ the fundamental form is in general a rational function, which is not a polynomial. By Remark 3.6 singularities of the fundamental form could occur out of the set $\mathcal{R}_D$, i.e., at the points, where the weight of the corresponding Jacobi curve is not constant.

First, take some curve $\Lambda(t)$ in the Grassmannian of half-dimensional subspaces $G_m(W)$ and suppose that in some punctured neighborhood of $\bar{t}$ the curve $\Lambda(t)$ has a constant weight k, while at $\bar{t}$ it has the weight $k+1$. In this case we will say that $\bar{t}$ is the *point of the weight jump one* of $\Lambda(t)$.

Lemma 5.1. *If the curve $\Lambda(t)$ in $G_m(W)$ has the point of the weight jump one at $\bar{t}$, then the generalized Ricci curvature has a pole of order 2 at $t=\bar{t}$. If in addition the weight of $\Lambda(t)$ in the punctured neighborhood of $\bar{t}$ is greater than 1, then the density of the fundamental form of this curve has a pole of order 4 at $t=\bar{t}$.*

The proof of this lemma follows by direct computations from formula (2.11), the definition of generalized Ricci curvature, and formula (2.10).

Lemma 5.2. *The point $\bar{t}$ is the point of the weight jump one of rank 1 curve $\Lambda(t)$ in the Lagrange Grassmannian $L(W)$, $\dim W=2m$, iff the following relations hold*

$$\dim\mathcal{D}^{(m-1)}\Lambda(\bar{t})=\dim\mathcal{D}^{(m)}\Lambda(\bar{t})=2m-1,\qquad \dim\mathcal{D}^{(m+1)}\Lambda(\bar{t})=2m. \tag{5.1}$$

The proof of this lemma can be easily obtained by application of some formulas and statements of Sections 6 and 7 of [3] (for example, formulas (6.15), (6.16), (6.18), (6.19), Lemma 6.1 and Proposition 3 there).

Let us apply Lemmas 5.1 and 5.2 to the distribution D. For this let the subspace $\mathcal{J}^{(i)}(\lambda)$ be as in (3.6). Set

$$\begin{aligned}S_D^0(q)=\bigl\{\lambda\in(D^2)^\perp(q)\backslash(D^3)^\perp(q)\colon\ &\dim\mathcal{J}^{(n-4)}(\lambda)=\dim\mathcal{J}^{(n-3)}(\lambda)=2n-5,\\ &\dim\mathcal{J}^{(n-2)}(\lambda)=2n-4\bigr\}.\end{aligned} \tag{5.2}$$

By Lemma 5.2 the set $S_D^0(q)$ coincides with the subset of $(D^2)^\perp(q)\backslash(D^3)^\perp(q)$, consisting of points, in which the corresponding reduced Jacobi curves have the weight jump one. Also, from Proposition 3.3 one has

Proposition 5.1. *The reduced Jacobi curve of the regular abnormal extremal γ has the weight jump one at a point λ iff*

$$\dim\mathcal{T}_\xi^{(n-4)}(q)=\dim\mathcal{T}_\xi^{(n-3)}(q)=n-2,\qquad \dim\mathcal{T}_\xi^{(n-2)}(q)=n-1, \tag{5.3}$$

where $\xi=\pi(\gamma)$ is the abnormal trajectory corresponding to γ and $q=\pi(\lambda)$.

Further note that by Proposition 3.2 the weight of these curves in the punctured neighborhoods of these points is equal to $(n-3)^2$ and therefore it is greater than 1 in the considered cases. As a direct consequence of Lemma 5.1 and Proposition 3.7 we obtain the following

Proposition 5.2. *If the sets $\mathcal{R}_D(q)$ and $S_D^0(q)$ are not empty, then the fundamental form of the distribution D at the point q is a rational function, which is not a polynomial: all points of $S_D^0(q)$ are the points of discontinuity of it.*

Example. Consider the distribution $\widetilde{D}$ in $\mathbb{R}^6$ spanned by the following vector fields

$$X_1 = \frac{\partial}{\partial x_1}, \qquad X_2 = \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3} + \frac{x_1^2}{2} \frac{\partial}{\partial x_4} + \left(\frac{x_1^4}{4!} + \frac{x_1^2 x_2}{2}\right) \frac{\partial}{\partial x_5} + x_1 x_2 \frac{\partial}{\partial x_6}. \tag{5.4}$$

Distribution $\widetilde{D}$ has the maximal possible small growth vector $(2, 3, 5, 6)$ at any point. We claim that its fundamental form at 0 is a rational function, which is not a polynomial. Indeed, by Proposition 3.6, the set $\mathcal{R}_{\widetilde{D}}(0)$ is not empty. It is not hard to show that the curve $(x_1, 0, \dots, 0)$ is a regular abnormal trajectory of corank 1. Moreover, from Proposition 5.1 it follows that the reduced Jacobi curve of any its lift γ has the weight jump one at the point of intersection of γ with $(\widetilde{D}^2)^\perp(0)$, which implies that $S^0_{\widetilde{D}}(0)$ is not empty. Now our claim follows from Proposition 5.2.

In general the set $S^0_D(q)$ could be empty, but it turns out that for $n > 5$, after an appropriate complexification of the fibers of the cotangent bundle, a natural complex analogue of the set $S^0_D(q)$ is not empty for generic germ of $(2, n)$-distribution at q. Let us describe this complex analogue of $S^0_D(q)$. First note that the mappings $\lambda \mapsto \mathcal{J}^{(i)}(\lambda)$, $\lambda \in (D^2)^\perp(q)$, depend rationally on λ and therefore can be rationally continued to $(D^2)^\perp(q)^{\mathbb{C}}$ (after this continuation we look on $\mathcal{J}^{(i)}(\lambda)$ as on complex linear spaces).

Let

$$S^0_D(q)^{\mathbb{C}} = \left\{\lambda \in (D^2)^\perp(q)^{\mathbb{C}}\colon\ \dim \mathcal{J}^{(n-4)}(\lambda) = \dim \mathcal{J}^{(n-3)}(\lambda) = 2n - 5,\ \dim \mathcal{J}^{(n-2)}(\lambda) = 2n - 4\right\} \tag{5.5}$$

(here all dimensions are over $\mathbb{C}$). Then, using a complex analogous of Lemma 5.1, one can prove that the statement of Proposition 5.2 remains true if one replace $S^0_D(q)$ by $S^0_D(q)^{\mathbb{C}}$. Moreover, similarly to the proof of Proposition 3.4, one can prove that in the case $n > 5$ for a generic germ of $(2, n)$-distribution D at q_0 the set $S^0_D(q_0)^{\mathbb{C}}$ is not empty. Due to the limit of space, we omit the proof of this fact, referring the reader to the preprint [21] (Proposition 4.5 there). Finally we obtain the following

Theorem 5. *In the case $n > 5$ a generic germ of $(2, n)$-distribution D at q has the fundamental form, which is a nonpolynomial rational function on $(D^2)^\perp(q)$.*

Acknowledgements

I would like to thank professor A. Agrachev for his constant attention to this work, stimulating discussions, and valuable remarks, professor B. Doubrov for introducing me to the Tanaka theory. Also I am very grateful to professors R. Bryant and R. Montgomery for critical remarks and useful advises concerning the exposition.

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